JOUR. FAC. SCI., SHNSHU UNIV., vol. 8, pp. 1—26, June, 1973

Generalized Derivatives and Their Integrations, I.

By AKIRA ASADA

Department of Mathematics, Faculty of Science,
Shinshu University
(Received April 28, 1963)

Introduction.

In our previous papers [1], [2], [2]′, the author introduced the notion of the generalized tangent of a curve. The motivation of this paper is to consider the following problem. Under what condition, the graph of a function (of one variable) has the generalized tangent ?

To expose the problem precisely, we first review the definition of the generalized tangent. A function f at a, $a \in \boldsymbol{R}^n$, is said to be (right) Gâteaux differentiable at a, or $C(S^{n-1})$-differentiable at a, if the limit

$$d_p f(a, y) = \lim_{t\to 0}.\frac{1}{t}(f(a+ty)-f(a))$$

exists for any y, $||y||=1$, and $d_p f(a, y)$ becomes a continuous function on S^{n-1}. Then, for a curve γ of $\boldsymbol{R}^n$ given by $\gamma : \boldsymbol{I}\to\boldsymbol{R}^n$ such that $\gamma(0)=a$, we call γ has the $C(S^{n-1})$-tangent, or the generalized tangent at a if the limit

$$<\mathfrak{X}\gamma(a),\ d_p f(a)> = \lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{f(\gamma(t))-f(a)}{t}dt$$

exists. Here, $\mathfrak{X}\gamma(a)$ is an element of $C(S^{n-1})^*$, the dual space of $C(S^{n-1})$. It is shown that if γ is smooth at a, then $\mathfrak{X}\gamma(a)=c\delta_y$, where c is a constant and δ_y is the Dirac measure on S^{n-1} concentrated at y. Moreover, in $\boldsymbol{R}^2$, the curve $r\theta=1$ has the generalized tangent $(1/2\pi)d\theta$ at the origin and the graph of $x\sin(1/x)$, $x>0$ has the generalized tangent $(1/\pi\cos^2\theta\sqrt{\cos 2\theta})\ d\theta$, $-(\pi/4)<\theta<(\pi/4)$ at the origin ([1], [2]′).

To consider the above problem, we note

$$\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{f(\gamma(t))-f(a)}{t}$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s d_p f\left(a, \frac{\gamma(t)}{||\gamma(t)||}\right)\frac{||\gamma(t)||}{t}dt.$$

Hence, if γ is the graph of a function φ, then

$$\langle \mathfrak{X}\gamma(a), d_p f(a)\rangle$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s d_p f\left(a, \tan^{-1}\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)\right)\frac{\sqrt{t^2+(\varphi(a+t)-\varphi(a))^2}}{t}dt.$$

Therefore, the graph of φ has the generalized tangent at a if the limit

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(g)$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s g\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt$$

exists for those g such that g is continuous on $\boldsymbol{R}^1$ and $\lim._{x\to\pm\infty} g(x)/(1+|x|)$ both exist. In the above notation, $\mathscr{F}(\boldsymbol{R}^1)$ means the space of those functions.

By this reason, for a fixed function space $\mathscr{F}(\boldsymbol{R}^1)$ over $\boldsymbol{R}^1$ such that whose elements are locally integrable and whose topology is not weaker than that of $L^1_{loc.}(\boldsymbol{R}^1)$, if the limit

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f)$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt$$

exists, then we call $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$, an element of $\mathscr{F}(\boldsymbol{R}^1)^*$, to be the (right) $\mathscr{F}(\boldsymbol{R}^1)$-derivate of φ at a, or the generalized derivate of φ at a. For example, if we take $\mathscr{F}(\boldsymbol{R}^1)$ to be $\boldsymbol{R}\{x\}$, the 1-dimensional vector space generated by the function x, then

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(cx)$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{k\to 0}.\int_h^s c\frac{\varphi(a+t)-\varphi(a)}{t}dt.$$

Therefore, we may consider $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ to be the right Borel derivate $B_r\varphi(a)$ of φ at a ([6], [9], [11], cf. also [4]). But our main interest is in the case that $\mathscr{F}(\boldsymbol{R}^1)$ is dense in $C(R^1)$, the space of continuous functions on $\boldsymbol{R}^1$. In this case, $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ is a measure on $\boldsymbol{R}^1\cup\{\pm\infty\}$, the 2-points compactification of $\boldsymbol{R}^1$, and it is a probabilistic distribution on $\boldsymbol{R}^1\cup\{\pm\infty\}$. If $\mathscr{F}(\boldsymbol{R}^1)$ is equal to $C_b(\boldsymbol{R}^1)$, the space of bounded continuous functions on $\boldsymbol{R}^1$, then $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(\mathrm{a})$ is a probabilistic distribution on $\boldsymbol{R}$. Explicitly, this measure is given by

$$\mathfrak{d}^{+}\varphi(a)(E)=\lim_{\delta\to 0} m(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in E\}\cap(0,\ \delta])/\delta,$$

where m is the Lebesgue measure and E is an open set of $\boldsymbol{R}\cup\{\pm\infty\}$. We note that it is shown that to set

$$\mathfrak{d}^{+}\varphi(a)(E)$$

$$=\lim_{\delta\to 0}\ sup.\ \overline{m}(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in E\}\cap(0,\ \delta])/\delta,$$

where $\overline{m}$ is the Lebesgue's outer measure and E is an open set of $\boldsymbol{R}\cup\{\pm\infty\}$, from $\mathfrak{d}^{+}\varphi(a)$ we can defines a (Caratheodry's) outer measure of $\boldsymbol{R}\cup\{\pm\infty\}$.

