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# *A Note on the Groups $K_G(X)$, $K(X/G)$.*

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***Introduction.*** Let $G$ be a topological group and $X$ a compact $G$-space. The aim of this paper is to observe Kernel $\Pi^*$ and Kernel $\mathscr{F}$, where $\Pi^*$ is a homomorphism from $\widetilde{K}(X/G)$ to $\widetilde{K}(X)$ induced by the projection $\Pi : X \longrightarrow X/G$, and $\mathscr{F}$ is a forgetful homomorphism from $\widetilde{K}_G(X)$ to $\widetilde{K}(X)$.

1. Let $G$ be a topological group, $X$ a $G$-space and $U(\mathrm{n})$ the classical unitary group. Consider the space

$$\mathbf{B}_G{}^n = Map(G, e;\ U(n),\ I_n)$$

*of maps* $\theta : G \longrightarrow U(n)$ with $\theta(e) = I_n$ with compact open topology, where $e$ and $I_n$ are unit and unit matrix. Let $G$ act on this by

(1.1) $$(g\theta)(h) = \theta(hg)\,\theta(g)^{-1}$$

for $g, h, \in G$ and $\theta \in B_G{}^n$. Let $i_r : B_G{}^n \longrightarrow B_G{}^{n+r}$ be a natural inclusion induced by a natural inclusion $U(n) \subset U(n+r)$. We define the G-action on $E_G{}^n = B_G{}^n \times \boldsymbol{C}^n$ by

(1.2) $$g(\theta,\ v) = (g\theta,\ \overline{\theta(g)}v),$$

where $\overline{\theta(g)}$ is an conjugate matrix of $\theta(g)$. Then $E_G{}^n$ is a $G$-vector bundle and we have

(1.3) $$i_r{}^*E_G{}^{n+r} = E_G{}^n \oplus r$$

where $r$ is a product $G$-bundle $B_G{}^n \times \boldsymbol{C}^n$ with trivial $G$-module $\boldsymbol{C}^n$.

2 ***Definition.*** Two equivariant maps $f_i : X \longrightarrow B_G{}^n$ $(i = 0, 1)$ are equivalent, in symbols

$$f_0 \sim f_1$$

if there is a map $F : X \longrightarrow U(n)$ such that

(2.1) $$F(gx) = f_1(x)(g)\,F(x)\,f_0(x)(g)^{-1}$$

for any $x \in X$ and $g \in G$. Two equivariant maps $f_i : X \longrightarrow B_G{}^{n_i}$ $(i = 0, 1)$ are stably equivalent, in symbols

$$f_0 \underset{s}{\sim} f_1$$

if there are two integers $r_i$ such that

$$i_{r_0} f_0 \sim i_{r_1} f_1 \tag{2.2}$$

Moreover $f_i$ are stably equivalent with respect to the representation of $G$, in symbols

$$f_0 \underset{s.G}{\sim} f_1,$$

if there are two $G$-invariant points $\theta_i \in B_G^{r_i}$ such that

$$f_0 \oplus \theta_0 \sim f_1 \oplus \theta_1, \tag{2.3}$$

where $(f_i \oplus \theta_i)(x)(g) = f_i(x)(g) \oplus \theta_i(g)$. We shall write $\{f\}$ for the stable equivalence class of $f$. The set of such stable equivalence classes is denoted by

$$\{X, B_G\}, \tag{2.4}$$

where $B_G = \text{limit } B_G^n$. For the relation $\underset{s.G}{\sim}$ we shall write

$$\{f\}^G, \quad \{X, B_G\}^G . \tag{2.5}$$

Clearely the direct sum in matrices induces the semi group structures of $\{X, B_G\}$ and $\{X, B_G\}^G$.

