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Alexander-Spanier Cochains of Degree $\infty - p$

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Introduction

The main purpose of this paper is to consider the volume elements of infinite dimensional spaces. In fact, we show the possibility of the construction of a volume element v on E_0, a separable infinite dimensional real Banach space with a normalized monotone basis $\{e_1, e_2, \cdots\cdots\}$ such that

$$\sum_{n=1}^{\infty} x_n e_n \in E_0, \quad if\ |x_n| \leqq c_n,$$

where $\{c_n\}$ is a series of (non zero) positive numbers with $lim._{n\to\infty} c_n = 0$, such that to set

$$[\sigma, \ \{c_n\}] = \{x \mid x = \sum_{n=1}^{\infty} x_n e_n, \ \ 0 \leqq x_n \leqq c_n\},$$

$v([\sigma, \ \{c_n\}]) = \int_{[\sigma, \ \{c_n\}]} v$ takes non zero finite value. (A Banach space $\boldsymbol{B}$ is called to have a basis if there is a countable set $\{b_n\}$ of $\boldsymbol{B}$ such that any element x of $\boldsymbol{B}$ can expressed uniquely as $x = \sum_n x_n b_n$. Much of Banach spaces such as $C(\Omega)$, $L^p(\Omega)$, $1 \leqq p < \infty$, etc, have basis. But there exist Banach spaces which have no basis ([8]). For the details about the bases of Banach spaces, we refer ([16]). We note that, of cause we have for this v,

$$v([\sigma, \ \{c'_n\}]) = \infty, \ \ if\ c_n = o(\{c'_n\}),$$
$$v([\sigma, \ \{c''_n\}]) = 0, \ \ if\ c''_n = o(\{c_n\}).$$

Since the volume element of a finite dimensional space M can be dfined by an Alexander-Spanier n-cochain on M, $n = dim.\ M$ (cf. [4]), we first define the Alexander-Spanier cochain of degree $\infty - p$ for a Banach manifold M modeled by

E_0 for this purpose. In this definition, first we note that since there are many possibilities of the definition of topology of the infinite product $E_0\times E_0\times\cdots\cdots$, there are many types of $(\infty-p)$-Alexander-Spanier cochains of M. Moreover, unlike in the finite degree case (cf. [1], [17]), the product space topology of $E_0\times E_0\times\cdots\cdots$ seems to be not appropriate in the definition of $(\infty-p)$-cochains. By this reason, we denote by $C^{\infty-p}{}_{U(\Delta(E))}(M)$ the space of $(\infty-p)$-cochains on M by a fixed topology of $E_0\times E_0\times\cdots\cdots$determined by $U(\Delta(E))$. (The meaning of $U(\Delta(E))$ is as follows: First we consider $E_0\times E_0\times\cdots\cdots$to be the space of linear maps from $\bigcup_{n=1}^{\infty}E_n$, E_n is the space spanned by $\{e_0, e_1,\cdots\cdots, e_n\}$ and it is considered to be a subspace of E, the space spanned by $\{e_0, e_1, e_2\cdots\cdots\}$, into E_0. Then, by this correspondence, the diagonal element $(a, a, a,\cdots\cdots)$ of $E_0\times E_0\times\cdots\cdots$corresponds to the operator Δ_a defined by

$$\Delta_a(x)=\Big(\sum_i x_i\Big)a,\quad x=\sum_i x_ie_i,$$

and to set $\Delta(E)=\{\Delta_a\,|\,a\in E_0\}$, $U(\Delta(E))$ is a suitable subset of $L\Big(\bigcup_nE_n,\ E_0\Big)$contains $\Delta(E)$. The examples of $U(\Delta(E))$ are as follows:

$$U(\Delta(E))=\{T+\Delta_a\,|\,T\in l^p(E,\ E_0)\},\quad 1\leqq p<\infty,$$
$$U(\Delta(E))=\{T+\Delta_a\,|\,T\in L(E,\ E_0)\},$$

where $L(E,\ E_0)$ is the space of bounded operators from E into E_0 and $l^p(E,\ E_0)$ is given by

$$l^p(E,\ E_0)=\Big\{\mathrm{T}\,|\sum_{i,j}|c_{ij}|^p<\infty,\ \ T(e_i)=\sum_j c_{ij}e_j\Big\}.$$

We denote the above two types of $U(\Delta(E))$ by $U_p(\Delta(E))$ and $U_b(\Delta(E))$. For some types of $U(\Delta(E))$ such as $U_p(\Delta(E))$ or $U_b(\Delta(E))$, we can define the coboundary maps for the elements of $C^{\infty-p}{}_{U(\Delta(E))}(M)$ by means of Abelian sum (cf. [12]). For example, if $f\in C^{\infty-1}{}_{U(\Delta(E))}(M)$, then we set

$$\delta_a f(\lambda)=\sum_{n=1}^{\infty}\lambda^a(-1)^ne^{-n\lambda}f(x_0,\ x_1,\cdots\cdots,\ x_{n-1},\ x_{n+1},\cdots\cdots),$$

where λ is a complex number with Re. $\lambda>0$ and α is a fixed complex number such that $0\leqq\mathrm{Re}.\,\alpha\leqq1$, and define the coboundary map δ_a by

$$\lambda_a f=\lim_{\lambda\to0}.\ \delta_a f(\lambda),$$

if the limit exists.

The definition of the integral of an $(\infty-p)$-cochain f is done along the same line as in [4]. But, although $f \in C^{\infty}{}_{U(\Delta(E))}(E_0)$ and the integral is considered on an cube of $\boldsymbol{E}_0$, such as $[\sigma, \{c^n\}]$, the set of partıtions $\{J\}=\{(j_1, j_2, \cdots\cdots)\}$ should be an infinite set although for each k, $0<x_{1,k}<\cdots\cdots<x_{m_k,k}<c_k$ is a finite partition. By this reason, we define $\int_{[\sigma, \{c_n\}]} f$ by the limit

$$\int_{[\sigma, \{c_n\}]} f$$
$$=\lim_{|x_{k, p_q}-x_{k, p_q+1}|\to 0} \lim_{m\to\infty} \sum_{k=1}^{m} \Big(\sum_{J\in(J)_k} f(x_j, x_{j+1_1}, x_{j+1_2}, \cdots\cdots)\Big),$$
$$(J)_k=[J]_k-[J]_{k-1}, \quad [J]_k=\{J \mid J=(j_1, \cdots\cdots, j_k, 0, \cdots\cdots)\},$$
$$J+1_i=(j_1, \cdots\cdots, j_{i-1}, j_i+1, j_{i+1}, \cdots\cdots), \quad x_{m_i+1, i}=c_i,$$
$$x_J=\sum_n x_{i_n, n} e_n,$$

(cf. [10], [14], [19]). To snow the existence of this limit, we assume $f=f(x_0, x_1, x_2, \cdots\cdots)$ is Frécnet-derivable for each x_k, $k\geqq 1$, and assume

$$|f_m(x_0, x_1, x_2, \cdots\cdots, t)|\leqq t^{-m}\frac{M}{H(\frac{1}{t})}, \quad t>0,$$

$$H(t)=\sum_{n=1}^{\infty} c_1 c_2 \cdots\cdots c_n t^n,$$
$$f_m(x_0, x_1, x_2, \cdots\cdots, t)$$
$$=<d_{x_m}<d_{x_{m-1}}<\cdots\cdots<d_{x_1} f(x_0, \cdots\cdots, x_0, x_0+t(x_{m-1}-x_0), \cdots\cdots),$$
$$\cdots\cdots>, (x_1-x_0)>, (x_{m-1}-x_0)>, (x_m-x_0)>,$$

where d_{x_k} means the Fréchet derivation in x_k. Then, to have the meaning of this inequality for small t, the series $\{c_n\}$ should be tend to 0, and ın this case, we have $|\int_{[\sigma, \{c_n\}]} f|<2M$. Moreover, if each $f_m>0$ and

$$f_m(x_0, x_1, x_2, \cdots\cdots, t)\geqq t^{-m}\frac{M'}{H(\frac{1}{t})}, \quad t>0,$$

is hold for each m for some $M'>0$, we also get $\int_{[\sigma, \{c_n\}]} f>M'$. Therefore, in this case, we get a volume element of E_0 and by this volume element, $[\sigma, \{c_n\}]$ has finite non-zero volume. But, under the same assumption, we also have

$$\int_{[\sigma, \{c'_n\}]} f=\infty, \quad if\ c_n=o(\{c'_n\}),$$
$$\int_{[\sigma, \{c''_n\}]} f=0, \quad if\ c''_n=o(\{c_n\}),$$

although $[\sigma, \{c'_n\}]$ and $[\sigma, \{c''_n\}]$ are both defined and bounded open sets of $\boldsymbol{E}_0$

and both homeomorphic to $[\sigma, \{c_n\}]$ (*cf.* [2], [13]). The integration of $f \in C^{\infty-p}_{U(\Delta(E))}(M)$ on an $(\infty-p)$-chain of M is also done similarly as in [4].

Since $\partial[\sigma, \{c_n\}]$ is, if defined, an infinite chain, and therefore $\int_{\partial[\sigma, \{c_n\}]} f$ can not be defined directly, we set

$$\partial_a[\sigma, \{c_n\}](\lambda)$$
$$=\sum_{m=0}^{\infty} \lambda^a(-1)^m e^{-m\lambda} ([\sigma, \{c_n\}]_{x_{m+1}=0} - [\sigma, \{c_n\}]_{x_{m+1}=c_{m+1}}).$$

Then, by virtue of Stokes' theorem for the integrals of alternative Alexander-Spanier cochains of finite degree ([4]), we can show for the alternative f, the Stokes' theorem

$$\lim_{\lambda\to 0} \int_{\partial_a[\sigma, \{c_n\}](\lambda)} f = \lim_{\lambda\to 0} \int_{[\sigma, \{c_n\}]} \delta_a f(\lambda),$$

if the limits of both sides exist. Similarly, to define $\partial_a \gamma(\lambda)$ by using $\partial_a[\sigma, \{c_n\}](\lambda)$, we also get

$$\lim_{\lambda\to 0} \int_{\partial_a\gamma(\lambda)} f = \lim_{\lambda\to 0} \int_{\gamma} \delta_a f(\lambda).$$

Simbolically, we may also write these Stokes' theorems as follows:

$$\int_{\partial[\sigma, \{c_n\}]} f = (\int_{[\sigma, \{c_n\}]} \delta_a f)\lambda^{-a} + o(|\lambda|^{\mathrm{Re.}\, a}),$$
$$\int_{\partial\gamma} f = (\int_{\gamma} \delta_a f)\lambda^{-a} + o(|\lambda|^{\mathrm{Re.}\, a}).$$

Here, $\partial[\sigma, \{c_n\}]$ means the formal sum $\sum_{m-0}^{\infty}(-1)^m([\sigma, \{c_n\}]_{x_{m+1}=0} - [\sigma, \{c_n\}]_{x_{m+1}=c_{m+1}})$. If $\alpha=0$, the above simbolical expression may be reduced to

$$\int_{\partial[\sigma, \{c_n\}]} f = \int_{[\sigma, \{C_n\}]} \delta_0 f,$$
$$\int_{\partial\gamma} f = \int_{\gamma} \delta_0 f.$$

We note that by this Stokes' theorem, there should be exist non-exact closed $(\infty-1)$-cochain on $\partial[\sigma, \{c_n\}]$ (with respect to some $U(\Delta(E))$) although $\partial[\sigma, \{c_n\}]$ is homeomorphic to $[\sigma, \{c_n\}]$ ([6], [7]).

We also note that, unlike in the finite degree case, the $(\infty-p)$-Alexander-Spanier cochains, or more general, the $(\infty-p)$-cohomologies of Banach manifolds (cf. [3], [7], [11], [15]), are not topological objects. Or, in other word, the geometry of Banach manifolds seems to be not based on topology.

The outline of this paper is as follows: In § 1, we define $(\infty-p)$ -cochains. The coboundary operators and related topics are stated in § 2. The integration of ∞ -cochains is defined in § 3. § 4 is devoted to the definition of the integration of $(\infty-p)$ -cochains. In § 5, we prove Stokes' theorem.

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§ 1. Definition of $(\infty-p)$ -cochains

1. Let $\boldsymbol{E}$ be a separable Banach space (over $\boldsymbol{R}$ or $\boldsymbol{C}$) with a normalized monotone basis $\{e_0, e_1, e_2, \cdots\cdots\}$, that is, for every $x \in \boldsymbol{E}$, there is unique series of scalar $\{x_k\}$ such that

$$x=\sum_{k=0}^{\infty} x_k e_k,$$

$$\left|\left|\sum_{k=0}^{n} x_k e_k\right|\right| \leqq \left|\left|\sum_{k=0}^{m} x_k e_k\right|\right|, \quad n<m,$$

and $||e_k||=1$ for each k. We denote by $\boldsymbol{E}_n$, $n \geqq 1$, the subspace of $\boldsymbol{E}$ spanned by $\{e_0, e_1, \cdots\cdots, e_n\}$, and the subspace of $\boldsymbol{E}$ spanned by $\{e_1, e_2, \cdots\cdots\}$ is denoted by $\boldsymbol{E}_0$.

We note that under this assumption, if T is a bounded linear operator from $\boldsymbol{E}$ into $\boldsymbol{E}$, then

$$||T||=\lim_{n\to\infty}. \sup_{||x||\leqq 1}. \left|\left|\sum_{k=0}^{n} x_k T(e_k)\right|\right|, \quad x=\sum_k x_k e_k.$$

Definition. *We denote by $L(\bigcup \boldsymbol{E}_n, \boldsymbol{E}_0)$ the space of all linear maps (not necessarilly bounded) from $\bigcup \boldsymbol{E}_n$, the subspace of $\boldsymbol{E}$ consisted by those x that $x_k=0$ with finite excption, into $\boldsymbol{E}_0$ with the compact open topology. Then we set $\Delta(\boldsymbol{E})$ the subspace of $L(\bigcup \boldsymbol{E}_n, \boldsymbol{E}_0)$ given by*

$$\Delta(\boldsymbol{E})=\left\{\Delta_a \,|\, \Delta_a(x)=(\sum_k x_k)a, \quad x=\sum_k x_k e_k, \quad a\in \boldsymbol{E}_0\right\}.$$

By definition, we may identify $\Delta(\boldsymbol{E})$ and a by the correspondence $a \to \Delta_a$. We note that if $a \neq 0$, then Δ_a may not be defined on $\boldsymbol{E}$ unless $\boldsymbol{E}=l^1$ and $\{e_0, e_1\ e_2, \cdots\cdots\}$ is its natural basis.

We take a subset $U(\Delta(\boldsymbol{E}))$ of $L(\bigcup \boldsymbol{E}_n, \boldsymbol{E}_0)$ with a fixed topology such that (i). *The topology of* $U(\Delta(\boldsymbol{E}))$ *is not weaker than the induced topology of* $U(\Delta(\boldsymbol{E}))$ *from* $L\ (\bigcup \boldsymbol{E}_n,\ \boldsymbol{E}_0)$.

(ii). $\Delta(\boldsymbol{E}) \cap U(\Delta(\boldsymbol{E}))$ *is dense in* $\boldsymbol{E}_0$ *by the (strong) topology of* $\boldsymbol{E}_0$.

Example 1. We set $T(e_i)=\sum_j c_{ij}\ e_j$ and set

$$l^p(\boldsymbol{E},\ \boldsymbol{E}_0)=\left\{T\,|\sum_{i,\ j}|c_{ij}|^p<\infty\right\},\ 1\leqq p<\infty.$$

Then we set $U(\Delta(\boldsymbol{E}))$ as the subset of $L(\bigcup \boldsymbol{E}_n,\ \boldsymbol{E}_0)$ to be

$$U(\Delta(\boldsymbol{E}))=\{S\,|\,S=T+\Delta_a,\ T\in l^p(\boldsymbol{E},\ \boldsymbol{E}_0),\ a\in \boldsymbol{E}_0\}.$$

By definition, we get $U(\Delta(\boldsymbol{E}))=l^p(\boldsymbol{E},\ \boldsymbol{E}_0)\times\Delta(\boldsymbol{E})$ because Δ_a does not contained in $l^p(\boldsymbol{E},\ \boldsymbol{E}_0)$ unless $a=0$. Since we can consider $l^p(\boldsymbol{E},\ \boldsymbol{E}_0)$ to be the Banach space by the l^p-norm and $\Delta(\boldsymbol{E})$ is the Banach space by the norm of $\boldsymbol{E}_0$, we define the topology of $U(\Delta(\boldsymbol{E}))$ by the Banach space topology given by the product structure. This $U(\Delta(\boldsymbol{E}))$ is denoted by $U_p\ (\Delta(\boldsymbol{E}))$.

By definition, if $p=2$ and $\boldsymbol{E}$ is a Hilbert space, then $U_2(\Delta(\boldsymbol{E}))$ is also a Hilbert space.