For the above $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)$, or $\mathfrak{d}^{+}\varphi(a)$, we can show the followings.

(i). $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)$ *is a probabilistic distribution on* $\boldsymbol{R}\cup\{\pm\infty\}$. *Conversely, for any probabilistic distribution* P *on* $\boldsymbol{R}\cup\{\pm\infty\}$, *there exists a (right) continuous function* φ *at* a *such that*

$$\mathfrak{d}^{+}\varphi(a)=P$$

(§3, Theorem 6, cf. [1], Theorem 3, [2], §5).

(ii). *car* $\mathfrak{d}^{+}\varphi(a)\subset[L,\ K]$ *if and only if for any* $\varepsilon>0$

$$\lim_{\delta\to 0} m(\{t\,|\,L+\varepsilon\leqq\frac{\varphi(a+t)-\varphi(a)}{t}\leqq K+\varepsilon\}\cap(0,\ \delta])/\delta=1,$$

or in other word, φ *is (right) Lipschitz continuous at* a *with the bounds* L, K *in the sence of Lèvy* ([7]) (§2, Theorem 4′).

(ii)′. $\mathfrak{d}^{+}\varphi(a)=\delta_c$, *the Dirac measure at* c, *if and only if* φ *is (right) approximately derivable at* a (cf. [10]) *and*

$$A^{+}D\varphi(a)=c,$$

where $A^{+}D\varphi(a)$ *means the (right) approximately derivate of* φ *at* a (§2, Theorem 3).

(iii). *If* φ_1 *and* φ_2 *are both* $\mathscr{F}(\boldsymbol{R}^1)$*-derivable at* a *and* $\mathscr{F}(\boldsymbol{R}^1)\subset C(\boldsymbol{R}^1)$ *with the condition that if* $f\in\mathscr{F}(\boldsymbol{R}^1)$, *then* f_a, $f_a(x)=f(a+x)$, $\in\mathscr{F}(\boldsymbol{R}^1)$, *then*

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}(\varphi_1+\varphi_2)(a)=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi_1(a)*\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi_2(a),$$

*where *means the convolution* (§3, Lemma 12).

(iv). *If* $\mathscr{F}(\boldsymbol{R}^1)=C_{|x|}(\boldsymbol{R}^1)=\{f\,|\,f(x)/(1+|x|)$ *is bounded on* $\boldsymbol{R}^1\}$, *and* φ *is (right)* $\mathscr{F}(\boldsymbol{R}^1)$*-derivable at* a, *or, in other word, if* $\mathfrak{d}^{+}\varphi(a)$ *has the expectation, then*

$$\lim_{c\to 0}\frac{1}{c}\frac{-1}{2\pi\sqrt{-1}}\log(\mathscr{F}\ [\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}(c\varphi)(a)])=B_r\varphi(a)t.$$

Here, $\mathscr{F}$ means the Fourier transformation and $B_r\varphi(a)$ is the right Borel derivate of φ at a (§3, Theorem 7).

The above (ii′) shows that the (right) $\mathscr{F}(\boldsymbol{R}^1)$-derivability at a is weaker than (right) approximately derivability at a. For the existence of $\mathscr{F}(\boldsymbol{R}^1)$-derivate, it is shown by Prof. Uchiyama that if $t(\varphi(a+1/t)-\varphi(a))$ is almost periodic in the sence of Besicovič ([3]), then φ is (right) $C(\boldsymbol{R}^1\cup\{\pm\infty\})$-derivable at a (§1, Theorem 1). It is also shown that if φ is Lipschitz continuous near a, then φ is $C^1(\boldsymbol{R}^1)$-derivable at a, where $C^1(\boldsymbol{R}^1)$ means the space of C^1-class functions on $\boldsymbol{R}^1$ (§2, Theorem 5). On the other hand, in the case $\mathscr{F}(\boldsymbol{R}^1)=C(\boldsymbol{R}^1\cup\{\pm\infty\})$, the space of those continuous functions f, such that $lim._{x\to\pm\infty} f(x)$ both exist, we get $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}(\sqrt{t})(0)=\delta_\infty$, she Dirac measure at ∞, or in other word, $\delta_\infty(f)=lim._{x\to\infty} f(x)$. Moreover, if $\varphi(t)$ is the Weierstrass' example, that is

$$\varphi(t)=\sum_n b^n\cos(c^n\pi t),\quad c \text{ is odd},\ 0<b<1,\ bc>1+\frac{3}{2}\pi,$$

we get

$$\mathfrak{d}_{C(R^1\cup\{\pm\infty\})}{}^{+}\varphi(a)=\delta_{-\infty},\quad \text{if } a=\frac{m}{c^k},\ m,\ k \text{ are integers},$$

where $\delta_{-\infty}$ is given by $\delta_{-\infty}(f)=lim._{x\to-\infty} f(x)$.