**3.** Let $E$ be an $n$-dimentional complex vector bundle over $X$ such that $E$ is trivial as vector bundle, then we set

$$\Gamma_\varphi(x) = \begin{pmatrix} \varphi(x, e_1) \\ \vdots \\ \varphi(x, e_n) \end{pmatrix}$$

for some isomorphism $\varphi : X \times C^n \longrightarrow E$ and canonical base $e_1 \cdots\cdots e_n$ of $C^n$. We define a map

$$f_\varphi : X \longrightarrow B_G^n \quad \text{by} \quad f_\varphi(x)(g)\, g\Gamma_\varphi(x) = \Gamma_\varphi(gx) . \tag{3.1}$$

then we have

**4 Lemma.** *$f_\varphi$ is an equivariant map, Moreover if $\varphi' : X \times C^n \longrightarrow E$ is an another isomorphism then $f_\varphi \sim f_{\varphi'}$ .*

***Proof.*** From (1.1) and (3.1), we have

$$\begin{aligned} f_\varphi(gx)(h)\, h\, \Gamma_\varphi(gx) &= \Gamma_\varphi(hgx) = f_\varphi(x)(hg)\, hg\, \Gamma_\varphi(x) \\ &= f_\varphi(x)(hg) f_\varphi(x)(g)^{-1} h\Gamma_\varphi(gx) = (g f(x))(h)\, h\Gamma_\varphi(gx), \end{aligned}$$

so $f_\varphi$ is an equivariant map. Now for the last part, we define a map $F : X \longrightarrow U(n)$ by

$$\Gamma_{\varphi'}(x) = F(x)\Gamma_{\varphi}(x) \tag{4.1}$$

then we have

$$\begin{aligned} f_{\varphi'}(x)(g)F(x)f_{\varphi}(x)(g)^{-1}\Gamma_{\varphi}(gx) &= f_{\varphi'}(x)(g)F(x)g\Gamma_{\varphi}(x) \\ = f_{\varphi'}(x)(g)g\Gamma_{\varphi'}(x) &= F(gx)\Gamma_{\varphi}(gx), \end{aligned}$$

so $f_{\varphi} \sim f_{\varphi'}$ .

**5 *Lemma.*** *If $f_i : X \longrightarrow B_G{}^n\,(i=0,1)$ are equivalent, then $f_0{}^*E_G{}^n$ and $f_1{}^*E_G{}^n$ are $G$-*isomorphic.

***Proof.*** From **2**, there is a map $F : X \longrightarrow U(n)$ such that (2.1). Now we define $\varphi : f_0{}^*E_G{}^n \longrightarrow f_1{}^*F_G{}^n$ by

$$\varphi(x,\ v) = (X,\ \overline{F(x)v}) \tag{5.1}$$

Then $\varphi(g(x,\ v)) = \varphi(gx\ ,\ \overline{f_0(x)(g)v}) = (gx\ ,\ \overline{F(gx)f_0(x)(g)v}) = (gx,\ \overline{f_1(x)(g)F(x)v})$ $= g(x,\ \overline{F(x)v}) = g\varphi(x,\ v)$, so $\varphi$ is an equivariant isomorphism.

**6 *Lemma.*** *Let $E_i$ $(i=0,1)$ be $n$-dimentional complex $G$-vector bundles over $X$ such that $E_i$ are trivial as vector bundle. If there is a $G$-isomorphism $\psi : E_0 \longrightarrow E_1$, then*