Example 2. We denote by $L(\boldsymbol{E},\ \boldsymbol{E}_0)$ the Banach space of bounded linear operators from $\boldsymbol{E}$ into $\boldsymbol{E}_0$. Then, since $||\Delta_a||=(sup._{||x||=1}|\sum_k x_k|)||a||$, we have

$$L(\boldsymbol{E},\boldsymbol{E}_0)\ \cap\ \Delta(\boldsymbol{E})=\{\Delta_0\},\ \text{or}\ \Delta(\boldsymbol{E}).$$

In the first case, we set

$$U(\Delta(\boldsymbol{E}))=\{S\,|\,S=T+\Delta_a,\ T\in L(\boldsymbol{E},\ \boldsymbol{E}_0),\ a\in \boldsymbol{E}_0\},$$

and give the product space topology of $L(\boldsymbol{E},\ \boldsymbol{E}_0)\times\Delta(\boldsymbol{E})$ to $U(\Delta(\boldsymbol{E}))$ and in the second case, we set $U(\Delta(\boldsymbol{E}))=L(\boldsymbol{E},\ \boldsymbol{E}_0)$ as the topological space. These $U(\Delta(\boldsymbol{E}))$ are denoted by $U_b(\Delta(\boldsymbol{E}))$.

Note. These examples, $U(\Delta(\boldsymbol{E}))$ satisfy the stronger condition

(ii)$'$. $U(\Delta(\boldsymbol{E}))$ *contains* $\Delta(\boldsymbol{E})$.

2. ***Definition.*** *To fix* $U(\Delta(\boldsymbol{E}))$, *a germ of continuous function* f *defined on some dense subset of a neighborhood of* $\Delta(\boldsymbol{E})$ *in* $U(\Delta(\boldsymbol{E}))$ *(by the topology of* $U(\Delta(\boldsymbol{E}))$*) such that*

$$f(\Delta_a)=0,\ a\in \boldsymbol{E}_0,\ f \text{ is defined at } \Delta_a,$$

at $\Delta(\boldsymbol{E})$ *is called an Alexander-Spanier cochain of degree* ∞ *of* $\boldsymbol{E}_0$ *with respect to* $U(\Delta(\boldsymbol{E}))$.

We denote by $\overline{f}$ or f the germ of f. The set of Alexander-Spanier cochains of degree ∞ of $\boldsymbol{E}_0$ with respect to $U(\varDelta(\boldsymbol{E}))$ is denoted by $C^{\infty}{}_{U(\varDelta(\boldsymbol{E}))}(\boldsymbol{E}_0)$.

Definition. *An Alexander-Spanier cochain of degree* ∞ *of* $\boldsymbol{E}_0$ *with respect to* $U(\varDelta(\boldsymbol{E}))$, $\overline{f}$ *is called standard if* $U(\varDelta(\boldsymbol{E}))$ *satisfies the condition* (ii)' *and* f, *a representative of* $\overline{f}$, *is defined on some neighborhood of* $\varDelta(\boldsymbol{E})$ *in* $U(\varDelta(\boldsymbol{E}))$.

The set of standard Alexander-Spanier cochains of degree ∞ of $\boldsymbol{E}_0$ with respect to $U(\varDelta(\boldsymbol{E}))$ is denoted by $s\text{-}C^{\infty}{}_{U(\varDelta(\boldsymbol{E}))}(\boldsymbol{E}_0)$, or simply, by $C^{\infty}{}_{U(\varDelta(\boldsymbol{E}))}(\boldsymbol{E}_0)$.

Definition. *We use following terminologies. Where* f *is a representative of* $\overline{f}$ *and* T, *etc., appeared in the definitions, are assumed to belong in* $U(\varDelta(\boldsymbol{E}))$.

(i). $\overline{f}$ *is* 0-*normal if* $f(T)=0$, *where* $T(e_k)=T(e_0)$ *for some* $k\neq 0$.

(ii). $\overline{f}$ *is normal if* $f(T)=0$, *where* $T(e_i)=T(e_j)$ *for some* $i\neq j$.

(iii). $\overline{f}$ *is regular if* $f(S)=0$, *where* S *is written uniquley* $T+\varDelta_a$ in $U(\varDelta(\boldsymbol{E}))$ *and* $ker.\ T\neq\{0\}$.

(iv). $\overline{f}$ *is alternative it* $f(\mathfrak{t}_i T)=-f(T)$ *for any* i, *where*

$$\mathfrak{t}_i T(e_j)=T(e_j),\quad j\neq i,\quad i+1,$$
$$\mathfrak{t}_i T(e_i)=T(e_{i+1}),\quad \mathfrak{t}_i T(e_{i+1})=T(e_i).$$

(v). $\overline{f}$ *is differentiable if* $U(\varDelta(\boldsymbol{E}))$ *allows a (fixed) differential structure and* f *is differentiable by this structure. Similarly, Lipschitz continuous Alexandes-Spanier cochain is also defined.*

(vi). $\overline{f}$ *is positive if* f *is real valued and* $f\geqq 0$.

***Lemma* 1.** *If* $S=T+\varDelta_a$, *then*

(1) $S(e_i)=S(e_j)$ *for some* $i\neq j$ *if and only if* $T(e_i)=T(e_j)$ *for some* $i\neq j$.

(2) $\mathfrak{t}_i S=\mathfrak{t}_i T+\varDelta_a$.

***Lemma* 2.** *There exists non-trivial standard regular (or normal or* 0-*normal) Alexander-Spanier cochain of degree* ∞ *of* $\boldsymbol{E}_0$ *with respect to* $U_2(\varDelta(\boldsymbol{E}))$.

Proof. *Since* $U_2(\varDelta(\boldsymbol{E}))=l^2(\boldsymbol{E},\ \boldsymbol{E}_0)\times\boldsymbol{E}_0$ and Whitney extension theorem is hold for real valued differentiable functions on $l^2(\boldsymbol{E},\ \boldsymbol{E}_0)$ ([18]), we have the lemma.

***Theorem* 1.** *There exists non-trivial standard alternative regular Lipschitz continuous Alexander-Spanier cochain of degree* ∞ *of* $\boldsymbol{E}_0$ *with respect to* $U_2(\varDelta(\boldsymbol{E}))$ *if* $\boldsymbol{E}$ *is a Hilbert space.*

Proof. We denote the space of bounded C^1-functions with bounded derivatives on $U_2(\varDelta(\boldsymbol{E}))$ *by* $C^1(U_2(\varDelta(\boldsymbol{E})))$ and its closure in $C(U_2(\varDelta(\boldsymbol{E}))$, the Banach space of bounded continuous functions on $U_2(\varDelta(\boldsymbol{E}))$ with $\|f\|=sup._T|f(T)|$, by $\overline{C}^1(U_2(\varDelta(\boldsymbol{E})))$. $\overline{C}^1(U_2(\varDelta(\boldsymbol{E}))$ is an infinite dimensional Banach space by a theorem *of Wells* ([18]).

On $\overline{C}^1(U_2(\varDelta(\boldsymbol{E})))$, we define operators A_n, $n=2,\ 3,\cdots\cdots$, by

$$A_n f(T)=\frac{1}{n!}\sum_{\mathfrak{s}\in\mathfrak{S}_n}\operatorname{sgn}(\mathfrak{s})f(\mathfrak{s}T).$$

Here $\mathfrak{S}_n$ is the symmetric group of n-letters $\{0, 1, \cdots\cdots, n\text{-}1\}$ and $\mathfrak{s}T$ is given by

$$\mathfrak{s}T(e_i)=T(e_{\mathfrak{s}(i)}),\ 0\leqq i\leqq n-1,$$
$$\mathfrak{s}T(e_i)=T(e_i),\ i\geqq n.$$

By definition $\mathfrak{s}$ maps $U_2(\varDelta(\boldsymbol{E}))$ onto $U_2(\varDelta(\boldsymbol{E}))$ because $\mathfrak{s}\varDelta_a=\varDelta_a$ and each A_n is a bounded linear operator with $||A_n||=1$. Moreover, since

$$\begin{aligned}&|A_nf(T_1)-A_nf(T_2)|\\ &\leqq\frac{1}{n!}\sum_{\mathfrak{s}\in\mathfrak{S}_n}|f(\mathfrak{s}T_1)-f(\mathfrak{s}T_2)|\\ &\leqq\frac{1}{n!}\sum_{\mathfrak{s}\in\mathfrak{S}_n}M||\mathfrak{s}T_1-\mathfrak{s}T_2||\\ &=\frac{1}{n!}\sum_{\mathfrak{s}\in\mathfrak{S}_n}M||T_1-T_2||=M||T_1-T_2||,\end{aligned}$$

where M is the bound of the derivative of f if $f\ni C^1(U_2(\varDelta(\boldsymbol{E})))$, $\{A_nf\}$ is equicontinuous if $f\in C^1(U_2(\varDelta(\boldsymbol{E})))$. Hence by the theorem of Ascoli-Arzéla, there exists a subsepuence $\{A_{n'}f\}$ of $\{A_nf\}$ such that $\{A_{n'}f\}$ converges in $\overline{C^1}(U_2(\varDelta(\boldsymbol{E})))$. Then, since $C^1(U_2(\varDelta(\boldsymbol{E})))$ is dense in $\overline{C^1}(U_2(\varDelta(\boldsymbol{E})))$ and $C_1(U_2(\varDelta(\boldsymbol{E})))$ is separable, we can choose a subsequence $\{A_{n''}\}$ of $\{A_n\}$ such that $\{A_{n''}f\}$ converges in $\overline{C^1}(U_2(\varDelta(\boldsymbol{E})))$ if f belongs in some dense subset of $\overline{C^1}(U_2(\varDelta(\boldsymbol{E})))$. Then, by the theorem of Banach-Steinhauss ([5]), $\{A_{n''}\}$ converges to a bounded operator A on $\overline{C^1}(U_2(\varDelta(\boldsymbol{E})))$, because $||A_{n''}||=1$ for each n''.

By the definition of A_n, for this A, we get

(3) $\quad Af(\mathfrak{s}T)=\operatorname{sgn}(\mathfrak{s})f(T),\ \mathfrak{s}\in\mathfrak{S}_n,$ *n is arbitraly,*

(4) $\quad |Af(T_1)-Af(T_2)|\leqq M||T_1-T_2||,$ *if* $|f(T_1)-f(T_2)|\leqq M||T_1-T_2||.$

To show $A\neq 0$, we set

$$e(\mathfrak{s})=\max_k.\ \{(0,\cdots\cdots, k-1)\mid \mathfrak{s}(i)=i,\ 0\leqq i\leqq k-1\},$$

for $\mathfrak{s}\in\mathfrak{S}_n$. Then for $T\in l^2(\boldsymbol{E}, \boldsymbol{E}_0)$ given by

$$T(e_k)=\frac{1}{k+1}e_{k+1},$$

we define a (conitnuous) function on $\{\mathfrak{s}T|\mathfrak{s}\in\bigcup_{n\geqq 1}\mathfrak{S}_n\}$ by

$$A_n f(T)$$
$$=\frac{1}{n!}((n!-(n-1)!)+\frac{1}{2}((n-1)!-(n-2)!)+\cdots\cdots+\frac{1}{n^2}),$$

we have $Af(T)=1$. Hence $A \neq 0$ and we have the theorem.

3. We denote by $\boldsymbol{I}=\boldsymbol{I}_p$, $\boldsymbol{J}=\boldsymbol{J}_q$, etc., the index sets $\{i_1, \cdots\cdots, i_p\}$ $(0 \leq i_1 < \cdots\cdots < i_p)$, $\{j_1, \cdots\cdots, j_q\}$ $(0 \leq j_1 < \cdots\cdots < j_q)$, *etc.*, and set

(5) $$\boldsymbol{E}_{\boldsymbol{I}}=\{x \mid x \in \boldsymbol{E},\ x=\sum_k x_k e_k,\ x_{i_1}=\cdots\cdots=x_{i_p}=0\},$$

etc.. For $\boldsymbol{E}_{\boldsymbol{I}}$, we define a (continuous) isomorphism $\tau_{\boldsymbol{I}}=\tau_{\boldsymbol{I}_p} : \boldsymbol{E}_{\boldsymbol{I}} \to \boldsymbol{E}$ by

(6) $$\tau_{\boldsymbol{I}}(e_0)=e_0, \cdots\cdots,\ \tau_{\boldsymbol{I}}(e_{i_1-1})=e_{i_1-1},\ \tau_{\boldsymbol{I}}(e_{i_1+1})=e_{i_1}, \cdots\cdots,$$
$$\tau_{\boldsymbol{I}}(e_j)=e_{j-r},\ (i_r<j<i_{r+1}), \cdots\cdots,\ \tau_{\boldsymbol{I}}(e_{i_p-1})=e_{i_p-p},$$
$$\tau_{\boldsymbol{I}}(e_{i_p+1})=e_{i_p-p+1}, \cdots\cdots,\ \tau_{\boldsymbol{I}}(e_j)=e_{j-p}, (j>i_p).$$

If $p=q$, then we define $\iota^{\boldsymbol{I}}{}_{\boldsymbol{J}}=\iota^{\boldsymbol{I}_p}{}_{\boldsymbol{J}_p}$ by

(7) $$\iota^{\boldsymbol{I}_p}{}_{\boldsymbol{J}_p}=\tau_{\boldsymbol{I}_p}(\tau_{\boldsymbol{J}_p})^{-1} : \boldsymbol{E}_{\boldsymbol{I}_p} \to \boldsymbol{E}_{\boldsymbol{J}_p}.$$

By definition, we have $\iota^{\boldsymbol{I}}{}_{\boldsymbol{J}}\,\iota^{\boldsymbol{J}}{}_{\boldsymbol{K}}=\iota^{\boldsymbol{I}}{}_{\boldsymbol{K}}$ and the following diagram is commutative.

$$\begin{tikzcd}
 & \boldsymbol{E} \arrow[dd, "\tau_{\boldsymbol{J}}"] \arrow[dr, "\tau_{\boldsymbol{K}}"] & \\
\boldsymbol{E}_{\boldsymbol{I}_p} \arrow[ur, "\tau_{\boldsymbol{I}}"] \arrow[rr, "\iota^{\boldsymbol{I}}{}_{\boldsymbol{K}}"] \arrow[dr, "\iota^{\boldsymbol{I}}{}_{\boldsymbol{J}}"'] & & \boldsymbol{E}_{\boldsymbol{K}_p} \\
 & \boldsymbol{E}_{\boldsymbol{J}_p} \arrow[ur, "\iota^{\boldsymbol{J}}{}_{\boldsymbol{K}}"'] &
\end{tikzcd}$$

We set $\tau_{\boldsymbol{I}}^{-1}(\boldsymbol{E}_n)=\boldsymbol{E}_{\boldsymbol{I},\,n}$. Then $\tau_{\boldsymbol{I}}$ induces the (continuous) isomorphism $\tau_{\boldsymbol{I}}{}^* : L(\bigcup \boldsymbol{E}_n,\ \boldsymbol{E}_0) \to L(\bigcup \boldsymbol{E}_{\boldsymbol{I},\,n},\ \boldsymbol{E}_0)$.

Definition. *To fix $U(\Delta(\boldsymbol{E}))$, a germ of continuous funcion f defned on some dense subset of a neighbotrhood of $(\tau_{\boldsymbol{I}_p}{}^*)^{-1}$ $(\Delta(\boldsymbol{E}))$ in $(\tau_{\boldsymbol{I}_p}{}^*)^{-1}$ $(U(\Delta(\boldsymbol{E})))$ (by the topology of $U(\Delta(\boldsymbol{E})))$ such that*

$$f((\tau_{I_p}{}^*)^{-1}(\Delta_a))=0,\ a\in E_0,\ f \text{ is defined at } (\tau_{I_p}{}^*)^{-1}(\Delta_a),$$

at $\Delta_{I_p}(E)=(\tau_{I_p}{}^*)^{-1}(\Delta(E))$ *is called an Alexander-Spanier cochain of degree* $(\infty-p)$ *of* E_0 *with respect to* $U(\Delta(E))$ *(or* $U(\Delta_{I_p}(E))$ *and* I_p.

We denote the set of Alexander-Spanier cochains of degree $(\infty-p)$ of E_0 with respect to $U(\Delta(E))$ and I_p by $C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$ or $C^{\infty-p}{}_{U(I_{I_p}(E)),\ I_p}(E_0)$.

For the element of $C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$, we define standard, normal, alternative, differentiable, *etc.*, similarlly as ∞-cochain. Especially, the set of standard cochains of E_0, denoted by $s\text{-}C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$, or simplly, $C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$, form a module.