In this first part of the paper, we only state the results about derivation. In the second part of this paper, we state the results about corresponding integration of this derivation. It is related to the stochastic process and the several types of integrals (cf. [4], [5], [7], [8], [9], [10]). The several variables case will be treated in another paper.

I would like to thank Prof. Uchiyama and Dr. Kano for their kind advices and encouragements.

§1. $\mathscr{F}(\boldsymbol{R}^1)$-derivatives

1. We denote by $(\boldsymbol{R}^1)$ a locally convex function space over $\boldsymbol{R}^1$, 1-dimensional real vector space, which satisfies the following (*a*), (*b*).

(*a*). *$\mathscr{F}(\boldsymbol{R}^1)$ is contained in $L^1{}_{loc.}(\boldsymbol{R}^1)$ and the topology of $(\boldsymbol{R}^1)$ is not weaker than that of $L^1{}_{loc.}(\boldsymbol{R}^1)$.*

(*b*). *If $f\in\mathscr{F}(\boldsymbol{R}^1)$, then $\rho_a f$ given by*

$$\rho_a f(x)=f(ax),\quad a\in\boldsymbol{R},$$

belongs in $\mathscr{F}(\boldsymbol{R}^1)$ for any $a\in\boldsymbol{R}$.

As usual, we denote by $C(\boldsymbol{R}^1)$ and $C_0(\boldsymbol{R}^1)$ the spaces of continuous (resp. compact carrier continuous) functions on $\boldsymbol{R}^1$. By the compact open topology, they both satisfy (a), (b). Other examples of $\mathscr{F}(\boldsymbol{R}^1)$, which are specially used later, are the following spaces.

(i). $C_b(\boldsymbol{R}^1)$: *Banach space of bounded continuous functions on* $\boldsymbol{R}^1$ *where* $||f||$ *is given by* $sup._{x\in\boldsymbol{R}^1}|f(x)|$.

(ii). $C(\boldsymbol{R}^1\cup\{\pm\infty\})$: *The subspace of* $C_b(\boldsymbol{R}^1)$ *consisted by those* f *that* $lim._{x\to\infty}f(x)$ *and* $lim._{x\to-\infty}f(x)$ *both exist.*

(iii). $C_{|x|}(\boldsymbol{R}^1)$: *Banach space of those continuous functions* f *on* $\boldsymbol{R}^1$ *such that* $|f(x)|=O(|x|)$ *at infinity with the norm* $||f||=sup._{x\in\boldsymbol{R}^1}|f(x)|/(1+|x|)$.

By the assumption (i) of $\mathscr{F}(\boldsymbol{R}^1)$, we have

***Lemma* 1.** *Let* φ *be a function of* 1*-variable defined on* $a<t<a+\varepsilon$ *for some* $\varepsilon>0$, *and if the limit*

$$\lim_{s\to0}.\frac{1}{s}\lim_{h\to0}.\int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt$$

exists for any $f\in\mathscr{F}(\boldsymbol{R}^1)$, *then there exists an element* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ *of* $\mathscr{F}(\boldsymbol{R}^1)^*$, *the dual space of* $\mathscr{F}(\boldsymbol{R}^1)$, *suh that*

$$\begin{aligned}(1)\qquad &\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f)\\ &=\lim_{s\to0}.\frac{1}{s}\lim_{h\to0}.\int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt.\end{aligned}$$

Similarly, if $lim._{s\to0}(1/s)lim._{h\to0}\int_h^s f((\varphi(a)-\varphi(a-t))/t)$ dt exists for any $f\in\mathscr{F}(\boldsymbol{R}^1)$, then there exists an element $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)$ of $\mathscr{F}(\boldsymbol{R}^1)^*$ such that

$$\begin{aligned}(1)'\qquad &\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)(f)\\ &=\lim_{s\to0}.\frac{1}{s}\lim_{h\to0}.\int_h^s f\left(\frac{\varphi(a)-\varphi(a-t)}{t}\right)dt.\end{aligned}$$

Definition. *We call* φ *to be right (resp. left)* $\mathscr{F}(\boldsymbol{R}^1)$*-derivable at* a *if* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ *(resp.* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)$*) exists and* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ *(resp.* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)$*) is called right (resp. left)* $\mathscr{F}(\boldsymbol{R}^1)$*-derivate or generalized derivate of* φ *at* a. *If* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ *and* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)$ *both exist and coincide, then we call* φ *to be* $\mathscr{F}(\boldsymbol{R}^1)$*-derivable at* a. *In this case, we set*

$$(2)\qquad \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}\varphi(a)=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)\ (=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^-\varphi(a)).$$