$$f_{\varphi_0} \sim f_{\varphi_1}$$

*for any isomorphisms $\varphi_i : X \times C^n \longrightarrow E_i$ $(i=0,1)$.*

***Proof.*** We define a map $F : X \longrightarrow U(n)$ by

$$\psi_x\ \Gamma_{\varphi_1}(x) = F(x)\Gamma_{\varphi_0}(x) \tag{6.1}$$

Then we have

$$\begin{aligned} & f_{\varphi_1}(x)(g)F(x)f_{\varphi_0}(x)(g)^{-1}\Gamma_{\varphi_0}(gx) = f_{\varphi_1}(x)(g)F(x)g\Gamma_{\varphi_0}(x) \\ &= f_{\varphi_1}(x)(g)gF(x)\Gamma_{\varphi_0}(x) = f_{\varphi_1}(x)(g)g\psi_x\Gamma_{\varphi_1}(x) = gf_{\varphi_1}(x)(g)\psi_x\Gamma_{\varphi_1}(x) \\ &= g\psi_x f_{\varphi_1}(x)(g)\Gamma_{\varphi_1}(x) = \psi_{gx}gf_{\varphi_1}(x)(g)\Gamma_{\varphi_1}(x) = \psi_{gx}f_{\varphi_1}(x)(g)g\Gamma_{\varphi_1}(x) \\ &= \psi_{gx}\Gamma_{\varphi_1}(gx) = F(gx)\Gamma_{\varphi_0}(gx)\ , \end{aligned}$$

So $f\varphi_0 \sim f_{\varphi_1}$ .

**7 *Theorem.*** (M.F. ATIYAH [1]) *The category of $G$-vector bundles over free $G$-space $X$ is equivalent the category of vector bundles over $X/G$ under the projection $\Pi : X \longrightarrow X/G$.*

**8 *Theorem.*** *If $X$ is a free $G$-space, then there is a bijection*

$$\nu\ :\ \{X,\ B_G\} \longrightarrow Kernel\, \Pi^*,$$

*where $\Pi^*$ is a homomorphism from $\widetilde{K}(X/G)$ to $\widetilde{K}(X)$ induced by the projection $\Pi : X \longrightarrow X/G$.*

***Proof.*** *We define the map $\nu$ by*

(8.1) $$\nu(\{f\}) = \{f^*E_G{}^n/G\},$$

if $f$ is an equivariant map to $B_G{}^n$. From 5 and 7, $\nu$ is well defined, If $\{f_0{}^*E_G{}^n/G\} = \{f_1{}^*E_G{}^n/G\}$ for $f_i : X \longrightarrow B_G{}^n$ $(i = 0, 1)$, *then we have*

(8.2) $$f_0{}^*E_G{}^n/G \oplus \underline{\underline{m}} \cong f_1{}^*E_G{}^n/G \oplus \underline{\underline{m}} \text{ (as vector bundle)}$$

for some integer $m$. From (8.2), we have

$$\Pi^*(f_i{}^*E_G{}^n/G) \oplus \underline{\underline{m}} \cong f_i{}^*E_G{}^n \oplus \underline{\underline{m}} \cong f_i{}^*i_m{}^*E_G{}^{n+m},$$

as $G$-vector bundle. So, from 6, $i_m f_0 \sim i_m f_1$. Hence the map $\nu$ is one to one. Let $\{L\}$ be an element of $\widetilde{K}(X/G)$ such that $\Pi^*(\{L\}) = 0$. Then there are two integeres $m$ and $n$, and we have

$$\Pi^*L \oplus \underline{\underline{m}} \cong \underline{\underline{n}},$$

as vector bundle. Since $\Pi^*L \oplus m$ is a $G$-vector bundle, from 3 and 4, there is an equivariant map $f_\varphi : X \longrightarrow B_G{}^n$ for some isomorphism $\varphi : X \times C^n \longrightarrow \Pi^*L \oplus m$. Now we define
$\widetilde{\varphi} : f^*E_G{}^n \longrightarrow \Pi^*L \oplus \underline{\underline{m}}$ by $\widetilde{\varphi}(x, v) = \varphi(x, v)$. We set $f_\varphi(x)(g) = (f_{ij}(x, g))$, then

(8.3) $$\widetilde{\varphi}(g(x, e_i)) = \varphi(gx, \overline{f(x)(g)}e_i) = \varphi(gx, \sum_{l=1}^{n} \overline{f_{li}(x, g)}\, e_l)$$
$$= \sum_{l=1}^{n} \overline{f_{li}(x, g)}\, \varphi(gx, e_l)$$