By definition, τ_I induces the isomorphism $\tau_I{}^{\#}: C^{\infty}{}_{U(\Delta(E))}(E_0)\to C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$. Hence we have the isomorphism $\iota^I{}_J{}^{\#}: C^{\infty-p}{}_{U(\Delta(E)),\ J}(E_0)\to C^{\infty-p}{}_{U(\Delta(E)),\ J}(E_0)$. Then, since

$$\iota^{I}{}_{J}{}^{\#}\iota^{J}{}_{K}{}^{\#}=\iota^{I}{}_{K}{}^{\#},$$

we can classify $\bigcup_{I_p} C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$ by

(8) $$\overline{f}\sim\overline{g} \text{ if and only if } \overline{g}=\iota^{I}{}_{J}{}^{\#}(\overline{f}),$$

$$\overline{f}\in C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0),\quad \overline{g}\in C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0),$$

and the set of this equivalence class can be represented by $C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0)$ for a (fixed) I_p.

Definition. *The above equivalence class of* $\{\overline{f}\}$, *may be denoted by* $\overline{f}$ *or* f, *is called an Alexander-Spanier cochain of degree* $(\infty-p)$ *of* E_0 *with respect to* $U(\Delta(E))$ *(or* $U(\Delta_{I_p}(E))$*) and the set of Allexander-Spanier cochains of degree* $(\infty-p)$ *of* E_0 *with respect to* $U(\Delta(E))$ *is denoted by* $C^{\infty-p}{}_{U(\Delta(E))}(E_0)$.

For the element of $C^{\infty-p}{}_{U(\Delta(E))}E_0$, we define standard, normal, alternative, differentiable, *etc.*, similarly as for the element of $C^{\infty-p}{}_{U(\Delta(E)),\ I}(E_0)$. Then the set of standard cochains of E_0, denoted by $s\text{-}C^{\infty-p}{}_{U(\Delta(E))}(E_0)$, or $C^{\infty-p}{}_{U(\Delta(E))}(E_0)$, form a module.

Note. Similarly, although $I_\infty=\{i_1,\ i_2,\cdots\cdots\}$ is an infinite set, if the complement of I_∞ in $\{0,\ 1,\ 2,\cdots\cdots\}$ is also an infinite set, we can define Alexander-Spanier cochains of degree $(\infty-\infty)$ of E_0 with respect to $U\Delta(E))$ and I_∞ and Alexander-Spanier cochain of degree $(\infty-\infty)$ of E_0 with respect to $U(\Delta(E))$. Their sets are denoted by $C^{\infty-\infty}{}_{U(\Delta(E)),\ I_\infty}(lE_0)$ and $C^{\infty-\infty}{}_{U(\Delta(E))}(E_0.)$

By the same way, we can also define $(\infty+p)$—cochain or $(\infty+\infty)$—cochain.

4. Let $M=\{(U_\alpha, h_\alpha)\}$ be a (paracompact) Banach manifold modeled by $\boldsymbol{E}_0$, that is, $\{U_\alpha\}$ is an open covering of M and for each α, h_α is an homeomorphism from U_α onto $\boldsymbol{E}_0$. Then we can define a homeomorphism $h_\alpha^*: U_\alpha\times U_\alpha\times\cdots\cdots \to L(\boldsymbol{E}_n, \boldsymbol{E}_0)$ (the topology of the infinite product $U_\alpha\times U_\alpha\times\cdots\cdots$is the weak topology) by

$$(9)\qquad h_\alpha^*((\xi_0, \xi_1, \cdots\cdots))=\{T,\ T(e_i)=h_\alpha(\xi_i),\ i=0,\ 1, \cdots\cdots\}.$$

By (9), to set $g_{\alpha\beta}^*=h_\alpha^*(h_\beta^*)^{-1}$, $g_{\alpha\beta}^*$ is a (continuous) isomorphism defined in $L(\bigcup E_n, E_0)$.

***Lemma* 3.** *$g_{\alpha\beta}^*$ maps $\Delta(\boldsymbol{E})$ into $\Delta(\boldsymbol{E})$.*

Proof. Since we know

$$T\Delta_a=\Delta_{T(a)},$$

we have the lemma. In fact, we have $h_\alpha^*((\xi_0, \xi_1, \cdots\cdots))=\Delta_a$ if and only if $\xi_0=\xi_1=\cdots\cdots=\xi$ and $a=h_\alpha(\xi)$. Then we get

$$g_{\alpha\beta}^*(\Delta_{h\beta(\xi)})=\Delta_{h\alpha(\xi)}.$$

Definition. *The collection $\{(U_\alpha\times U_\alpha\times\cdots\cdots, h_\alpha^*)\}$ is called local $\boldsymbol{E}$-product of M and denoted by $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$.*

Definition. *We set $\Delta(U_\alpha)=\{(\xi, \xi, \cdots\cdots) \mid \xi\in U_\alpha\}$* and call the element of $\{(\Delta_{U\alpha}, h_\alpha^*)\}$ in $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ *the diagonal element of* $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ *and the set of diagonal elements is denoted by $\Delta_{\boldsymbol{E}}(M)$.*

By lemma 3, *we may define $\Delta_{\boldsymbol{E}}(M)$* by

$$(10)\qquad \Delta_{\boldsymbol{E}}(M)=\bigcup_\alpha (h_\alpha^*)^{-1}(\Delta(\boldsymbol{E})).$$

Definition. *We fix $U(\Delta(\boldsymbol{E}))$. If g^* maps $U(\Delta(\boldsymbol{E}))$ into $U(\Delta(\boldsymbol{E}))$ for any (α, β), then we call $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ has the $U(\Delta(\boldsymbol{E}))$-structure.*

Example 1. $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ always have $U_b(\Delta(\boldsymbol{E}))$-structure.

Example 2. If M is a Hilbert manifold and $(e_1, e_2, \cdots\cdots)$ is an O. N.-basis of $\boldsymbol{E}_0$, then $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ has $U_2(\boldsymbol{E}))$-structure.

We assume $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ has the $U(\Delta(\boldsymbol{E}))$-structure. Then we set

$$(11)\qquad U(\Delta_{\boldsymbol{E}}(M))=\bigcup_\alpha (h_\alpha^*)^{-1}(U(\Delta(\boldsymbol{E}))).$$

We that note by (10), we have $\Delta_{\boldsymbol{E}}(M)\subset U(\Delta_{\boldsymbol{E}}(M))$.

Definition. *A germ of continuous function f defined on some dense subset of a* *neighborhood of* $\Delta_E(M)$ *in* $U(\Delta_E(M))$ *such that* f *vanishs on* $\Delta_E(M)$ *is called an Allexander-Spanier cochain of degree* ∞ *of* M *with respect to* $U(\Delta(\boldsymbol{E}))$ *and the set of Alexander-Spanier cochains of degree* ∞ *of* M *with respect to* $U(\Delta(\boldsymbol{E}))$ *is denoted by* $C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$.

As in n°2, we define standard, normal, alternative, differentiable (if M is a Banach differentiable manifold), *etc.*, for the element of $C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$ and the set of standard cochains of M, denoted by $s\text{-}C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$ or $C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$, form a module.

Similarlly as theorem 1, we have

Theorem **1'**. *If* M *is a differentiable Hilbert manifold and* $\mathfrak{U}(\Delta_E(M))$ *has the* $U_2(\Delta(\boldsymbol{E}))$*-structure, then there exists non-trivial alternative regular Lipschitz continuous Alexander-Spanier cochain of degree* ∞ *of* M *with respect to* $U_2(\Delta(\boldsymbol{E}))$.

Similarlly, to define $h_{I,\alpha}{}^* : U_\alpha \times U_\alpha \times \cdots\cdots \to L(\bigcup E_{I,n}, \boldsymbol{E}_0)$ by

$$h_{I,\alpha}{}^*((\xi_0, \xi_1, \cdots\cdots)) = T, \quad T(\tau_I{}^{-1}(e_i)) = h_\alpha(\xi_i), \tag{9$'$}$$

we define $\mathfrak{U}(\Delta_{E_I}(M))$ as the collection of $\{(U_\alpha \times U_\alpha \times \cdots\cdots, h_{I,\alpha}{}^*)\}$ and $\Delta_{E_I}(M)$ and $U(\Delta_{E_I}(M))$ are also defined similarlly as above. Then using $U(\Delta_{E_{I_p}}(M))$, we define Alexander-Spanier cochain of degree $(\infty - p)$ of M with respect to $U(\Delta(\boldsymbol{E}))$ and I_p as in n°3. Then since $h_{\alpha,\alpha}{}^* = h_\alpha{}^* \tau_I{}^{-1}$ and the diagram

$$\begin{tikzcd}
 & U_\alpha \times U_\alpha \times \cdots\cdots \arrow[dr, "h_{J,\alpha}{}^*"] & \\
L(\bigcup E_{I,n}, E_0) \arrow[ur, "h_{I,\alpha}{}^*"] \arrow[rr, "\iota^I_J"] \arrow[dr, "\tau_I{}^{-1}"'] & & L(\bigcup E_{J,n}, E_0) \\
 & L(\bigcup E_n, E_0) \arrow[uu, "h_\alpha{}^*"] \arrow[ur, "\tau_J{}^{-1}"'] &
\end{tikzcd}$$

is commutative, we can define $\iota^I p_{J_p} : \mathfrak{U}(\Delta_{E_{I_p}}(M)) \to \mathfrak{U}(\Delta_{E_{J_p}}(M))$, and therefore we can define Alexander-Spanier cochain of degree $(\infty - p)$ with respect to $U(\Delta(\boldsymbol{E}))$ as in n°3. The sets of Alexander-Spanier cochain of degree $(\infty - p)$ of M with respect to $U(\Delta(\boldsymbol{E}))$ and I_p or with respect to $U(\Delta(\boldsymbol{E}))$ are denoted by $C^{\infty - p}{}_{U(\Delta(\boldsymbol{E})),\, I_p}(M)$ or $C^{\infty - p}{}_{U(\Delta(\boldsymbol{E}))}(M)$.

As ∞-cochain, we define standard, normal, alternative, differentiable, *etc.*, for $(\infty - p)$-cochains and the sets of standard cochains, denoted by $s\text{-}C^{\infty - p}{}_{U(\Delta(\boldsymbol{E})),\, I}(M)$ and $s\text{-}C^{\infty - p}{}_{U(\Delta(\boldsymbol{E}))}(M)$, or $C^{\infty - p}{}_{U(\Delta(\boldsymbol{E})),\, I}(M)$ and $C^{\infty - p}{}_{U(\Delta(\boldsymbol{E}))}(M)$, form modules.

Note. As in n°3, we can also define Alexander-Spanier cochain of degree $\infty - \infty$ of M with respect to $U(\Delta(\boldsymbol{E}))$ and I_∞ or with respect to $U(\Delta(\boldsymbol{E}))$ if I_∞ is an infinite set and the complement of I_∞ in $\{0, 1, \cdots\cdots\}$ is also an infinite set. These sets are denoted by $C^{\infty-\infty}{}_{U(\Delta(E)), I}(M)$ or $C^{\infty-\infty}{}_{U(\Delta(E))}(M)$.

Similarlly, we can also define $(\infty+p)$-cochain or $(\infty+\infty)$-ochain, *etc.*.

§ 2. Operations on $C^{\infty-p}{}_{U(\Delta(E))}(M)$

5. By definition, we can consider the addition and the (scalar) product of the continuous functions on M for the elements of $C^{\infty-p}{}_{U(\Delta(E))}(M)$. Moreover, by the proof of theorem 1, we have

***Lemma* 4.** *If M is a differentiable Hilbert manifold, then the alternation operator A is defined on $D^{\infty-p}{}_{U(\Delta(E))}(M)$ and non-trivial if $U(\Delta(E))=U_2(\Delta(E))$. Here $D^{\infty-p}{}_{U(\Delta(E))}(M)$ means the germ of those continuous functions of $U(\Delta_{E_I}(M))$ at $\Delta_E(M)$ that can be uniformly approximated by C^1-class functions with bounded derivatives of $U(\Delta_{E_I}(M))$.*

To define the product of $(\infty-p)$-cochains *etc.*, first we define

Definition. *Let $I=\{i_1, i_2, \cdots\cdots\}$ and $J=\{j_1, j_2, \cdots\cdots\}$ be two index sets tuch that $I \cap J=\emptyset$. Then to set $E^I=\{x \mid x \in E,\ x=\sum_k x_k e_k,\ x_k=0,\ k \in I\}$, $E^I{}_n=\{x \mid x=\sum_{j=1}^{n} x_{ij} e_{ij},\}$ etc., for $T_1 \in L(\bigcup E^I{}_n, E_0)$, $T_2 \in L(\bigcup E^J{}_n, E_0)$, we define an element $T_1 \vee T_2$ of $(L\bigcup E^{I\cup J}{}_n, E_0)$ by*

$$(12)\qquad (T_1\vee T_2)(e_i)=T_1(e_i),\ i \in I,\ T_1\vee T_2)(e_j)=T_2(e_j),\ j \in J.$$

We note that if $T \in L(\bigcup E^{I\cup J}{}_n, E_0)$, then there exist $T_1 \in L(\bigcup E^I{}_n, E_0)$ and $T_2 \in L(\bigcup E^J{}_n, E_0)$ both uniquely, such that

$$T=T_1\vee T_2.$$

Starting from E^I, we can define the set of Alexander-Spanier cochains $C^I{}_{U(\Delta(E))}(M)$ by the same method as in § 1. If I is a finite set, then $U(\Delta^I(E))$ must be equal to $L(E^I, E_0)$ and therefore we write $C^d(M)$ in this case. Here $\Delta^I(E)$ is defined similarly as $\Delta^I(E)$.

Definition. *Let I and J be two index sets such that $I \cap J=\emptyset$* and $U(\Delta^{I\wedge J}(E))$, *$U(\Delta^I(E))$ and $U(\Delta^J(E))$ are given to satisfy*

$$(p)\qquad T_1\vee T_2 \in U(\Delta^{I\vee J}(E)) \text{ if } T_1 \in U(\Delta^I(E)),\ T_2 \in U(\Delta^J(E)),$$

$$(p)'\qquad T=T_1\vee T_2,\ T_1 \in U(\Delta^I(E)),\ T_2 \in U(\Delta^J(E)) \text{ if } T \in U(\Delta^{I\cup J}(E)).$$

Then we define $\overline{fg} \in C^{I\cup J}U(\Delta^{I\vee J}(E))(M)$ for $\overline{f} \in C^I U(\Delta^I(E))(M)$ *and $\overline{g} \in C^I U(\Delta^J(E))(M)$ by*

(13) $$\overline{fg}=\bar{f}\bar{g},\ (fg)(T)=f(T_1)\,g(T_2)\quad if\ T=T_1\vee T_2.$$

Similarly, if M is a differentiable Hilbert manifold, we define the inner product $\bar{f}\wedge\bar{g}\in AD^{I\cup J}U(\Delta^{I\cup J}(E))(M)$ by

(13)′ $$\bar{f}\wedge\bar{g}=\overline{f\wedge g},\ f\wedge g=A(fg).$$

Example. If I is a finite set, then for $U_p(\Delta^{I\cup I}(\boldsymbol{E}))$ and $U_p(\Delta^{I}(\boldsymbol{E}))$, or for $U_b(\Delta^{I\cup J}(\boldsymbol{E}))$ and $U_b(\Delta^{I}(\boldsymbol{E}))$, $U(\Delta^{I}(E))$ satisfies the assumption of the above definition.

Definition. *Let I and J be two index sets such that I is a finite set, J contains I and the complement of J in $\{0,\ 1,\ 2,\cdots\cdots\}$ is an infinite set. Then we call $U(\Delta(\boldsymbol{E}))$ satisfies the condition (P) if $U(\Delta_J(\boldsymbol{E}))$ and $U(\Delta_{J-I}(\boldsymbol{E}))$ are given by*

$$U(\Delta_J(\boldsymbol{E}))=(\tau_I{}^*)^{-1}\,(U(\Delta(E))),$$

$$U(\Delta_{J-I}(\boldsymbol{E}))=(\tau_{J-I}{}^*)^{-1}\,(U(\Delta(E))),$$

and the triple $U(\Delta^I(\boldsymbol{E}))$, $U(\Delta_J(\boldsymbol{E}))$ and $U(\Delta_{J-I}(\boldsymbol{E}))$ are satisfy (p) and $(p)'$.

By definition, if $U(\Delta(\boldsymbol{E}))$ satisfies (P), then for any $f\in C^p(M)$, the set of Alexander-Spanier cochain of degree p of M, $p<\infty$, and $g\in C^{\infty-q}{}_{U(\Delta(\boldsymbol{E}))}(M)$, $q\geqq p$ *or* $q=\infty$, we can define the product $fg\in C^{\infty-q+p}{}_{U(\Delta(\boldsymbol{E}))}(M)$. *Espesially, since* $U_2(\Delta(\boldsymbol{E}))$ satisfies (P), $f\wedge g\in AD^{\infty-q+p}{}_{U(\Delta(\boldsymbol{E}))}(M)$ can be defined if f *is* differentiable.