Example 1. If φ is differentiable in the usual sence at a and $\mathscr{F}(\boldsymbol{R}^1)$ is contained in $C(\boldsymbol{R}^1)$, then, since

$$f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)=f(\varphi'(a))+o(1),$$

we get

(3) $$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}\varphi(a)=\delta_{\varphi'(a)}.$$

Example 2. If $\mathscr{F}(\boldsymbol{R}^1)=\{x\}$, the 1-dimensional vector space generated by the function x, $x(t)=t$, then to identify $\{x\}^*=R$, we get

$$\mathfrak{d}_{\{x\}}{}^+\varphi(a)=\lim_{s\to 0}.\frac{1}{S}\lim_{h\to 0}.\int_h^s\frac{\varphi(a+t)-\varphi(a)}{t}dt.$$

This right hand is known to be the (right) Borel derivate of φ at a. For the Borel derivatives, it is known that φ has the Borel derivate at a if and ony if the indefinite integral Φ of φ has the second de la Valleè-Poussin derivative (or Peano derivative) at a ([6], [9], [11]). Here, de la Vallée-Poussin derivative is defined as follows: *If f is written at a*

$$f(a+t)=c_0+c_1t+\frac{c_2}{2!}t^2+\cdots\cdots+\frac{c_k}{k!}t^k+o(t^k),$$

then c_k is called the k-th de la Vallée-Poussin derivative of f at a.

Note. If $\mathscr{F}(\boldsymbol{R}^1)=C(\boldsymbol{R}^1\cup\{\pm\infty\})$, then we have

(4) $$\mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^+\varphi(a)=\delta_{\pm\infty},$$

if we get $lim._{t\to 0}(\varphi(a+t)-\varphi(a))/t=\pm\infty$. Here δ_∞ (or $\delta_{-\infty}$) is given by

$$\delta_\infty f=\lim_{x\to\infty}.f(x)\quad(\delta_{-\infty}f=\lim_{x\to-\infty}.f(x)).$$

2. The following existence criterion of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ due to Prof. Uchiyama.

Lemma 2. *If $h(x)$ is measurable and essentially bounded on $x\geqq c$, then $lim._{S\to\infty}S\int_S^\infty h(x)(dx/x^2)$ exists if and only if $lim._{T\to\infty}(1/T)\int_a^{a+T}h(x)dx$ exists, and we have for any $a\geqq 0$,*

(5) $$\lim_{S\to\infty}.S\int_S^\infty h(x)\frac{dx}{x^2}=\lim_{T\to\infty}.\frac{1}{T}\int_a^{a+T}h(x)dx.$$

Proof. First we assume $lim._{T\to\infty}(1/T)\int_a^{a+T}h(x)dx$ exists and set it by $M_a(h)$. Then, since

$$\int_{a+s}^{a+t}\frac{h(x)}{x^2}dx=\left[\frac{1}{(x+a)^2}\int_a^{a+x}h(\xi)\,d\xi\right]_s^t+2\int_s^t\left(\frac{1}{(x+a)^3}\int_a^{a+x}h(\xi)d\xi\right)dx,$$

we get

$$\int_{a+S}^{\infty}\frac{h(x)}{x^2}dx=-\frac{1}{(S+a)^2}\int_a^{a+S}h(\xi)d\xi+2\int_S^{\infty}\left(\frac{1}{(x+a)^3}\int_a^{a+x}h(\xi)d\xi\right)dx,$$

because $M_a(h)$ exists. Then, since $1/(x+a)\int_a^{a+x}h(\xi)d\xi\leqq M_a(h)+\varepsilon$ for any $\varepsilon>0$ if x is sufficiently large, we get

$$\lim_{S\to\infty}.\ sup.\ (a+S)\int_{a+S}^{\infty}\frac{h(x)}{x^2}dx$$

$$\leqq -M_a(h)+\lim_{S\quad\infty}.\ sup.\ 2(a+S)\int_S^{\infty}\frac{1}{(x+a)^2}(M_a(h)+\varepsilon)dx=M_a(h)+2\varepsilon.$$

Similarly, we obtain $lim.\ inf._{S\to\infty}(a+S)\int_{a+S}^{\infty}h(x)(dx/x^2)\geqq M_a(h)-2\varepsilon$. Therefore $lim._{S\to\infty}S\int_S^{\infty}h(x)(dx/x^2)$ exists and it is equal to $M_a(h)$. On the other hand, since

$$\frac{1}{T}\int_a^{a+T}h(x)dx=-\frac{(a+T)^2}{T}\int_{a+T}^{\infty}\frac{h(\xi)}{2}d\xi+$$

$$+\frac{a^2}{T}\int_a^{\infty}\frac{h(\xi)}{2}d\xi+\frac{2}{T}\int_a^{a+T}dx\int_x^{\infty}\frac{h(\xi)}{2}d\xi,$$