From (3.1), we have

(8.4) $$\sum_{k=1}^{n} f_{lk}(x, g)\, g\varphi(x, e_k) = \varphi(gx, e_l)$$

and since $f_\varphi(x)(g)$ is in $U(n)$,

(8.5) $$\sum_{l=1}^{n} f_{lk}(x, g)\, \overline{f_{li}(x, g)} = \delta_{ki}.$$

Then, from (8.3), (8.4) and (8.5), we have

$$\widetilde{\varphi}(g(x, e_i) = \sum_{k=1}^{n} (\sum_{l=1}^{n} f_{lk}(x, g)\, \overline{f_{li}(x, g)}\, g\varphi(x, e_k))$$
$$= \sum_{k=1}^{n} \delta_{ki}\, g\varphi(x, e_k) = g\varphi(x, e_i) = g\widetilde{\varphi}(x, e_i).$$

Hence $\widetilde{\varphi}$ is an equivariant isomorphism. So we have

$$\nu(\{f\}) = \{f^*E_G{}^n/G\} = \{\Pi^*L \oplus m/G\} = \{\Pi^*L/G\} = \{L\},$$

therefor the map $\nu$ is onto.

**9 Corollary.** *If $X$ is a free $G$-space, then $\{X, B_G\}$ is an abelian group and $\nu$ is an isomorphism of abelian groups.*

10 **Theorem.** *There is an bijection*

$$\nu_0 : \{X, B_G{}^n\}^G \longrightarrow \textit{Kerenel } \mathscr{F} ,$$

*where $\mathscr{F}$ is a forgetful homomorphism from $\widetilde{K}_G(x)$ to $\widetilde{K}(X)$.*

***Proof.*** We define the map $\nu_0$ by

$$\nu_0(\{f\}^G) = \{f^*E_G{}^n\} , \tag{10.1}$$

if $f$ is an equivariant map to $B_G{}^n$. For any $G$-invariant point $\theta$ of $B_G{}^m$, we can write $(f \oplus \theta)^*E_G{}^{n+m} = f^*E_G{}^n \oplus X \times M$, where $M$ is a $G$-module defined by $\theta \in \boldsymbol{H}\mathrm{om}(G, U(m))$. So $\nu_0$ is well defined. Hence the same argument with 8 gives that $\nu_0$ is a bijection.

**11 Collorary.** *$\{X, B_G\}^G$ is an abelian group and $\nu_0$ is an isomorphism of abelian groups.*

**12 Collorary.** *If $G$ act freely on the $2n$-1 dimensional sphere $S^{2n-1}$, then*

$$\nu : \{S^{2n-1}, B_G{}^n\} \longrightarrow \widetilde{K}(S^{2n-1}/G) \tag{12.1}$$

*is an isomorphism.*

***Proof.*** Since $\widetilde{K}(S^{2n-1}) = 0$, it is trivial.

**13 *Theorem*** (Jon. FOLKMAN [2]) Let $Z_m$ be cyclic group of order $m$ with generater $T$. If $Z_m$ act on $S^{2n-1}$ by $T(\boldsymbol{Z}_1, \cdots, \boldsymbol{Z}_n) = (e(m)\boldsymbol{Z}_1, \cdots, e(m)\boldsymbol{Z}_n)$, where $e(m) = \exp 2\pi i/m$, then any equivariant map $f : S^{2n-1} \longrightarrow B^k{}_{Z_m}$ is stably equivalent to a constant equivariant map $\theta \in B^r{}_{Z_m}$

**14 Corollary.** *Under the condition in* 13, *We have*

$$\widetilde{K}_{Z_m}(S^{2n-1}) = 0 \textit{ and } \widetilde{K}(S^{2n-1}/\boldsymbol{Z}_m) = \{S^{2n-1} , (B_{Z_m})^{Z_m}\} .$$

***Proof.*** From 11, 12 and 14, it is trivial.

## References

[1]. M. F. ATIYAH and G. B. SEGAL, *Equivariant K-theory*, Lectures at Oxford univ. 1965,

[2]. JON FOLKMAN, *Equivariant maps of spheres into the classical group*, Memoirs of the American Mathematical Society Number 95. 1971,