6. By the definition of $C^{\infty-p}{}_{U(\Delta(\boldsymbol{E})),\ I_p}(M)$, we have the isomorphism $\tau_I{}^{\#}:C^{\infty-p}{}_{U(\Delta(\boldsymbol{E})),\ I}(M)\to C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$. For $I=I_p\subset J=J_q$, we set

(14) $$\pi^I{}_J=\tau_I{}^{\#}\,(\tau_J{}^{\#})^{-1}:C^{\infty-p}{}_{U(\Delta(\boldsymbol{E})),\ I}(M)\to C^{\infty-q}{}_{U(\Delta(\boldsymbol{E})),\ J}(M).$$

By definition, we have for $I\subset J\subset K$,

(15) $$\pi^I{}_J\pi^J{}_K=\pi^I{}_K$$

In the case $q=p+1$, we have $J=[I,\ k]=\{i_1,\cdots\cdots,\ i_p,\ k\}$ $(I=\{i_1,\cdots\cdots,\ i_p\})$ (k may be smallar than i_j for some j) and we denote

(14)′ $$\pi_{I,\ K}=\pi^I{}_{[I,\ k]}.$$

Similarly, we denote

(14)′′ $$\pi_{I,\ k,\ k'}=\pi^I{}_{[I,\ k,\ k']}.$$

Here $[I, k, k']$ means $[[I, k], k']$ but assumed $k<k'$.

On the other hand, we denote the cannonical isomorphism from $C^{\infty-p}{}_{U(\Delta(E))}(M)$ onto $C^{\infty-p}{}_{U(\Delta(E)),\,I}(M)$ by ρ_I. Summarising these, we have the following commutative diagram. Here $\rho_{I,\,k}$ means $\rho_{[I,\,k]}$.

$$
\begin{tikzcd}
C^{\infty-q}{}_{U(\Delta(E))}(M) \arrow[d, "\rho_J"] & & C^{\infty-p}{}_{U(\Delta(E))}(M) \arrow[d, "\rho_I"] \arrow[dr, "\iota^{I}_{I'}", "\rho_{I'}"'] & (q>p) \\
C^{\infty-q}{}_{U(\Delta(E)),\,J}(M) & & C^{\infty-p}{}_{U(\Delta(E)),\,I}(M) \arrow[ll, "\pi^{I}_{J}"'] \arrow[d, "\pi_{I,\,k}"] & C^{\infty-p}{}_{U(\Delta(E)),\,I'}(M) \arrow[l] \\
C^{\infty}{}_{U(\Delta(E))}(M) \arrow[u, "\tau^{\#}_{J}"] \arrow[urr, "\tau^{\#}_{I}"] \arrow[drr, "\tau^{\#}_{[I,\,k,\,k']}"'] & & C^{\infty-(p+1)}{}_{U(\Delta(E)),\,[I,\,k]}(M) \arrow[ll, "\tau^{\#}_{[I,\,k]}"] \arrow[d, "\pi_{I,\,k,\,k'}"] & C^{\infty-(p+1)}{}_{U(\Delta(E))}(M) \arrow[l, "\rho_{I,\,k}"'] \\
& & C^{\infty-(p+2)}{}_{U(\Delta(E)),\,[I,\,k,\,k']}(M) &
\end{tikzcd}
$$

For $k\in I$, we define an integer $v(k)=v_I(k)$ by

$$(16)\qquad \tau_I(e_k)=e_{v_I(k)}.$$

Definition *For* $f\in C^{\infty-p-1}{}_{U(\Delta(E))}(M)$, *we define a function* $\delta_{I,\alpha}(f)(\lambda)$, $\alpha\in C$, Re. $\lambda>0$, by

$$(17)\qquad \begin{aligned}&\delta_{I,\,\alpha}(f)(\lambda)(T)\\ &=\sum_{k\notin I}\lambda^{\alpha}(-1)^{v(k)}\,e^{-v(k)\lambda}\pi_{I,\,k}(\rho_{I,\,k}f)\ (T).\end{aligned}$$

***Lemma* 5.** *If* $U(\Delta(E))$ *is contained in* $U_b(\Delta(E))$ *and the inclusion* $U(\Delta(E))\to U_b(\Delta(E))$ *is continuous, then* $\delta_{I,\,\alpha}(f)(\lambda)(T)$ *is holomorphic for* Re $\lambda>0$ *and*

$$\lim_{\lambda\to 0}.\ \delta_{I,\,\alpha}(f)(\lambda)\ (T)=0 \ \textit{for any } T \textit{ if } \mathrm{Re.}\ \alpha>1.$$

Proof. By assumption, $|\pi_{I,\,k}(\rho_{I,\,k}f)(T)|$ is uniformly bounded (in k). Hence we have for some $M>0$,

$$(18)\qquad \begin{aligned}|\delta_{I,\,\alpha}(f)(\lambda)(T)| &\leqq \sum_{n\geqq 0}|\lambda|^{\mathrm{Re.}\,\alpha}e^{-n\mathrm{Re.}\,\lambda}M\\ &=\frac{M|\lambda|^{\mathrm{Re.}\,\alpha}}{1-e^{-\mathrm{Re.}\,\lambda}}, \qquad \mathrm{Re.}\ \lambda>0.\end{aligned}$$

Therefore we obtain the lemma.

In the rest, we assume $0\leqq \mathrm{Re.}\,\alpha\leqq 1$.

Note. By (18) and Fatou's lemma ([9]), if $\mathrm{Re.}\,\alpha=1$, then $\lim._{\sigma\to 0}\ \delta_{I,\,\alpha}(f)(\sigma+it)(T)$ exists for almost all t.

Definition. *If* $\lim._{\lambda\to 0}\delta_{I,\alpha}(f)(\lambda)(T)$ *exists, then we define* $\delta_{I,\alpha}(f)$ *by*

$$\delta_{I,\alpha}(f)(T)=\lim_{\lambda\to 0}\delta_{I,\alpha}(f)(\lambda)(T). \tag{19}$$

Since $\delta_{I,\alpha}(f)=\lambda^{\alpha-\beta}\delta_{I,\beta}(f)$ we have

$$\delta_{I,\beta}(f)(T)=0,\ \textit{if } \delta_{I,\alpha}(f)(T) \textit{ exists and } \mathrm{Re.}\ (\beta-\alpha)>0.$$

Lemma 6. *For any* I_p, J_p, *we have*

$$\rho_I^{-1}\delta_{I,\alpha}(f)(\lambda)(T)=\rho_J^{-1}\delta_{J,\alpha}(f)(\lambda)(T). \tag{20}$$

Proof. By the definitions of ρ_I and $\pi_{\alpha,k}$, we have

$$\rho_I^{-1}\pi_{I,k}\,\rho_{I,k}=\rho_J^{-1}\pi_{J,k'}\rho_{J,k'},\ \textit{if}\ v_I(k)=v_J(k').$$

Hence we have the lemma.

Definition. *We set*

$$\delta_\alpha(f)(\lambda)\ (T)=\rho_I^{-1}\delta_{I,\alpha}(f)(\lambda)(T), \tag{21}$$

$$\delta_\alpha(f)(T)=\lim_{\lambda\to 0}\ \delta_\alpha(f)(\lambda)(T). \tag{21$'$}$$

By definition, δ_α is defined in $C^{\infty-p-1}{}_{U(\Delta(E))}(M)$ and maps it into $C^{\infty-p}{}_{U(\Delta(E))}($ $M)$.

Theorem 2. $\delta_1\delta_\alpha$ *is equal to* 0. *That is, if* $\delta_\alpha(f)(T)$ *exists, then*

$$\lim_{\lambda\to 0}\ \lim_{\mu\to 0}\ \delta_1(\delta_\alpha(f)\ )\mu))\ (\lambda)\ (T)=0, \tag{22}$$

if $U(\Delta(E))$ *is contained in* $U_b\,(\Delta(E))$.

Proof. By definition, we have

$$\begin{aligned}
&\delta_{I,1}(\delta_\alpha\ f(\mu))\ (\lambda)\ (T)\\
&=\lambda\sum_{k\notin I}(-1)^{v(k)}\ e^{-v(k)\lambda}\ \pi_{I,k}(\rho_{I,k}\ \delta_\alpha\ f(\mu))\ (T)\\
&=\lambda\mu^\alpha\sum_{k\notin I}(-1)^{v_I(k)}e^{-v_I(k)\lambda}\pi_{I,k}(\rho_{I,k}(\sum_{j\notin[I,k]}(-1)^{v[I,k](j)}\\
&\quad e^{-v[I,k](j)\mu}\ \pi_{I,k,j}\ (\rho_{I,k,j}\ f)\ (T))\\
&=\lambda\mu^\alpha\sum_{k,\ j\notin I,\ k<j}(-1)^{v_I(k)+v[I,k](j)}\ (\mathrm{e}^{-v_I(k)\lambda-v[I,k](j)\mu}\\
&\quad -\mathrm{e}^{-(v[I,k](j)-1)\lambda-v_I(k)\mu})\ \pi_{I,k,j}\ (\rho_{I,k,j}\ f)\ (T)
\end{aligned}$$

$$=\lambda\mu^{\alpha}\Big(\sum_{k\leq k}(-1)^{k}\Big(\sum_{p\leq k} \mathrm{e}^{-((k-p)\lambda+p\mu)}-\mathrm{e}^{-(p\lambda+(k-p+1)\mu)}\Big)$$
$$\pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f\Big)(T)$$
$$+\sum_{k>k}(-1)^{k}\Big(\sum_{p\leq k} \mathrm{e}^{-((k-p)\lambda+p\mu}-\mathrm{e}^{-(p\lambda+(k-p+1)\mu)}\Big)$$
$$\pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f\Big)(T)\Big).$$

Then, since $\delta_{\alpha}(f)(T)$ exists, we have by (18),

$$\Big|\sum_{k>K}(-1)^{k}\Big(\sum_{p\leq K}\mathrm{e}^{-((k-p)\lambda+p\mu)}-\mathrm{e}^{-(p\lambda+(k-p+1)\pi)}\Big)$$
$$\pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f(T)\Big|$$
$$<\varepsilon\,|\lambda|^{-1}\,|\mu|^{-\mathrm{Re.}\ \alpha},$$

for given $\varepsilon>0$ if K is sufficiently large. On the other hand, since

$$\lim_{\mu\to 0}.\ \sum_{p\leq k}\Big(\mathrm{e}^{-((k-p)\lambda+p\mu)}-\mathrm{e}^{-(p\lambda+(k-p+1)\mu)}\Big)$$
$$\pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f(T)$$
$$=\sum_{p\leq k}\Big(\mathrm{e}^{-(k-p)\lambda}-\mathrm{e}^{-p\lambda)}\ \pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f(T)=0,$$

we get

$$\Big|\sum_{k\leq k}(-1)^{k}\Big(\sum_{p\leq k}\mathrm{e}^{-((k-p)\lambda+p\mu)}-\mathrm{e}^{-(p\lambda+(k-p+1)\mu)}\Big)$$
$$\pi_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, \rho_{I,\, v_I^{-1}(k),\, v_{I,\,k}^{-1}(k-p+1)}\, f(T)\Big|<\varepsilon,$$

for the above ε if $|\mu|$ is sufficiently small. Hence we obtain

$$|\delta_{I,\,1}(\delta_{\alpha}\mathrm{f}(\mu))(\lambda)(T)|<\varepsilon+\varepsilon\,|\lambda|\,|\mu|^{\mathrm{Re.}\ \alpha},$$

for given $\varepsilon>0$ if $|\mu|$ is sufficiently small. This shows (22).

7. The following lemma owe to Dr. Matsugu.

***Lemma* 7.** *If $\{a_n\}$ is a series such that $\lim._{n\to\infty} a_n=0$, then*

$$\lim_{z\to 1}.\ (z-1)\Big(\sum_{n=0}^{\infty} a_n z^n\Big)=0,\quad |z|<1.$$

Proof. Since $(z-1)(\sum a_n z_n)=-a_0+\sum_{n\geq 1}(a_{n-1}-a_n)z^n$, and we know

$$\lim_{z\to 1}. \left(-a_0+\sum_{n=1}^{k}(a_{n-1}-a_n\ z_n)\right)=-a_k,$$

we have the lemma by Abel's continuity theorem.

Corollary. *If $f \in C^{\infty - q}{}_{U_p(\Delta(E))}(M)$, then $\delta_1 f$ is equal to 0. In general, if $U(\Delta(E))$ satisfies*

$$(C)\qquad \lim_{k\to\infty} T(e_k)=0\ \ if\ \ T \in U(\Delta(E)),$$

or more general,

$$(C)'\qquad \lim_{k\to\infty} T(e_{2k}-e_{2k-1})=0,\ \ or\ \lim_{k\to\infty} T(e_{2k}-e_{2k+1})=0,$$

then $\delta_1 f=0$ if $f \in C^{\infty-p}{}_{U(\Delta(E))}(M)$.

Proof. We set

$$\begin{aligned}
&\delta_1 f(\lambda)\ (T)\\
&=\lambda\Big(\sum_{k=0}^{\infty}(-1)^k\ e^{-k\lambda}\ \pi_{I,\ v_I}{}^{-1}(k)\ \rho_{I,\ v_I}{}^{-1}(k)\ f(T)\Big)\\
&=\Big(\lambda\sum_{k=0}^{\infty} e^{-2k\lambda}\ (1-e^{\lambda})\ \pi_{I,\ v_I}{}^{-1}(2k)\ \rho_{I,\ v_I}{}^{-1}(2k)\ f(T)+\sum_{k=1}^{\infty} e^{-(2k-1)\lambda}\\
&\quad(\pi_{I,\ v_I}{}^{-1}(2k)\ \rho_{I,\ v_I}{}^{-1}(2k)\ f(T)-\pi_{I,\ v_I}{}^{-1}(2k-1)\ \rho_{I,\ v_I}{}^{-1}(2k-1)\ f\ (T))\Big).
\end{aligned}$$

Then by (18), $\left|(1-e^{\lambda})\sum_{k=0}^{\infty} e^{-2k\lambda}\ \pi_{I,\ v_I}{}^{-1}(2k)\ {}_{I,\ v_I}{}^{-1}(2k)\ f(T)\right|$ is bound in λ. On the other hand, to set $\lambda=\log z$, $|z|<1$, Re. $z>0$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\lambda\Big(\sum_{k=1}^{\infty} e^{-(2k-1)\lambda}\ \pi_{I,\ v_I}{}^{-1}(2k)\ \rho_{I,\ v_I}{}^{-1}(2k)\ f(T)\\
&\quad-\pi_{I,\ v_I}{}^{-1}(2k-1)\ \rho_{I,\ v_I}{}^{-1}(2k-1)\ f\ (T)\Big)\Big)\\
&=0
\end{aligned}$$

by $(C)'$ and lemma 7. Hence we have the corollary.

We assume $U(\Delta(E))$ satisfies the condition (P). Then to denote $\Delta_{n,\ a}$ the operator in $L(E^{\{e_n\}},\ E_0)$ given by

$$\Delta_{n,\ a}\ (e_n)=a,\ \ a \in E_0,$$

we define the map $k_{I,\ n,\ a}: C^{\infty-p}{}_{U(\Delta(E)),\ I_p}(E_0) \to C^{\infty-p-1}{}_{U(\Delta(E)),\ [I,\ n]}(E_0)$, $n \in I_p$, by

(23) $$(k_{I,\,n,\,a}\ f)\ (T)=f(T\vee\Delta_{n,\,a}),$$

$$f\in C^{\infty-p}{}_{U(\Delta(E)),\ I}\ (E),\quad T\in(\tau_{[I,\ n]^*})^{-1}(U(\Delta(E))).$$

Since we know

$$k_{I,\,n,\,a}\ \rho_I\ f=k_{J,\,n',\,a}\ \rho_J f,$$

if $n\in I$, $n'\in J$ and $v_I(n)=v_J(n')$, we define the map $k_{m,\,a}:C^{\infty-p}{}_{U(\Delta(E))}\ (E_0)\to C^{\infty-p-1}{}_{U(\Delta(E))}\ (E_0)$ by

(24) $$k_{m,\,a}\ f=k_{I,\ v_I^{-1}(m),\ a}\ \rho_I\ f.$$

***Theorem* 3.** *If $U(\Delta(E))$ satisfies the condition $(C)'$, then*

(25) $$\lim_{\lambda\to 0}\ \lambda^{-\alpha}(k_{1,\,a}\ (\delta_a f(\lambda))\ (T)\ +\delta_\alpha\ (k_{1,\,a}f)\ (\lambda)\ (T))=f(T).$$

Proof. By definition, we have

$$\begin{aligned}
&\lambda^{-\alpha}(k_{1,\,a}\ (\delta_{I,\,\alpha}\ f(\lambda))\ (T)+\delta_{I,\,\alpha}\ (k_{1,\,a}f)\ (\lambda)\ (T))\\
&=k_{1,\,a}\ (\sum_{n\geqq 0}\ (-1)^n\ \mathrm{e}^{-n\lambda}\ \pi_{I,\ v_I^{-1}(n)}\ \rho_{I,\ v_I^{-1}(n)}f(T))\\
&\quad+\sum_{n\geqq 0}\ (-1)\ {}^n\mathrm{e}^{-n\lambda}\ \pi_{[I,\ 1],\ v_{[I,\ 1]}^{-1}(n)}\ \rho_{[I,\ 1],\ v_{[I,\ 1]}^{-1}(n)}\ k_{1,\,a}\ f(T)\\
&=(\tau_I^{\#})^{-1}\ f(T)+(1-\mathrm{e}^{-\lambda})\ (\sum_{n\geqq 0}\ (-1)^n\ \mathrm{e}^{-n\lambda}\ k_{1,\,a}\ (\pi_{I,\ v_I^{-1}(n)}\ (\rho_{I,\ v_I^{-1}(n)}f)\ (T)).
\end{aligned}$$

Then, since $(1-\mathrm{e}^{-\lambda})=O\ (\lambda)$, $\lambda\downarrow 0$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\ (1-\mathrm{e}^{-\lambda})\ (\sum_{n\geq 0}\ (-1)^n\ \mathrm{e}^{-n\lambda}\ k_{1,\,a}\ (\pi_{I,\ v_I^{-1}(n)}\ (\rho_{I,\ I^{-1}(n)}\ f)\ (T))\\
&=0,
\end{aligned}$$

by the corollary of lemma 7, we obtain the theorem.