$M_a(h)$ exists and coincides to $lim._{S\to\infty}S\int_S^{\infty}h(x)(dx/x^2)$. Because by the essential boundedness of h, we get

$$\int_S^{\infty}\frac{h(x)}{x^2}dx=o\left(\frac{1}{S}\right),\ S\uparrow\infty,\ \left|\int_a^{\infty}\frac{h(x)}{x^2}dx\right|<\infty,\ a>0.$$

We note that by lemma 2, if $M_a(h)$ exists, then it does not depend on a. Hence in the rest, we denote $M(h)$ instead of $M_a(h)$.

Since we know

$$\lim_{s\to0}.\frac{1}{s}\lim_{h\to0}.\int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt$$

$$=\lim_{S\to\infty}.\ S\int_S^{\infty}f(u(\varphi(a+\frac{1}{u})-\varphi(u)))\frac{du}{u^2},$$

we get by lemma 2,

***Theorem* 1.** *If the elements of $\mathscr{F}(\boldsymbol{R}^1)$ are essentially bounded at infininity, then we have*

(6) $$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)(f)=M(f(\varphi_a{}^{+})).$$

Here $\varphi_a{}^{+}$ is given by

(7) $$\varphi_a{}^+(t)=t(\varphi(a+\frac{1}{t})-\varphi(a)),$$

and $f(\varphi_a{}^+)$ means the composition of f and $\varphi_a{}^+$.

***Corollary* 1.** *If $\mathscr{F}(\boldsymbol{R}^1)$ is contained in $C(\boldsymbol{R}^1\cup\{\pm\infty\})$ and $\varphi_a{}^+$ is almost periodic in the sence of Besicovič ([3]), that is, for any $\varepsilon>0$, there exists $A=A(\varepsilon)>0$ and a periodic, essentially bounded mesurable function $\varphi_A(x)$ such that*

$$\lim_{T\to\infty} \sup. \frac{1}{T}\int_0^T |\varphi_a{}^+(x)-\varphi_A(x)|\,dx<\varepsilon,$$

then $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ exists.

***Proof*.** Since an almost periodic function h in the sence of Besicovič has the mean value $M(h)$, we have the corollary.

***Corollary* 2.** *To define $\mathscr{F}(\boldsymbol{R}^1)$ by $\{f\,|\,f\in L^1{}_{loc}(\boldsymbol{R}^1),\ f(\varphi_a{}^+)$ is almost periodic in the sence of Besicovič$\}$,* φ is *$\mathscr{F}(\boldsymbol{R}^1)$-derivable at a.*

Example. If $\varphi(t)$ is given by a trigonometrical series $\sum_n(a_n\sin\alpha_{nt}+b_n\cos\beta_n t)$, then

$$\varphi_a{}^+(t)=2t\sum_n(a_n\sin\alpha_n(a+\frac{1}{2t})\cos\frac{\beta_n}{2t}-b_n\sin\beta_n(a+\frac{1}{2t})\sin\frac{\alpha_n}{2t}).$$

Hence, if $\mathscr{F}(\boldsymbol{R}^1)=C(\boldsymbol{R}^1\cup\{\pm\infty\})$ and $\varphi(t)$ is the Cellérier's example

$$\varphi(t)=\sum_{n=0}^{\infty}\frac{\sin(c^n x)}{c^n},\quad c \text{ is even},$$

then we get

$$\mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^+\varphi(a)=\delta_\infty,\quad \mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^-(a)=\delta_{-\infty},$$

if $a=m/c^k$, m, k are integers.

Similarly, if $\varphi(t)$ is the Weierstass' examle

$$\varphi(t)=\sum_{n=0}^{\infty}b^n\cos(c^n\pi x),\quad c \text{ is odd},\quad 0<b<1,\quad bc\geqq 1+\frac{3}{2}\pi,$$

then we get

$$\mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^+\varphi(a)=\delta_{-\infty},\quad \mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^-\varphi(a)=\delta_\infty,$$

if $a=m/c^k$, m, k are integers.

3. In our previous papers [1], [2], [2]$'$, the authour introduce the notion of generalized tangent, or $C(S^{n-1})$-tangent $\mathfrak{X}_a\gamma$ of a curve γ at a as follows: *Suppose γ is given by $\gamma: \boldsymbol{I}\to\boldsymbol{R}^n$* with $\gamma(0)=a$, *then*

$$<\mathfrak{X}_a\gamma,\ d_\rho g(a)> = \lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{g(\gamma(t))-g(a)}{t}dt,$$

where g is a $C(S^{n-1})$-differentiable function at a and $d_\rho g(a)$ is its Gâteaux differential at a given by

$$d_\rho g(a, y) = \lim_{t\to 0}.\frac{1}{t}(g(a+ty)-g(a)),\quad ||y||=1.$$

Since we know

$$\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{g(\gamma(t))-g(a)}{t}dt$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s d_\rho g(a, \frac{\gamma(t)-a}{||\gamma(t)-a||})||\gamma(t)-a||dt,$$

if $\mathfrak{X}_a\gamma$ exists, we get

(8)′ $$<\mathfrak{X}_a\gamma,\ d_\rho g(a)>$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s d_\rho g(a,\ \tan^{-1}(\frac{\varphi(a+t)-\varphi(a)}{t}))\frac{\sqrt{t^2+(\varphi(a+t)-\varphi(a))^2}}{t}dt,$$

if γ is the graph of φ starts from $(a,\ \varphi(a))$, that is, γ is given by

$$\varphi_a^+ : \boldsymbol{I}\to\boldsymbol{R}^2,\quad \varphi_a^+(t)=(a+t,\ \varphi(a+t)).$$

We note that in (8)′, S^1 is parametrized by, $-\pi<\theta\leq\pi$.

By (8)′, if γ is the graph of φ and to have the generalized tangent $\mathfrak{X}_a\gamma$, then

(9) $$car.\ \mathfrak{X}_a\gamma\subset[-\frac{\pi}{2},\frac{\pi}{2}].$$

Moreover, to define the map $\tan^{-1}\#: C(S^1)\to C(\boldsymbol{R}^1)$ by

$$(\tan^{-1}\# f)(x)=f(\tan^{-1}x),\quad x\in\boldsymbol{R}^1,\quad f\in C(S^1),$$

we get

(10) $$\tan^{-1}\#(C(S^1))=C(\boldsymbol{R}^1\cup\{\pm\infty\}).$$

By the definition of $\tan^{-1}\#$ and (10), we get

(9)′ $$\tan^{-1}\#^*(C(\boldsymbol{R}^1\cup\{\pm\infty\})^*)=\{\xi\,|\,\xi\in C(S^1)^*,\ car.\ \xi\subset[-\frac{\pi}{2},\frac{\pi}{2}]\}.$$

By (8)′, (9) and (9)′, we obtain

(8) $$<(\tan^{-1}\#^*)^{-1}\mathfrak{X}_a\gamma,\ (\tan^{-1}\#^*)^{-1}d_\rho g(a)>$$

$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s((\tan^{-1}\#)^{-1}d_\rho g(a))(\frac{\varphi(a+t)-\varphi(a)}{t})\frac{\sqrt{t^2+(\varphi(a+t)-\varphi(a))^2}}{t}dt.$$

Hence if $(\tan^{-1}\#^*)^{-1}\mathfrak{X}_a\gamma$ can be considered to be an element of $C_{|x|}(\boldsymbol{R}^1)$, then we have

$$\text{(11)}\qquad \mathfrak{d}_{C_{|x|}(\boldsymbol{R}^1)}{}^+\varphi(a)(f(x)\sqrt{1+x^2})$$
$$=\langle(\tan^{-1}\#^*)^{-1}\mathfrak{X}_a\gamma,\ f\rangle,\quad f\in C(\boldsymbol{R}^1\cup\{\pm\infty\}).$$

Hence we obtain

***Theorem* 2.** *$\mathfrak{d}_{C_{|x|}(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^+\varphi(a)$ exists if and only if the graph of φ has the $C(S^1)$-tangent at $x=a$. Here $C_{|x|}(\boldsymbol{R}^1\cup\{\pm\infty\})$ is the subspace of $C_{|x|}(\boldsymbol{R}^1)$ such that*

$$C_{|x|}(\boldsymbol{R}^1\cup\{\pm\infty\})$$
$$=\{f\mid \lim_{x\to\infty}.\frac{f(x)}{x}\ \text{and}\ \lim_{x\to-\infty}.\frac{f(x)}{x}\ \text{both exist}\}.$$

Note. Starting from another function space $\mathscr{F}(S^1)$, we can obtain similar conclusion. For example, we get

***Theorem* 2.** *$\mathfrak{d}_{L^2{}_{|x|^4}(\boldsymbol{R}^1)}{}^+\varphi(a)$ exists if and only if the graph of φ has the $L^2(S^1)$-tangent at $x=a$. Here, $L^2{}_{|x|^4}(\boldsymbol{R}^1)$ means the Hilbert space over $\boldsymbol{R}^1$ snch that whose norm is given by*

$$||f||^2=\int_{\boldsymbol{R}^1}\frac{|f(x)|^2}{(1+|x|)^4}dx.$$

§ 2. Carriers of $\mathscr{F}(\boldsymbol{R}^1)$-derivatives.