Note. Since we know

$$\sum_{n\geqq 0}\ n^a r^{-n}=O(\frac{1}{(1-r)^{1+\mathrm{Re.}\ \alpha}}),\ \mathrm{Re.}\ \alpha>-1,\ r\downarrow 1,$$

if $U(\Delta(E))$ is taken to satisfy such that if $\mathrm{T}\in U(\Delta(E))$, then

$(C)'$ $$|\ T(e_{2k}-e_{2k-1})\ |=O(k^{\mathrm{Re.}\ \alpha}),\ \ or\ |\ T(e_{2k}-e_{2k-1})\ |=O(k^{\mathrm{Re.}\ \alpha}),$$
$$-1<\mathrm{Re.}\ \alpha<0,$$

we have for $f\in C^{\infty-p}{}_{U(\Delta(E))}\ (E)$,

$$\delta_\beta\ (f)\ (T)=0,\ \text{Re.}\ \beta > 1+\alpha,$$
$$\delta_{1+\alpha}\ (\delta_\gamma f)\ (T)=0,\ \textit{if}\ \delta_\gamma f\ \textit{exists.}$$

We also have that by Fatou's lemma, $lim_{\sigma\to+0}\ \delta_{1+\alpha}\ f(\sigma+it)\ (T)$ exists for almost all t in this case.

§ 3. Integration of Alexander-Spanier cochain of degree ∞.

8. We denote by E_0^∞ the infinite product $E_0\times E_0\times\cdots\cdots=\{(x0,\ x_1,\ x_2,\cdots\cdots)\mid x_i\in E_0\}$ of E_0. The diagonal of E_0^∞ is denoted by $\Delta(E_0^\infty)$. In E_0^∞, we denote by D a subset such that

(i). $D\cap\Delta\ (E_0^\infty)\neq\emptyset$.

(ii). *If* $x+a\in D,\ a\in\Delta(E_0^\infty)$, *then* $tx+a\in D,\ 0\leqq t\leqq 1$.

(iii). *D can be considered to be a C^∞-class Banach differentiable manifold and the topology of D by this structure is not weaker than that of the induced topology of (the weak topology of) E_0^∞.*

Then, if f is Fréchet derivable on D, and satisfies

$$f\ x_0,\ x_1,\ x_2,\cdots\cdots)=0\ \textit{if}\ x_i=x_0\ \textit{for some}\ i, \tag{26}$$

then we have

$$\begin{aligned}&f(a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)\\ &=<d_{x_1}\ f(a,\ a,\ a+t(x_2-a),\cdots\cdots),\ t(x_1-a)>+o(|\ t\ |)\\ &=<d_{x_2}<d_{x_1}\ f(a,\ a,\ a,\ a+t(x_3-a),\cdots\cdots),\ t(x_1-a)>,\ t(x_2-a)>+o(|\ t^2\ |)\\ &=\cdots\cdots\cdots\cdots\\ &=t^k<d_{x_k}<d_{x_{k-1}}<\cdots\cdots<dx_1\ f(a,\cdots\cdots,\ a,\ a+t(x_{k+1}-a),\cdots\cdots),\ (x_1-a)>,\cdots\cdots>,\\ &\quad (x_{k-1}-a)>,\ (x_k-a)>+o(|\ t^{k+1}\ |),\end{aligned}$$

for any k if $f(a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)$ is C^∞-class in t. Here d_{x_i} means Fréchet derivation in x_i. For simple, we set

$$\begin{aligned}&f_k(a,\ x_1,\ x_2,\cdots\cdots,\ t)\\ &=<d_{x_k}<d_{x_{k-1}}<\cdots\cdots<d_{x_1}\ f(a,\cdots\cdots,\ a,\ a+t(x_{k+1}-a),\ (x_1-a)>,\cdots\cdots>,\\ &\quad (x_{k-1}-a)>,\ (x_k-a)>.\end{aligned}\tag{27}$$

By (27), we get

$$\begin{aligned}&f(a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)\\ &=t^k f_k(a,\ x_1,\ x_2,\cdots\cdots,\ t)+o(|\ t^{k+1}\ |).\end{aligned}\tag{28}$$

On the other hand, since

$$| f_k(a,\ x_1,\ x_2, \cdots\cdots,\ t) |$$
$$\leq \sup_{||c_i||\leq 1} || d_{x_k} < d_{x_{k-1}} < \cdots\cdots < d_{x_1} f(a, \cdots\cdots,\ a,\ a+t(x_{k+1}-a),\ c_1 > \cdots\cdots >,\ c_k > ||$$
$$|| x_1-a_1 || \cdots\cdots || x_k-a_k ||,$$

we may assume that there exists a continuous positive real valued function $g_k(a, x_{k+1}, \cdots\cdots, t)$ such that

$$(28)' \qquad | f(a,\ a+t(x_1-a),\ a+t(x_2-a), \cdots\cdots) | \leq t^k g_k(a,\ x_{k+1}, \cdots\cdots,\ t).$$

Moreover, to set

$$G_k(a,\ x_{k+1}, \cdots\cdots,\ t) = t^k g_k(a,\ x_{k+1}, \cdots\cdots,\ t),$$

we may assume that

$$(29) \qquad | f_m(a,\ x_1,\ x_2, \cdots\cdots,\ t) | \leq t^{-m}\, G_k(a,\ x_{k+1}, \cdots\cdots,\ t),\ 0 < t < 1,$$

is hold for any m if $m \leq k$.

Definition. *Let $G(x, t)$ be a positive real valued function on $U(a) \times [0, 1,]$, where $U(a)$ is a neighborhood of a in $D \cap \Delta(E_0^\infty)$ and $\{c_m\}$ and $\{\varepsilon_m\}$ be serieses of positive numbers. Then we denote*

$f(b,\ b+t(x_1-b),\ b+t(x_2-b), \cdots\cdots) = O(G)$, with respect to $\{c_m\}$ and $\{\varepsilon_m\}$, on $U(a)$ if

$$(30) \qquad | f_m(b,\ x_1, \cdots\cdots x_2, \cdots\cdots,\ t) | \leq c_m t^{-m}\, G(b,\ t),\ 0 < t < \varepsilon_m,$$
$$(b,\ x_1,\ x_2, \cdots\cdots) \in D,\ b \in U(a),$$

is hold for all m, $m \geq 0$. Moreover, we denote

$f(b,\ b+t(x_1-b),\ b+t(x_2-b), \cdots\cdots) \sim G$, with respect to $\{c_m\}$ and $\{\varepsilon_m\}$, on $U(a)$ if (30) *and*

$$(30)' \qquad | f_m(b,\ x_1,\ x_2, \cdots\cdots,\ t) | \geq c'_m\, t^{-m}\, G(b,\ t),\ 0 < t < \varepsilon'_m,$$
$$c'_m = 0(\{c_m\}),\ \varepsilon'_m = 0(\{\varepsilon_m\}),$$

are both hold.

Example. We set $a = \sum_{k=1}^{\infty} a_k e_k$ and assume $a_1 \neq 0$. We also assume $(a,\ a+tx'_1,\ a+tx'_2, \cdots\cdots) \in D$ and to set

$$x'_i = \sum_{k=1}^{\infty} x'_{ik} e_k,\ i=1,\ 2, \cdots\cdots,$$

we can take a series $\{n(i)\}$ such that

$$0 < \xi_i < |x'_{i,\, n(i)}|, \quad (a,\ a+tx'_1,\ a+tx'_2, \cdots\cdots) \in D, \quad i=1,\ 2, \cdots\cdots$$
$$|a_1| > |a_1-t_i| \geqq |a_1+\varepsilon_i tx'_{i,\, n(i)}|, \quad t\neq 0, \quad |\varepsilon_i|=1.$$

Moreover, we assume that there is a series $\{r_i\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{r_i+1}\left(\frac{a_1-t\ \xi_i}{a_1}\right)^{(r_i+1)} < \infty.$$

Then, to set

$$f(a,\ a+tx'_1,\ a+tx'_2, \cdots\cdots)$$
$$=\sum_{i=1}^{\infty}\left(1-\frac{a_1+\varepsilon_i tx'_{i,\, n(i)}}{a_1}\right) e^{-g_i(\varepsilon_i tx'_{i,\, n(i)})},$$
$$g_i(\varepsilon_i tx'_{i,\ n\ i})=\sum_{m=1}^{r_i} \frac{1}{m}\left(\frac{a_1+\varepsilon_i tx'_{i,\, n(i)}}{a_1}\right)^{m},$$

$f(a,\ a+tx'_1,\ a+tx'_2, \cdots\cdots)=O(G)$ with respect to $\{c_m\}$ and $\{\varepsilon_m\}$, where G, $\{c_m\}$ and $\{\varepsilon_m\}$ are given by

$$G(a,\ t)=\sum_{i=1}^{\infty}\left(1-\frac{a_1-t\xi_i}{a_1}\right) e^{-g_i(-t\xi_i)},$$

$$c_m=\sup_{x'_m}.\quad 2\left|\frac{1}{x'_{i,\, n(i)}}-\log r_i\right|,$$

$$\varepsilon_m=\sup_{\varepsilon}.\quad \left\{\left|\frac{1}{x'_{i,\, n(i)}}-g_i'(\varepsilon_i\tau x'_{i,\, n(i)})\right| < c_m,\ 0 < \tau < \varepsilon\right\}.$$

Definition. *If we can take the above $G(b,\ t)$ to be*

(31) $$G(b,\ t)=\frac{1}{H(b,\ t^{-1})},\quad H(b,\ t) \text{ is an entire function in } t.$$

Then we call f to be class H with respect to $\{c_m\}$ and $\{\varepsilon_m\}$.

By definition, $H(b,\ t)$ has essential singularity at $t=\infty$ and to set $H(b,\ t)=\sum_{n=0}^{\infty} c_n(b)t^n$, we may consider $c_n(b) > 0$ for any n and in this case, we have

$$\lim_{n\to\infty}\frac{c_{n+1}(b)}{c_n(b)}=0.$$

9. By the map $j: L(\cup \boldsymbol{E}_n,\ \boldsymbol{E}_0) \to \boldsymbol{E}_0^{\infty}$ given by

$$j(T)=(T(e_0),\ T(e_1),\ T(e_2),\cdots\cdots),$$

there is a 1 to 1 correspondence between $L(\bigcup E_n,\ E_0)$ and E_0^∞ and we have

$$j(\Delta(E))=\Delta(E_0^\infty),\ j(\Delta_a)=(a,\ a,\cdots\cdots).$$

Hence $j(U(\Delta(E)))$ contains $\Delta(E_0^\infty)$. The conditions (ii) and (iii) of n°8 are changed to

(ii)′ $T+\Delta_a \in U(\Delta(E))$ *implies* $tT+\Delta_a \in U(\Delta(E)), 0 \leqq t \leqq 1$,

(iii)′ $U(\Delta(E))$ *has the structure of a Banach differentiable manifold,*

of the conditions of $U(\Delta(E))$. Moreover, $j^*(f)$ is 0 -normal if and only if f satisfies (26).

Definition. *We assume* $U(\Delta(E))$ *satisfies the above* (ii)′ *and* (iii)′. *Then a differerntiable standard* 0 *-normal Alexander-Spanier cochain of degree* ∞ *of* E *with respect to* $U(\Delta(E))$ *is denoted*

$$f(\Delta_b+tT)=O(G),\ \textit{with respect to}\ \{c_m\}\ \textit{and}\ \{\varepsilon_m\},$$

near a *if* $((j^*)^{-1}f)\ (j^*(\Delta_b+tT))=O(G)$ *with respect to* $\{c_m\}$ *and* $\{\varepsilon_m\}$. *Similarly, we define* $f(\Delta_b+tT) \sim G$ *and* $f(\Delta_b+tT)$ *is class* H *as above.*

Note. Although f is not differentiable, or more weaker, $U(\Delta(E))$ does not satisfy (iii)′, if for each m, there exists a series $\{g_m\}$ such that

$$\begin{aligned}(32)\qquad & |\ ((j^*)^{-1}f)\ (a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)\ | \\ & \leqq t^m g_m(a,\ x_{m+1},\ x_{m+2},\cdots\cdots,\ t),\ t \geqq 0,\end{aligned}$$

and for these $\{g_m\}$, we get

$$(32)'\qquad |\ g_m(b,\ x_{m+1},\ x_{m+2},\cdots\cdots,\ t)\ | \leqq c_m t^{-m}\ G(b,\ t),\ 0 < t < \varepsilon_m,$$

then we also denote $f=O(G)$ with respect to $\{c_m\}$ and $\{\varepsilon_m\}$. If G satisfies (31), then we call f to be class H.

By definition, we may consider $\{c_m\}=O(1)$ in this case. In fact, we have

$$|\ ((j^*)^{-1}f)\ (b,\ b+t(x_1-b),\cdots\cdots)\ | \leqq c_m\ G(b,\ t),\ 0 < t < \varepsilon_m,$$

for any m by (32) if (32)′ is hlod.

Lemma 8. *If* $\sup.\ (x_1,\ x_2,\cdots\cdots)\ |\ ((j^*)^{-1}f)\ (a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)\ |$ *exists for* $0 \leqq t \leqq \varepsilon$, *then to set*

$$G(a,\ t) = \sup_{(x_1,\ x_2,\cdots\cdots)}\ |\ ((j^*)^{-1}f)\ (a,\ a+t(x_1-a),\ a+t(x_2-a),\cdots\cdots)\ |,\ 0 \leqq t < \varepsilon,\ f=O(G)\ \textit{with}$$

respect to $\{1\}$ *and* $\{\varepsilon\}$.

Note. If $f(\Delta_b+tT)$ is C^∞-class in t and f is 0 -normal, then,

$$|f(\Delta_b+tT)|=o(t^m),\ t\to 0,\ \textit{for any}\ m>0,$$

it is not so restrictive that to assume the above $G(a,\ t)$ to be $o(t^m)$ for $t\downarrow 0$ for any $m>0$. Hence it is also not so restrictive that to assume there is an analytic function $H(a,\ t)$ such that

$$G(a,\ t)\leqq\frac{1}{H(a,\ t^{-1})},\quad H(a,\ t)=\sum_n c_n(a)t_n,\quad c_n(a)>0.$$

10. Let $\{c_n\}$ be a series of positive numbers such that $lim_{n\to\infty}c_n=0$. Then assuming $\boldsymbol{E}_0$ to be a real Banach space such that

$$\sum_{k=1}^{\infty}x_ke_k\in\boldsymbol{E}_0,\ \textit{if}\ |x_k|\leqq c_k,\ k=1,\ 2,\cdots\cdots,$$

we define a subset $[\sigma,\ \{c_n\}]$ of $\boldsymbol{E}_0$ by

$$[\sigma,\ \{c_n\}]=\{x\mid x=\sum_{k=1}^{\infty}x_ke_k,\ 0\leqq x_k\leqq c_k\}. \tag{33}$$

By definiton, *Int.* $[\rho,\ \{c_n\}]$ is non-void in $\boldsymbol{E}_0$.

For $\{c_n\}$, we define a power series $H=H\ \{c_n\}$ by

$$H(t)=\sum_{n=1}^{\infty}c_1\cdots\cdots c_nt^n.$$

By definition, $H(t)$ defines an entire function and along the real axis, we have $lim_{t\to\infty}H(t)=\infty$. More presisely, we get $lim_{t\to\infty}\ t^{-k}H(t)=\infty$ for any k.