4. By the definition of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+$, we obtain

***Lemma* 3.** *If φ satisfies*

$$\text{(12)}'\qquad L\leqq\frac{\varphi(a+t)-\varphi(a)}{t}\leqq K,\quad 0<t<\varepsilon,$$

for some $\varepsilon>0$, and if *$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ exists, then*

$$\text{(12)}\qquad car.\,\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)\subset[L,\ K].$$

Corollary. *If (right) Dini derivates $D_a{}^+\varphi=\lim.\sup._{t\to+0}(\varphi(a+t)-\varphi(a))/t$ and $d_a{}^+\varphi=\lim.\inf._{t\to+0}(\varphi(a+t)-\varphi(a))/t$ are given by $D_a{}^+\varphi=K$, $d_a{}^+\varphi=L$, then* (12) *is hold if $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+(a)$ exists.*

Note. In this corollary, K may be equal to ∞ and L may, be equal to $-\infty$.

***Lemma* 4.** *If $\mathscr{F}(\boldsymbol{R}^1)$ containes $C(\boldsymbol{R}^1\cup\{\pm\infty\})$ and $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ exists and* (12) *is hold, then for any* $\varepsilon>0$, *we have*

$$(13)\qquad \lim_{\delta\to 0}.\, m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \geqq K+\varepsilon\}\cap(0,\ \delta])/\delta=0,$$

$$(13)'\qquad \lim_{\delta\to 0}.\, m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \leqq L-\varepsilon\}\cap(0,\ \delta])/\delta=0,$$

where $m(E)$ *means the Lebesgue measure of* E.

Proof. By assumption, there exists a series of functions $\{f_n\}$ of $\mathscr{F}(\boldsymbol{R}^1)$ such that

$$f_n \uparrow \chi_{[K+\varepsilon,\ \infty)},$$

where χ_E means the characteristic function of E. Then, since $f_n \leqq 1$, by the definition of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$, we get

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_1) \leqq \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_2) \leqq \cdots\cdots \leqq \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_n) \leqq$$

$$\leqq \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_{n+1}) \leqq \cdots\cdots \leqq \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(1)=1.$$

Therefore $\lim._{n\to\infty}\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_n)$ exists and by the definition of $\{f_n\}$, we get

$$\lim_{n\to\infty}.\, \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f_n)$$

$$=\lim_{\delta\to 0}.\ m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \geqq K+\varepsilon\}\cap(0,\ \varphi])/\delta.$$

But since we can take each f_n to satisfy $car. f_n \subset [K+\varepsilon/2,\ \infty)$, we have (13) by assumption.

Similarly, since $C(\boldsymbol{R}^1 \cup \{\pm\infty\})^*$ is tha space of the (finte) measures on $\boldsymbol{R}^1 \cup \{\pm\infty\}$, we get

***Lemma* 4′.** *Under the same assumptions as in lemma* 4, *if* (13) *is hold, then* $car.\, \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a) \subset [-\infty,\ K]$. *Hence if* (13) *and* (13)′ *are hold, then we have* (12).

Note. In the case $L=\infty$ (or $K=-\infty$), we get $car.\, \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)=\{\infty\}$ (or $car.\, \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)=\{-\infty\}$).

***Theorem* 3.** *If* $\mathscr{F}(\boldsymbol{R}^1)$ *containes* $C(\boldsymbol{R}^1 \cup \{\pm\infty\})$, *then*

$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)=\delta_c$, *the Dirac measure concentrated at* c,

if and only if φ *is* (*right*) *approximately derivable at* a (*cf.* [10]) *and*

(14) $\qquad c=AD^+\varphi(a)$, *the* (*right*) *approximate derivate of* φ *at* a,

that is, for any $\varepsilon>0$, *we get*

$$\lim_{\delta\to 0} m(\{t \mid c-\varepsilon \leqq \frac{\varphi(a+t)-\varphi(a)}{t} \leqq c+\varepsilon\} \cap (0,\ \delta])/\delta = 1. \tag{15}$$

Proof. By lemma 4 and 4′, we need only to show the existence of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)$. But since we get by (15)

$$m(\{t \mid c-\varepsilon \leqq \frac{\varphi(a+t)-\varphi(a)}{t} \leqq c+\varepsilon\} \cap (0, s]) = s - o(s),$$

for any ε, we have

$$|\lim_{h\to 0} \int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right) dt - f(c)s| \leqq$$
$$\leqq \left(\max_{c-\varepsilon\leqq x,\ y\leqq c+\varepsilon} |f(x)-f(y)|\right)s + o(s),$$

for any $\varepsilon > 0$. Hence, since f is continuous, we get

$$\lim_{s\to 0} \frac{1}{s} \lim_{h\to 0} \int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right) dt = f(c).$$

Therefore $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)$ exists and it is equal to δ_c.

Note. We set $AD^{+}\varphi(a) = \infty$ if

$$\lim_{\delta\to 0} m(\{t \mid \frac{\varphi(a+t)+\varphi(a)}{t} \geqq c\} \cap (0,\ \delta])/\delta = 1, \tag{15$'$}$$

for any c. Then we get $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a) = \delta_\infty$ if and only if $AD^{+}\varphi(a) = \infty$. Similarly, $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a) = \delta_{-\infty}$ if and only if $AD^{+}\varphi(a) = -\infty$, that is $\lim_{\delta\to 0} m(\{t \mid (\varphi(a+t)-\varphi(a))/t \leqq c\} \cap (0,\ \delta])/\delta = 1$ for any c.