On each interval $[0,\ c_k]$, $k=1,\ 2,\cdots\cdots$, we consider a partition

$$0=x_{k,\,0}<x_{k,\,1}<\cdots\cdots<x_{k,\,m_k}<c_k,\quad k=1,\ 2,\cdots\cdots.$$

Then, for an index set $\boldsymbol{J}=(j_1,\ j_2,\cdots\cdots,\ j_k,\cdots\cdots)$, we set

$$x_{\boldsymbol{J}}=\sum_{k=1}^{\infty}x_{k,\,j_k}e_k\in\boldsymbol{E}_0,$$

$$x_{\boldsymbol{J}+1_i}=\sum_{k\neq i}x_{k,\,j_k}e_k+x_{i,\,j_{i}+1}e_i,\quad i=1,\ 2,\cdots\cdots,\quad x_{i,\,m_{i}+1}=c_i.$$

For the index set $\boldsymbol{J}$, we also set

$$[J]_k=\{J \mid J=(j_1,\ j_2,\cdots\cdots,\ j_k,\ 0,\ 0,\cdots\cdots)\},$$
$$(J)_k=[J]_k-[J]_{k-1},\ k \geqq 2,\ (J)_1=[J]_1.$$

Definition. *We call $U(\varDelta(\boldsymbol{E}))$ satisfy the condition $(I)_{[\sigma,\ \{c_n\}]}$, if there exists a series $\{\varepsilon_n\}$ such that $\{c_n\}=O(\varepsilon_n)$ and*

$$j^{-1}(x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots) \in U(\varDelta_E) \text{ for any } x_J \in [\sigma,\ \{c_n\}],$$

if $\mid x_{n,\ p_q}-x_{n,\ p_{q+1}} \mid < \varepsilon_n$ for any n and p_q.

Definition. *We assume $U(\varDelta(\boldsymbol{E}))$ satisfies $(I)_{[\sigma,\ \{c_n\}]}$. Let f be a standard Alexander-Spanier cochain of degree ∞ with respect to $U(\varDelta(\boldsymbol{E}))$ of $\boldsymbol{E}_0$ with the representation f. Then we define the integral of f on $[\sigma,\ \{c_n\}]$, denoted by $\int_{[\sigma,\ \{c_n\}]} f$, by*

$$\tag{34} \int_{[\sigma,\ \{c_n\}]} f = \lim_{|x_{k,\ p_q}-x_{k,\ p_q+1}|\to 0}\ \lim_{m\to\infty} \sum_{k=1}^{m} \Big(\sum_{J\in(J)_k} j^{-1}(f)\,(x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots)\Big).$$

Theorem 4. *$\int_{[\sigma,\ \{c_n\}]} f$ exists if f is class $(H_{\{c_n\}})$ with respect to $\{M\}$* and *$\{\varepsilon_n\}$.*

Proof. For each fixed k and the partition $0=x_{k,\,0}<x_{k,\,1}<\cdots\cdots<x_{k,\,m_k}<c_k$, we assume

$$\mid x_{k,\ p+1}-x_{k,\ p} \mid \leqq t,\ t < \varepsilon_n.$$

Then we have by assumption,

$$\begin{aligned}
&\Big|\sum_{J \in (J)_k} j^{-1}(f)\,(x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots)\Big| \\
&\leqq \sum_{J \in (J)_k} \mid j^{-1}(f)\,(x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots) \mid \\
&\leqq \sum_{J \in [J]_k} \mid j^{-1}(f)\,(x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots) \mid \\
&\leqq 2c_1c_2\cdots\cdots c_k\ \sup.\ \mid f_m\,(x_J,\ e_1,\ e_2,\cdots\cdots,\ t) \mid \\
&\leqq 2Mc_1c_2\cdots\cdots c_k\ t^{-k}\frac{1}{H(t^{-1})},\quad 0 < t < \varepsilon_k,
\end{aligned}$$

because $x_{J+1_k}-x_J=(x_{j,\ j_k+1}-x_{j,\ j_k})\,e_k$. Hence we get

$$\begin{aligned}
&\Big|\sum_{k=1}^{m} \Big(\sum_{J\in(J)_k} j^{-1}(f)\ (x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots)\Big)\Big| \\
&\leqq \sum_{k=1}^{m} \Big| \sum_{J\in(J)_k} j^{-1}(f)\ x_J,\ x_{J+1_1},\ x_{J+1_2},\cdots\cdots) \Big|
\end{aligned}$$

$$\leqq \sum_{k=1}^{m} 2Mc_1c_2\cdots\cdots c_k t^{-k}\frac{1}{H(t^{-1})},\ 0 < t < min.\ (\varepsilon_1, \cdots\cdots, \varepsilon_m),$$
$$\leqq 2M,$$

if the partitions $0=x_{k,0} < x_{k,1} < \cdots\cdots < x_{k,m_k} < c_k$, $k=1, \cdots\cdots, m$, are taken to satisfy

$$|x_{k,p+1} - x_{k,p}| < min.\ (\varepsilon_1, \cdots\cdots, \varepsilon_m),\ k=1, \cdots\cdots, m.$$

Hence we have the theorem.

Note 1. If $f_m \geqq 0$ for large m and $f \sim (H(1/t))^{-1}$ with respect to $\{M'\}$ and $\{\varepsilon_n\}$, then we have

$$\int_{[\sigma,\ \{c_n\}]} f \geqq M'.$$

Note 2. If f is class $(H\{c_n\})$ with respect to $\{M\}$ and $\{\varepsilon_n\}$, then we have

$$\int_{[\sigma,\ \{c_n\}]} f$$
$$= \lim_{|x_{k,p_q} - x_{k,p_q+1}| \to 0}\ \lim_{m\to\infty} \sum_{k=s}^{\infty} \left(\sum_{J \in (J)_k} j^{-1}(f)\ (x_J,\ x_{J+1_1},\ x_{J+1_2}, \cdots\cdots) \right),$$

for any s.

11. Since $[\sigma,\ \{c_n\}]$ has interior point in $\boldsymbol{E}_0$ by assumption, we can consider (continuous or) differentiable map φ from $[\sigma,\ \{c_n\}]$ into M if M is a Banach differentiable manifold.

Definition. *$\varphi : [\sigma,\ \{c_n\}] \to M$ is called non -degenerate at $x \in [\sigma,\ \{c_n\}]$ if for any (relative) neighborhood $U(x)$ of x in $[\sigma,\ \{c_n\}]$, there is a neighborhood $V(x)$ of x in $[\sigma,\ \{c_n\}]$* such that

$$V(x) \subset U(x),\ h_\alpha(\varphi(V(x)))\ \textit{has interior point in}\ \boldsymbol{E}_0.$$

Here the manifold structure of M is given by $\{(U_\alpha,\ h_\alpha)\}$ and $V(x)$ is assumed to be $\varphi(V(x)) \subset U$.

Definition. *Let M be given $U(\Delta(\boldsymbol{E}))$-structure by local $\boldsymbol{E}$-product $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$. Then we call $U(\Delta(\boldsymbol{E}))$ satisfy the condition $(I)_{\varphi[\sigma,\ \{c_n\}]}$ if there exists a series $\{\varepsilon_n\}$ such that $\{c_n\}=O(\{\varepsilon_n\})$ and*

$$(\varphi(x_J),\ \varphi(x_{J+1_1}),\ \varphi(x_{J+1_2}), \cdots\cdots) \in \mathfrak{U}(\Delta_{\boldsymbol{E}}(M)),$$

for some $\mathfrak{U}(\Delta_{\boldsymbol{E}}(M))$ and for any $x_J \in [\sigma,\ \{c_n\}]$ if

$$| x_{n, p_q} - x_{n, p_{q+1}} | < \varepsilon_n, \ \textit{for any } n \textit{ and } p_q.$$

Definition. *We assume* $U(\Delta(\boldsymbol{E}))$ *satisfies* $(I)_{\varphi[\sigma, \{c_n\}]}$. *Then for a standard cochain* $f \in C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M)$ *with representation* f, *we define the integral of* f *on* $\varphi[\sigma, \{c_n\}]$, *denoted by* $\int_{\varphi[\sigma, \{c_n\}]} f$, *by*

$$(34)' \int_{\varphi[\sigma, \{c_n\}]} f$$
$$= \lim_{|x_{k, p_q} - x_{k, p_q+1}| \to 0} \lim_{m \to \infty} \sum_{k=0}^{m} \Big(\sum_{J \in (J)_k} f(\varphi(x_J), \ \varphi(x_{J+1_1}), \ \varphi(x_{J+1_2}), \cdots\cdots) \Big).$$

Note. To set

$$\varphi^* f(x_0, \ x_1, \ x_2, \cdots\cdots) = f(\varphi(x_0), \ \varphi(x_1), \ \varphi(x_2), \cdots\cdots), \ f \in C^{\infty}{}_{U(\Delta(\boldsymbol{E}))}(M),$$

we have

$$\int_{\varphi[\sigma, \{c_n\}]} f = \int_{[\sigma, \{c_n\}]} \varphi^* f.$$

Here $\varphi^* f$ is identified to $j^*(\varphi^* f)$ and the integral of the right hand side is the integration of ∞ -cochain of $\boldsymbol{E}_0$ defined in n°10.

By the definition of the integral and theorem 4, we get

Theorem 4'. $\int_{\varphi[\sigma, \{c_n\}]} f$ *exists if* $\varphi^* f$ *is class* $(H_{\{c_n\}})$ *with respect to* $\{M\}$ *and* $\{\varepsilon_n\}$. *Moreover, if* $\varphi^* f$ *satisfies the assumption of note* 1 *of n°*10, *then* $\int_{\varphi[\sigma, \{c_n\}]} f \neq 0$.

As usual, we define the chain γ of M by the finite formal sum

$$\gamma = \sum_{i=1}^{s} a_i \varphi_i [\sigma, \ \{c_n\}].$$

Then we define the integral of f, a standard $\infty -$cochain of M with respect to $U(\Delta(\boldsymbol{E}))$, on γ, denoted by $\int_\gamma f$, by

$$(35) \qquad \int_\gamma f = \sum_{i=1}^{s} a_i \int_{\varphi_i[\sigma, \{c_n\}]} f.$$

Here we assume $U(\Delta(\boldsymbol{E}))$ satisfies $(I)_{\varphi_i[\sigma, \{c_n\}]}$ for each i.

On the other hand, we get by the definition of the integral

$$(36) \qquad \int_\gamma \Big(\sum_{i=1}^{r} b_i f_i \Big) = \sum_{i=1}^{r} b_i \int_\gamma f_i,$$

if $\int_\gamma f_i$ exists for each i.

Definition. *We call f is absolutely integrable on γ if γ is given by $\sum_i a_i\varphi_i\,[\sigma, \{c_n\}]$ and $\int_{\varphi_i[\sigma, \{c_n\}]} |f|$ exists for each i. Here $|f|$ means the ∞-cochain with representation $|f|$.*

Definition. *We call $U(\Delta(\boldsymbol{E}))$ satisfy the condition (S) if $U(\Delta(\boldsymbol{E}))$ satisfies $(I)_{[\sigma, \{c_n\}]}$ (or $(I)_{\varphi[\sigma, \{c_n\}]}$) and $\{c'_n\}=O(\{c_n\})$, then $U(\Delta(\boldsymbol{E}))$ also satisfies $(I)_{[\sigma, \{c'_n\}]}$ (or $(I)_{\varphi[\sigma, \{c'_n\}]}$).*

Definition. *For $\mathfrak{s} \in \mathfrak{S}_m$, considering $\mathfrak{s}$ to be a transposition of $\{0, 1, \cdots\cdots, m-1\}$, we set*

(37) $$\mathfrak{s}[\sigma, \{c_n\}]=\{x \mid x=\sum_{k=1}^{\infty} x_k e_k,\ 0 \leqq x_k \leqq c_{\mathfrak{s}(n)},\ \mathfrak{s}(n)=n,\ n \geqq m\}.$$

We note that by definition, we mayewrite

$$\mathfrak{s}[\sigma, \{c_n\}]=[\sigma, \{c_{\mathfrak{s}(n)}\}],\ \{c_{\mathfrak{s}(n)}\}=O(\{c_n\}).$$

For $\gamma=\sum_i a_i\varphi_i[\sigma, \{c_n\}]$, we set

(37)′ $$\mathfrak{s}(\gamma)=\sum_{i=1}^{s} a_i\varphi_i(\mathfrak{s}[\sigma, \{c_n\}]).$$

By theorem 2 of [4], we have

Lemma 9. *If f is absolutely integrable on γ and alternative, $U(\Delta(\boldsymbol{E}))$ satisfies the condition (S) and $(I)_{\varphi_i[\sigma, \{c_n\}]}$ for each i, then we have*

(38) $$\int_{\mathfrak{s}(\gamma)} f=\operatorname{sgn}(\mathfrak{s})\int_\gamma f,$$

for any $\mathfrak{s} \in \mathfrak{S}_m$, m *is* arbitraly.

§ 4. Integration of Alexander-Spanier cochain of degree $\infty-p$

12. ***Definition.*** *If $U(\Delta(\boldsymbol{E}))$satisfies $(I)_{[\sigma, \{c_n\}]}$ and f is a standard cochain of degree $(\infty-p)$ of $\boldsymbol{E}_0$ with respect to $U(\Delta(\boldsymbol{E}))$, then we define the integral of f on $\tau_I^{-1}[\sigma, \{c_n\}]$ by*

(39) $$\int_{\tau_I^{-1}[\sigma, \{c_n\}]} f=\int_{[\sigma, \{c_n\}]} (\tau_I^{\#})^{-1}f.$$

Here, the right hand side is the integration of $(\tau_I^{\#})^{-1}f \in C^\infty{}_{U(\Delta(\boldsymbol{E}))}(\boldsymbol{E}_0)$ on $[\sigma, \{c_n\}]$.

Note. Directly, $\int_{\tau_I^{-1}[\sigma, \{c_n\}]} f$ is defined as follows: We set

$$J_I=(j_{v_I}{}^{-1}(1),\ j_{v_I}{}^{-1}(2),\cdots\cdots,\ j_{v_I}{}^{-1}(k),\cdots\cdots),$$

$$(J+1_i)_I$$
$$=(j_{v_I}{}^{-1}(1),\ j_{v_I}{}^{-1}(2),\cdots\cdots,\ j_{v_I}{}^{-1}(i-1),\ j_{v_I}{}^{-1}(i)+1,\ j_{v_I}{}^{-1}(i+1),\cdots\cdots),$$

$$[J_I]_k$$
$$=\{J_I \mid J_I=(j_{v_I}{}^{-1}(1),\ j_{v_I}{}^{-1}(2),\cdots\cdots,\ j_{v_I}{}^{-1}(k),\ 0,\ 0,\cdots\cdots),$$

$$(J_I)_k=[J_I]_k-[J_I]_{k-1},$$

$$x_{I_J}=\sum_{k=1}^{\infty} x_{j_{v_I}{}^{-1}(k)}\, e_{v_I{}^{-1}(k)},$$

$$x_{(J+1_iI)}=\sum_{k\neq i} x_{j_{v_I}{}^{-1}(k)}\, e_{v_I{}^{-1}(k)}+x_{j_{v_I}{}^{-1}(i)+1}\, e_{v_I{}^{-1}(i)+1}.$$

Then, since we have

$$\tau_I{}^{1}([\sigma,\ \{c_n\}])$$
$$=\left\{x \mid x=\sum_{k=1}^{\infty} x_{v_I{}^{-1}(k)}\, e_{v_I{}^{-1}(k)},\ 0\leqq x_{v_I{}^{-1}(k)}\leqq c_k\right\},$$

we may define $\int_{\tau_I{}^{-1}[\sigma,\ \{c_n\}]} f$ by

(40)
$$\int_{\tau_I{}^{-1}[\sigma,\ \{c_n\}]} f$$
$$=\lim_{|x_{v_I{}^{-1}(k),\ p_q}-x_{v_I{}^{-1}(k),\ p_q+1}|\to 0}\ \lim_{m\to\infty}\sum_{k=1}^{\infty}\Big(\sum_{J_I\in(J_I)_k} j^{-1}(f)\ (x_{J_I},$$
$$x_{(J+1_1)I},\ x_{(J+1_2)I},\cdots\cdots)\Big).$$

Definition. *Let φ be a continuous map from $\tau_I{}^{-1}[\sigma,\ \{c_n\}]$ into M, a Banach manifold modeled by $\boldsymbol{E}_0$, and assume $U(\Delta(\boldsymbol{E}))$ satisfies $(I)_{\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}])}$. Then for a standard $(\infty-p)$-cochain with respect to $U(\Delta(\boldsymbol{E}))$ and $\boldsymbol{I}$ of M with the representation f, we define the integration of f on $\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}])$, denoted by $\int_{\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}])} f$, by*

(40)′
$$\int_{\varphi(\tau_I{}^{-1}[\sigma,\ \}c_n\}])} f$$
$$=\int_{\tau_I{}^{-1}[\sigma,\ \{c_n\}]} \varphi^* f$$
$$=\lim_{|x_{v_I{}^{-1}(k),\ p_q}-x_{v_I{}^{-1}(k),\ p_q+1}|\to 0}\ \lim_{m\to\infty}\sum_{k=1}^{\infty}\Big(\sum_{J_I\in(J_I)_k} j^{-1}(f)\ (\varphi(x_{J_I}),$$
$$\varphi(x_{(J+1_1)I}),\ \varphi(x_{(J+1)_2I}),\cdots\cdots)\Big).$$

By definition, we have

$$\int_{\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])} f = \int_{[\sigma,\ \{c_n\}]} (\tau_I^{\#})^{-1}(\varphi^* f), \tag{39$'$}$$

and if f is 0-normal and *codim.* $\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])$ is defined (for example, if M and φ are both differentiable), then

$$\int_{\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])} f = 0,\ \ if\ codim.\ \varphi(\tau_I^{-1}[\sigma,\ \{c_n\}]) \neq p.$$

By (39) and (39)$'$, we obtain

***Lemma* 10.** *If* $\varphi_I : \tau_I^{-1}[\sigma,\ \{c_n\}] \to M$ *and* $\varphi_J : \tau_J^{-1}[\sigma,\ \{c_n\}] \to M$ *are given to satisfy*

$$\varphi_I = \varphi_J \iota^I{}_J, \tag{41}$$

then we have

$$\int_{\iota^I{}_J(\tau_J^{-1}[\sigma,\ \{c_n\}])} f = \int_{\tau_J^{-1}[\sigma,\ \{c_n\}]} (\iota^I{}_J)^{\#} f, \tag{42}$$

$$\int_{\varphi_I(\tau_I^{-1}[\sigma,\ \{c_n\}])} f = \int_{\varphi_I(\tau_I^{-1}[\sigma,\ \{c_n\}])} (\iota^I{}_J)^{\#} f. \tag{42$'$}$$

By lemma 10, to define a singular $(\infty - p)$-simplex of M to be the equivalence class of $\{\varphi_I(\tau_I^{-1}[\sigma,\ \{c_n\}])\}$ by the equivalence relation $\varphi_I \sim \varphi_J$ if and only if φ_I and φ_J satisfy (41), and denote this equivalence class by $\varphi_{\infty-p}[\sigma,\ \{c_n\}]$, we can define the integral of a standard cochain $f \in C^{\infty-p}{}_{U(\Delta(E))}(M)$ on $\varphi_{\infty-p}[\sigma,\ \{c_n\}]$, denoted by $\int_{\varphi_{\infty-p}[\sigma,\ \{c_n\}]} f$, *by*

$$\int_{\varphi_{\infty-p}[\sigma,\ \{c_n\}]} f = \int_{\varphi_I[\sigma,\ \{c_n\}]} \rho_I f. \tag{43}$$

As usual, we define the singular $(\infty - p)$-cochain γ of M by the finite formal sum

$$\gamma = \sum_{k=1}^{s} a_k \varphi_k\ (\tau_{I(k)}{}^{-1}[\sigma,\ \{c_{n(k)}\}]),$$

Where $I(k) = \{i_{k,1}, \cdots\cdots, i_{k,p}\}$, $0 \leq i_{k,1} < i_{k,2} < \cdots\cdots < i_{k,p}$, for each k. Then if $U(\Delta(E))$ satisfies $(I)_{\varphi_k(\tau_{I(k)}{}^{-1}[\sigma,\ \{c_{n(k)}\}])}$ for each k, we define the integral of a standard cochain f of degree $(\infty-p)$ with respect to $U(\Delta(\boldsymbol{E}))$ on M, on γ, denoted by $\int_{\gamma} f$, *by*

$$(35)' \qquad \int_{\gamma} f=\sum_{k=1}^{s} a_k \int_{\varphi_k(\tau_{I(k)}{}^{-1}[\sigma,\ \{c_{n(k)}\}])} \rho_k f.$$

Then we also get

$$(36)' \qquad \int_{\gamma} \sum_{i=1}^{r} b_i f_i=\sum_{i=1}^{r} b_i \int_{\gamma} f_i,$$

$$(38)' \qquad \int_{\mathfrak{S}(\gamma)} f=\operatorname{sgn}(\mathfrak{S}) \int_{\gamma} f, \text{ if } f \text{ is alternative and } U(\Delta(\boldsymbol{E})) \text{ satisfies } (S).$$

13. For $[\sigma,\ \{c_n\}]$, we set

$$[\sigma,\ \{c_n\}]_k=\left\{x \mid x=\sum_{m=1}^{k} x_m e_m,\ 0 \leqq x_m \leqq c_m\right\},$$

$$[\sigma,\ \{c_n\}]_{\infty-k}=\left\{x \mid x=\sum_{m=k+1}^{\infty} x_m e_m,\ 0 \leqq x_m \leqq c_m\right\},$$

$$[\sigma,\ \{c_n\}]_{x_k=a_k}$$
$$=\left\{x \mid x=a_k e_k+\sum_{m\neq k} x_m e_m,\ 0 \leqq x_m \leqq c_m\right\}.$$

By definition, we have

$$[\sigma,\ \{c_n\}]=[\sigma,\ \{c_n\}]_k \times [\sigma,\ \{c_n\}]_{\infty-k},$$
$$(\partial[\sigma,\ \{c_n\}]_k) \times [\sigma,\ \{c_n\}]_{\infty-k}$$
$$=\sum_{i=1}^{k} (-1)^{i-1}([\sigma,\ \{c_n\}]_{x_i=0}-[\sigma,\ \{c_n\}]_{x_i=c_i}) \times [\sigma,\ \{c_n\}]_{\infty-k}.$$

Here $\partial[\sigma,\ \{c_n\}]_k$ means the usual boundary of k -cube $[\sigma,\ \{c_n\}]_k$.

***Lemaa* 11.** *Let $U(\Delta(\boldsymbol{E}))$ satisfy (S) and $(I)_{[\sigma,\ \{c_n\}]}$ and $f \in C^{\infty-1}{}_{U(\Delta(\boldsymbol{E}))}(\boldsymbol{E}_0)$ to be standard and $\rho_{\{k\}} f$ is class $(H_{\{c_n\}})$ with respect to (M) and $\{\varepsilon_n\}$, $\{\varepsilon_n\}=O(\{c_n\})$, on $[\sigma,\ \{c_n\}]_{x_k=0}$ and $[\sigma,\ \{c_n\}]_{x_k=c_k}$ for each k. Then to set*

$$(44)_k \qquad \partial_k[\sigma,\ \{c_n\}]\ (\lambda)$$

$$=\sum_{m=1}^{k} (-1)^{m-1}\, \mathrm{e}^{-(m-1)\lambda}\, ([\sigma,\ \{c_n\}]_{x_m=0}-[\sigma,\ \{c_n\}]_{x_m=c_m}),$$

$lim_{k\to\infty}\int_{\partial_k[\sigma,\ \{c_n\}](\lambda)} f$ *exists if* Re. $(\lambda) > 0$ *and we have*

(45) $$\lim_{k\to\infty}\int_{\partial_k[\sigma,\ \{c_n\}](\lambda)} f = O(\lambda^{-1}),\quad \text{Re.}\,\lambda \downarrow 0.$$

Proof. By assumption and theorem 4, we have

$$\begin{aligned}
&\left|\int_{\partial_k[\sigma,\ \{c_n\}](\lambda)} f\right| \\
&\leqq \sum_{m=1}^{k} e^{-(m-1)\,\text{Re.}\,\lambda}\left\{\left|\int_{[\sigma,\ \{c_n\}]_{x_m=0}} f\right| + \left|\int_{[\sigma,\ \{c_n\}]_{x_m=c_m}} f\right|\right\} \\
&\leqq \sum_{m=1}^{k} e^{-(m-1)\,\text{Re.}\,\lambda}\, 4M \\
&\leqq \frac{4M}{1-e^{-\text{Re.}\,\lambda}},
\end{aligned}$$

we have the lemma.

Definition. *We set* $\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)$ *to be the formal sum*

(44) $$\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda) = \sum_{m=1}^{\infty}(-1)^{m-1}\lambda^\alpha e^{-(m-1)\lambda}\left([\sigma,\ \{c_n\}]_{x_m=0} - [\sigma,\ \{c_n\}]_{x_m=c_m}\right).$$

By lemma 11, if f satisfies the assumptions of lemma 11, then we can define $\int_{\partial_\alpha[\sigma,\ \{c_n\}](\lambda)} f$ by

$$\int_{\partial_\alpha[\sigma,\ \{c_n\}](\lambda)} f = \lim_{k\to\infty}\lambda^\alpha\int_{\partial_k[\sigma,\ \}c_n\}](\lambda)} f,$$

if Re. $\alpha > 0$. Moreover, by Fatou's lemma, $lim_{t\to 0}\int_{\partial\alpha[\sigma,\ \{c_n\}](t+is)} f$ exists for almost all t if Re. $\alpha = 1$.

Definition. *For an* $(\infty - p)$ *singular simplex with the representation* $\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])$, *we set*

(44)′ $$\begin{aligned}
&\partial_\alpha\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])\ (\lambda) \\
&= \varphi(\tau_I^{-1}(\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)) \\
&= \sum_{m=1}^{\infty}(-1)^{m-1}\lambda^\alpha e^{-(m-1)\lambda}\left(\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])_{x_{v_I^{-1}(m)}=0} - \right.
\end{aligned}$$

$$-\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])_{x_{v_I^{-1}(m)}=c_m})).$$

Here, $(\tau_I^{-1}[\sigma,\ \{c_n\}])_{x_{v_I^{-1}(m)}=a}$ *is defined similarly as* $[\sigma,\ \{c_n\}]_{x_m=a}$.

Similarly, for $\gamma=\sum_k a_k\varphi_k(\tau_{I(k)})^{-1}[\sigma,\ \{c_n\}])$, we set

$$\partial_\alpha\gamma(\lambda)=\sum_{k=1}^{s} a_k\partial_\alpha\varphi_k(\tau_{I(k)})^{-1}[\sigma,\ \{c_n\}]\ (\lambda). \tag{44}''$$

***Lemma* 11′.** *Let γ be an $(\infty-p)$ singular chain of M and $U(\Delta(\boldsymbol{E}))$ satisfy (S) and $(I)_{\varphi_k(\tau_{I(k)}^{-1}[\sigma,\ \{c_n\}])}$ for each k, $f\in C^{\infty-p}{}_{U(\Delta(\boldsymbol{E}))}(M)$ be such that f is standard, $\rho_{I_k,m}\ f$ is class $(H_{\{c_n\}})$ with respect to $\{M\}$ and $\{\varepsilon_n\}$ on $\varphi_k(\tau_{I(k)})^{-1}[\sigma,\ \{c_n\}])_{x_{v_I^{-1}(m)}=0}$ and $\varphi_k(\tau_{I(k)})^{-1}[\sigma,\ \{c_n\}])_{x_{v_I^{-1}(m)}=c_m}$ for each k and m. Then $\int_{\partial_\alpha\gamma(\lambda)} f$ exists if* Re. *$\lambda>0$ and we have*

$$\int_{\partial_\alpha\gamma(\lambda)} f=O((\mathrm{Re.}\ \lambda)^{\mathrm{Re.}\ \alpha-1}),\ \ \mathrm{Re.}\,\lambda\downarrow 0. \tag{46}$$

§ 5. Stokes' theorem

14. We denote by $\boldsymbol{E}_{0,k}$ the subspace of $\boldsymbol{E}_0$ spanned by $e_1,\cdots\cdots,\ e_k$. The inclusion $\boldsymbol{E}_{0,k}\to\boldsymbol{E}_0$ is denoted by ι^k. Then $(\iota^k)^{-1}(x_{J+1_i})$ is defined if $J\in[J]_k$ and $1\leq i\leq k$ and we have

$$\begin{aligned}&j^{-1}(f)\ (x_J,\ x_{J+1_1},\cdots\cdots,\ x_{J+1_i},\cdots\cdots)\\&=j^{-1}(f)\ ((\iota^k)^{-1}(x_J),\ (\iota^k)^{-1}(x_{J+1_1}),\cdots\cdots,\ (\iota^k)^{-1}(x_{J+1^k}),\\&\quad(\iota^k)^{-1}(x_J)+x_{k+1}\,e_{k+1},\cdots\cdots,\ (\iota^k)^{-1}(x_J)+x_{m,1}e_m,\cdots\cdots),\end{aligned}$$

for any $f\in C^\infty{}_{U(\Delta(\boldsymbol{E}))}(\boldsymbol{E}_0)$.

We set

$$te_{\infty-k}(\{c_n\})=\sum_{m=k+1}^{\infty} tc_m e_m,\ \ 0\leq t\leq 1,$$

and define an Alexander-Spanier k- cochain $j^{-1}(f)_k\ (te_{\infty-k}(\{c_n\}))$, $te_{\infty-k}(\{c_n\})$ is a parameter, of $\boldsymbol{E}^k$ for $j^{-1}(f)$ by

$$\begin{aligned}&j^{-1}(f)_k\ (x_0,\ x_1,\cdots\cdots,\ x_k)\ (te_{\infty-k}\ (\{c_n\}))\\&=j^{-1}(f)\ (\iota^k(x_0),\ \iota^k(x_1),\cdots\cdots,\ \iota^k(x_k),\ \iota^k(x_0)+tc_{k+1}\ e_{k+1},\cdots\cdots,\\&\quad\iota^k(x_0)+tc_m\ e_m,\cdots\cdots).\end{aligned}\tag{47}$$

***Lemma* 12.** If $\int_{[\sigma,\ \{c_n\}]} f$ exists, then

$$(48)\qquad \int_{[\sigma,\ \{c_n\}]} f$$
$$=\lim_{t\to 0}\ \lim_{m\to\infty} \sum_{k=1}^{m} \Big\{\int_{[\sigma,\ \{c_n\}]_k} j^{-1}(f)_k\ (te_{\infty-k}(\{c_n\}))$$
$$-\int_{[\sigma,\ \{c_n\}]_{k-1}} j^{-1}(f)_{k-1}\ (te_{\infty-k+1}(\{c_n\}))\Big\}.$$

Here $\int_{[\sigma,\ \{c_n\}]_k} j^{-1}(f)_k\ (te_{\infty-k}(\{c_n\}))$ *is the integral of Alexander-Spanier* k *-cochain* $j^{-1}(f)_k$ *on* k *-simplex* $[\sigma,\ \{c_n\}]_k$ ([4]) *and we set* $\int_{[\sigma,\ \{c_n\}]_0} j^{-1}(f)_0=0$. *Moreover, if* f *satisfies the assumptions of theorem* 4, *then*

$$(49)\qquad \int_{[\sigma,\ \{c_n\}]} f$$
$$=\lim_{t\to 0}\ \lim_{m\to\infty} \sum_{k=1}^{m} \int_{[\sigma,\ \{c_n\}]_k} j^{-1}(f)_k\ (te_{\infty-k}(\{c_n\})).$$

Proof. By the definition of the integral, we have (48). On the other hand, since

$$\sum_{k=1}^{m} \Big|\int_{[\sigma,\ \{c_n\}]_{k-1}} j^{-1}(f)_{k-1}(te_{\infty-k+1}(\{c_n\}))\Big|$$
$$\leqq\sum_{k=1}^{m} Mc_1\cdots\cdots c_{k-1}t^{-k+1}\,\frac{1}{H(t^{-1})}$$
$$\leqq Mt,$$

we get (49).

Similarly, by (39), we have

***Lemma* 12′.** *Let* f *be an element of* $C^{\infty-p}{}_{U(\Delta(E)),\ I}\,(E_0)$ *such that* $\int_{\tau_I[\sigma,\ \{c_n\}]} f$ *exists, then*

$$(48)'\qquad \int_{\tau_I[\sigma,\ \{c_n\}]} f$$
$$=\lim_{t\to 0}\ \lim_{m\to\infty} \sum_{k=1}^{m} \Big\{\int_{\tau_I[\sigma,\ \{c_n\}]_k} j^{-1}((\tau_I^{\#})^{-1}(f))_k\ (te_{\infty-k}\ (\{cn\}))$$
$$-\int_{\tau_I([\sigma,\ \{c_n\}]_{k-1}} j^{-1}((\tau_I^{\#})^{-1}(f))_{k-1}\ (te_{\infty-k}\ (\{cn\}))\Big\}.$$

Moreover, if $(\tau_I^{\#})^{-1} f$ *satisfies the assumption of theorem* 4, *then*

(49)′
$$\int_{\tau_I[\sigma,\ \{c_n\}]} f$$
$$=\lim_{t\to 0}\lim_{m\to\infty}\sum_{k=1}^{m}\int_{\tau_I([\sigma,\ \{c_n\}]_k} j^{-1}((\tau_I^{\#})^{-1}(f)_k\ (te_{\infty-k}(\{c_n\})).$$

15. *Theorem* 5. *Let* $\{c_n\}$ *be monotone decreasing and* $f\in C^{\infty-1}{}_{U(\Delta(E))}(E_0)$ *be standard, alternative and satisfy the assumptions of lemma* 11. *Then, if* $lim_{\lambda\to 0}\int_{[\sigma,\ \{c_n\}]}\delta_\alpha f(\lambda)$ *and* $lim_{\lambda\to 0}\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)} f$ both *exist, we have*

(50)
$$\lim_{\lambda\to 0}\int_{[\sigma,\ \{c_n\}]}\delta_\alpha\ f(\lambda)=\lim_{\lambda\to 0}\int_{\partial_\alpha[\sigma,\ \{c_n\}](\lambda)} f.$$

Proof. First we note that, by assumption, $\delta_\alpha f(\lambda)$ satisfies the assumptions of theorem 4 for Re. $\lambda>0$. Hence $\int_{[\sigma,\ \{c_n\}]}\delta_\alpha f(\lambda)$ exists for Re. $\lambda>0$. Therefore by (49), we have

$$\int_{[\sigma,\ \{c_n\}]}\delta_\alpha f(\lambda)$$
$$=\lim_{t\to 0}\lim_{m\to\infty}\sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k} j^{-1}\ (\delta_\alpha f(\lambda))_k\ \ (te_{\infty-k}(\{c_n\})).$$

On the other hand, since we have

$$j^{-1}(\delta_\alpha f(\lambda))$$
$$=\lambda^\alpha\Big(\sum_{m=0}^{\infty}(-1)^m\ \mathrm{e}^{-m\lambda}\ j^{-1}(f)\ (x_0,\ x_1,\cdots\cdots,\ x_{m-1},\ x_{m+1},\cdots\cdots)\Big),$$

to set

$$\delta_\alpha j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))$$
$$=\lambda^\alpha\Big(\sum_{m=0}^{k}(-1)^m\ e^{-m\lambda}\ j^{-1}(f)_k\ (x_0,\ x1,\cdots\cdots,\ x_{m-1},\ x_{m+1},\cdots\cdots,\ x_k)\ te_{\infty-k}\ (\{c_n\})\Big),$$
$$\delta_\alpha j^{-1}(f)_{\infty-k}\ (\lambda)\ (x_0,\ x_1,\cdots\cdots,\ x_k)\ (te_{\infty-k}\ (\{c_n\}))$$
$$=\lambda^\alpha\Big(\sum_{m=k+1}^{\infty}(-1)^m\ e^{-m\lambda}\ j^{-1}\ (f)\ (\iota^k\ (x_0),\ \iota^k\ (x_1),\cdots\cdots,\ \iota^k\ (x_k),\ \iota^k\ (x_0)+tc_{k+1}\ e_{k+1},$$
$$\cdots\cdots,\ \iota^k(x_0)+tc_{m-1}\ e_{m-1},\ \iota^k\ (x_0)+tc_{m+1}\ e_{m+1},\cdots\cdots),$$

we get

(51)
$$j^{-1}(\delta_\alpha f\ (\lambda))_k\ (te_{\infty-k}\ (\{c_n\}))$$
$$=\delta_\alpha\ j^{-1}\ (f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))+\delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\})).$$

Hence we have by Stokes' theorem for the integrals of Alexander-Spanier cochains of finite degree ([4]),

$$\int_{[\sigma,\ \{c_n\}]_k} j^{-1}(\delta_\alpha f(\lambda))_k\ (te_{\infty-k}\ (\{c_n\}))$$

$$=\int_{[\sigma,\ \{c_n\}]_k} \delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}(\{c_n\}))$$

$$+\int_{[\sigma,\ \{c_n\}]_k} \delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))$$

$$=\int_{[\sigma,\ \{c_n\}]_k} \lambda^\alpha\ \delta(j^{-1}(f)_{k-1}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\})))$$

$$+\int_{[\sigma,\ \{c_n\}]_k} (\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))-\delta(j^{-1}(f)_{k-1}\ (\lambda)\ te_{\infty-k}\ (\{c_n\}))))$$

$$+\int_{[\sigma,\ \{c_n\}]_k} \delta_\alpha\ j^{-1}\ (f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))$$

$$=\int_{\partial[\sigma,\ \{c_n\}]_k} \lambda^\alpha\ j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))$$

$$+\int_{[\sigma,\ \{c_n\}]_k} (\delta^\alpha\ j^{-1}\ (f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))-\lambda^\alpha\ \delta(j^{-1}\ (f)_{k-1}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))))$$

$$+\int_{\{\sigma,\ \}c_n\}]_k} \delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))$$

$$=\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)\ {}_{k-1}} j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))$$

$$+\left\{\int_{\partial[\sigma,\ \{c_n\}]_k} \lambda^\alpha\ j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))\right.$$

$$\left.-\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)_k} j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))\right\}$$

$$+\int_{[\sigma,\ \{c_n\}]_k} (\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))-\lambda^\alpha\ \delta(j^{-1}(f)_{k-1}\ (\lambda)\ te_{\infty-k}\ (\{c_n\})))$$

$$+\int_{[\sigma,\ \{c_n\}]_k} \delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\})).$$

Here, $\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)_{k-1}$ is defined similarly as $[\sigma,\ \{c_n\}]_{k-1}$. Then, since we have by lemma 12 and the assumptions,

$$\int_{\partial_\alpha[\sigma,\ \{c_n\}](\lambda)} f$$

$$=\lim_{t\to0}\ \lim_{m\to\infty}\ \sum_{k=1}^{m} \int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)_k} j^{-1}(f)_k\ (te_{\infty-k}\ (\{c_k\})),$$

to show (50), it is sufficient to show

$$\text{(52)}\qquad \lim_{\lambda\to0}\ \lim_{t\to0}\ \lim_{m\to\infty}\ \sum_{k=2}^{m} \int_{\partial[\sigma,\ \{c_n\}]_k} \lambda^\alpha\ j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))$$

$$-\int_{\partial\alpha[\sigma,\ \{c_n\}]\ (\lambda)_k} j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))$$

$$=0,$$

(53) $$\lim_{\lambda\to 0}\lim_{t\to 0}\lim_{m\to\infty}\sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k}(\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}(\{c_n\}))$$
$$-\lambda^\alpha\ \delta(j^{-1}(f)_{k-1}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))))$$
$$=0,$$

(54) $$\lim_{\lambda\to 0}\lim_{t\to 0}\lim_{m\to\infty}\sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_{\infty-k}(\lambda)\ (te_{\infty-k}\ (\{c_n\}))=0.$$

But, since $\lim_{\lambda\to 0}\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)} f$ exists by assumption, we have

$$\lim_{\lambda,\ \mu\to 0}\lim_{t\to 0}\lim_{m\to\infty}\sum_{k=2}^{m}\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)_k} j^{-1}(f)_{k-1}\ (te_{\infty-k}(\{c_n\}))$$
$$-\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\mu)_k} j^{-1}(f)_{k-1}\ (te_{\infty-k}\ (\{c_n\}))$$
$$=0.$$

On the other hand, we also know

$$\int\lim_{\lambda\to 0}\int_{\partial_0[\sigma,\ \{c_n\}]\ (\lambda)_k} j^{-1}(f)_{k^-1}\ (te_{\infty-k}\ (\{c_n\}))$$
$$=\int_{\partial[\sigma,\ \{c_n\}]_k} j^{-1}(f)_{k^{-1}}\ (te_{\infty-k}\ (\{cn\})).$$

Hence we have (52).

To show (54), first we note that by the alternativity of f and the monotoneness of $\{cn\}$, we get

$$\left|\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))\right|$$
$$\leq c_1\cdots\cdots c_k t^{-k}\frac{M}{H(t^{-1})}\ t\left(\sum_{n=1}^{\infty}\ (c_{k+2n-1}\ \mathrm{e}^{-(2n-1)\mathrm{Re.}\ \lambda}-c_{k+2n}\ \mathrm{e}^{-2n\mathrm{Re.}\ \lambda})\right)\ \mathrm{e}^{-k\mathrm{Re.}\ \lambda}.$$

Hence we have

$$\left|\sum_{k=1}^{m}\int_{[\sigma.\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))\right|$$
$$\leq\sum_{k=1}^{m}c_1\cdots\cdots c_k t^{-k}\frac{M}{H(t^{-1})}\ t\left(\sum_{n=1}^{\infty}\ (_{k+2n-1}\ e^{-(2n-1)\mathrm{Re.}\ \lambda}\right.$$

$$-c_{k+2n}\ e^{-2n\ \mathrm{Re.}\ \lambda}))\ e^{-k\ \mathrm{Re.}\ \lambda}$$

$$\leqq Mt\ \left(\sum_{n=1}^{\infty}(c_{2n}\ e^{-2n\ \mathrm{Re.}\ \lambda}-c_{2n+m}\ e^{-(2n+m)\ \mathrm{Re.}\ \lambda})\right).$$

Therefore we get

$$\lim_{m\to\infty}\ \left|\ \sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_{\infty-k}\ (\lambda)\ (te_{\infty-k}(\{c_n\}))\ \right|$$

$$\leqq Mt\ \left(\sum_{n=1}^{\infty}c_{2n}\ e^{-2n\ \mathrm{Re.}\ \lambda}\right).$$

This shows (54).

Then, since we know

$$\int_{[\sigma,\ \{c_n\}]_k}j^{-1}(\delta_\alpha\ f(\lambda))_k$$

$$=\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_k\ \ (\lambda)+\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_{\infty-k},$$

for any $\varepsilon>0$, there exists $a\ \delta>0$ and $\varepsilon_k>0$, $k=1,\ 2,\cdots\cdots$ such that

$$\left|\ \int_{[\sigma,\ \{c_n\}]_k}j^{-1}(\delta_\alpha\ f(\lambda))_k\ (te_{\infty-k}\ (\{c_n\}))\right.$$

$$\left.-\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))\ \right|$$

$$<\varepsilon_k,\ \ if\ t<\delta,$$

$$\sum_{k=1}^{\infty}\varepsilon_k\leqq\varepsilon.$$

Therefore we have

$$\left|\ \lim_{m\to\infty}\ \left(\sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k}j^{-1}(\delta_\alpha\ f(\lambda))_k\ (te_{\infty-k}\ (\{c_n\}))\right.\right.$$

$$\left.\left.-\sum_{k=1}^{m}\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_k\ (te_{\infty-k}\ (\{c_n\})\right)\ \right|$$

$$<\varepsilon,\ if\ t<\delta.$$

Hence we obtain

$$\lim_{t\to0}\ \lim_{m\to\infty}\ \sum_{k=1}^{\infty}\int_{[\sigma,\ \{c_n\}]_k}\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))\tag{55}$$

$$=\int_{[\sigma,\ \{c_n\}]} \delta_\alpha f(\lambda).$$

Then, since $lim_{\lambda\to 0}\int_{[\sigma,\ \{c_n\}]} \delta_\alpha\ f(\lambda)$ exists by assumption, we get

$$\begin{aligned}&\lim_{\lambda,\ \mu\to 0}\ \lim_{t\to 0}\ \lim_{m\to\infty} \sum_{k=1}^{m} \int_{[\sigma,\ \{c_n\}]_k} (\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ te_{\infty-k}\ (\{c_n\}))\\ &\quad -\delta_\alpha\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\})))\\ &=0.\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}&\lim_{\lambda\to 0}\int_{[\sigma,\ \{c_n\}]_k} (\delta_0\ j^{-1}(f)_k\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))\\ &=\int_{[\sigma,\ \{c_n\}]_k} \delta(j^{-1}(f)_{k-1}\ (\lambda)\ (te_{\infty-k}\ (\{c_n\}))).\end{aligned}$$

Therefore we obtain (53).

16. ***Theorem* 5′.** *we assume γ, $U(\Delta(E))$ and f all satisfy the assumptions of lemma* 11′. *Moreover, we assume $\{c_n\}$ is monotone decreasing and f is alternative. Then, if $\lim_{\lambda\to 0}\int_\gamma \delta_\alpha f\ (\lambda)$ and $\lim_{\lambda\to 0}\int_{\partial_\alpha\ \gamma(\lambda)} f$ both exist, we have*

$$(50)' \qquad \lim_{\lambda\to 0}\int_\gamma \delta_\alpha\ f(\lambda)=\lim_{\lambda\to 0}\int_{\partial_\alpha\gamma(\lambda)} f.$$

Proof. By (44)″ and (35)′, it is sufficient to show (50)′ to prove

$$(50)'' \qquad \lim_{\lambda\to 0}\int_{\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}]} \delta_\alpha\ f(\lambda)=\lim_{\lambda\to 0}\int_{\partial_\alpha\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\})\ (\lambda)} f.$$

But since we know

$$(\tau_I^{\#})^{-1}\ (\varphi^*\delta_\alpha\ f(f))=\delta_\alpha((\tau_\kappa^{\#})^{-1}\ \varphi^*f)\ (\lambda),$$

we get by (39)′,

$$\int_{\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}])} \delta_\alpha\ f(\lambda)=\int_{[\sigma,\ \{c_n\}]} \delta_\alpha((\tau_I^{\#})^{-1}\ \varphi^*f)\ (\lambda).$$

Then by (50), (39)′ and (44)′, we have

$$\lim_{\lambda\to 0}\int_{\varphi(\tau_I{}^{-1}[\sigma,\ \{c_n\}])} \delta_\alpha\ f(\lambda)$$

$$=\lim_{\lambda\to 0}\int_{\partial_\alpha[\sigma,\ \{c_n\}]\ (\lambda)}((\tau_I^{\#})^{-1}\ \varphi^* f)$$

$$=\lim_{\lambda\to 0}\int_{\varphi(\tau_I^{-1}(\partial_\alpha[\tau,\ \{c_n\}]\ (\lambda))} f$$

$$=\lim_{\lambda\to 0}\int_{\partial_\alpha\varphi(\tau_I^{-1}[\sigma,\ \{c_n\}])\ (\lambda)} f.$$

This shows (50)″.

Example. If $f \in C^\infty{}_{U(\Delta(\boldsymbol{E}))}(\boldsymbol{E}_0)$ is standard and $j^{-1}(f)\sim H_{\{c_n\}}$, where f is positive and $U(\Delta(\boldsymbol{E}))$ and f both satisfy the assumptions of theorem 4, then we have

$$\int_{[\sigma,\ \{c_n\}]} f \gneqq 0. \tag{56}$$

In this case, we call f to be a volume element of $\boldsymbol{E}_0$ with respect to $[\sigma,\ \{c_n\}]$. We note that for this f, we have

$$\int_{[\sigma,\ \{c'_n\}]} f=\infty,\ \ if\ \{c_n\}=o(\{c'_n\}), \tag*{$(57)^\infty$}$$

$$\int_{[\sigma,\ \{c''_n\}]} f=0,\ \ if\ \{c''_n\}=o(\{c_n\}). \tag*{$(57)_0$}$$

We also note that starting from f, if $G(a,\ t)$ given by lemma 8 is same order as $1/H(t^{-1})$ for $t\ \downarrow\ 0$, where $H(t)=\sum_{n=1}^{\infty} h_n t^n$ with $h_n \gneqq 0$ for any n. Then to set $c_n=h_{n+1}/h_n$, we have

$$\int_{[\sigma,\ \{c_n\}]} f \gneqq 0,\ \ if\ f\ satisefis\ the\ assumptions\ of\ note\ 1\ of\ n^o 10, \tag*{$(56)'$}$$

if $\boldsymbol{E}_0$ satisfies $\sum_{h=1}^{\infty} x_n e_n \in \boldsymbol{E}_0$, if $|\ x_n\ | \leqq c_n$.

We assume $\boldsymbol{E}_0$ is a Hilbert space and $U(\Delta(\boldsymbol{E}))$ is contained in $U_2(\Delta(\boldsymbol{E}))$. Then by theorem 1, the alternation of f, Af is defined. Hence by theorem 3 and lemm 9, we get

$$\lim_{\lambda\to 0}\int_{\partial_0[\sigma,\ \{c_n\}]\ (\lambda)} k_{1,\ a}\ Af=\int_{[\sigma,\ \{c_n\}]} f \gneqq 0.$$

Therefore, there exists non-exact closed $(\infty-1)$ -Alexander-Spanier cochain in $C^{\infty-1}{}_{U(\Delta(\boldsymbol{E}))}(\partial[\sigma,\ \{c_n\}])$, although $\partial[\sigma,\ \{c_n\}]$ is homeomorphic to $[\sigma,\ \{c_n\}]$ (*cf.* [6], [7]).

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