5. ***Lemma.*** *To define* $\bar{\mathfrak{d}}^{-+}\varphi(a)(E)$ *by*

$$\bar{\mathfrak{d}}^{+}\varphi(a)(E) = \lim_{\delta\to 9} \sup \overline{m}(\{t \mid \frac{\varphi(a+t)\varphi\ \varphi(a)}{t} \in E\} \cap (0,\ \delta])/\delta, \tag{16}$$

where E is an (open) subset of $\boldsymbol{R}^1$ and $\overline{m}$ is the Lebesgue's outer measure, $\bar{\mathfrak{d}}^{+}\varphi(a)$ is a (Caratheodry's) outer measure of $\boldsymbol{R}^1$.

Proof. We need only to show

$$\bar{\mathfrak{d}}^{+}\varphi(a)(\bigcup_n E_n) \leqq \sum_n \bar{\mathfrak{d}}^{+}\varphi(a)(E_n).$$

But, since $\overline{m}$ is the Lebesgue's outer measure, we get

$$\overline{m}(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in \bigcup_n E_n\} \cap (0,\ \delta])$$

$$\leqq \sum_n \overline{m}(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E_n\} \cap (0, \ \delta]).$$

Hence we have

$$\mathfrak{d}^+\varphi(a)(\bigcup_n E_n)$$

$$\geqq \sum_n \lim_{\delta\to 0}. \sup. m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E_n\} \cap (0, \ \delta])/\delta$$

$$\leqq \sum_n \mathfrak{d}^+(a)(E_n).$$

***Lemma* 6.** *If E is Lebesgue measurable and φ is measurable, then E is $\mathfrak{d}^+\varphi(a)$-measurable if and only if $\lim._{\delta\to 0} m(\{t \mid \varphi(a+t)\text{-}\varphi(a)/t \in E\} \cap (0, \delta])/\delta$ exists. Here $\mathfrak{b}^+\varphi(a)$ means the corresponding measure of $\mathfrak{d}^+\varphi(a)$.*

***Proof*.** By assumption, we can set

$$\mathfrak{d}^+\varphi(a)(E) = \lim_{\delta\to 0}. \sup. m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E\} \cap (0, \ \delta])/\delta.$$

Then, Since

$$\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in X\} \cap (0, \ \delta]$$

$$= (\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E\} \cap (0, \ \delta]) \cup (\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in X-E\} \cap (0, \ \delta]),$$

$$(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E\} \cap (0, \ \delta]) \cap (\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in X-E\} \cap (0, \delta]) = \emptyset,$$

for any $X \supset E$, we get

$$\mathfrak{d}^+\varphi(\mathrm{a})(X)$$

$$= \lim_{\delta\to 0}. \sup. \overline{m}((\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E\} \cap (0, \ \delta]) \cup$$

$$\cup (\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in X-E\} \cap (0, \ \delta]))/\delta$$

$$= \lim_{\delta\to 0}. \sup. (\overline{m}(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in E\} \cap (0, \ \delta])/\delta +$$

$$+ \overline{m}(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \in X-E\} \cap (0, \ \delta]/\delta)$$

$$=\lim_{\delta\to 0}.\,\overline{m}(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in E\}\cap(0,\ \delta])/\delta+$$

$$+\lim_{\delta\to 0}.\,sup.\,\overline{m}(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in X-E\}\cap(0,\ \delta])/\delta$$

$$=\bar{\mathfrak{d}}^{+}\varphi(a)(E)+\bar{\mathfrak{d}}^{+}\varphi(a)(X-E),$$

if $lim._{\delta\to 0}m(\{t\,|\,(\varphi(a+t)-\varphi(a))/t\in E\}\cap(0,\ \delta])/\delta$ exists, because E is mesurable. Hence we have the first assertion. On the other hand, since

$$\lim_{\delta\to 0}.\,m(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in \boldsymbol{R}\}\cap(0,\ \delta])/\delta=1, \tag{17$'$}$$

by definition, we have

$$\lim_{\delta\to 0}.\,inf.\,m(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in E\}\cap(0,\delta])/\delta$$
$$=1-\bar{\mathfrak{d}}^{+}\varphi(a)(\boldsymbol{R}-E),$$

if E is (Lebesgue) measurable. Therefore, if $lim._{\delta\to 0}m(\{t\,|\,(\varphi(a+t)-\varphi(a))/t\in E\}\cap(0,\ \delta])/\delta$ does not exist, then E is not $\bar{\mathfrak{d}}^{+}\varphi(a)$-measurable.

Definition. *We call* $\mathfrak{d}^{+}\varphi(a)$ *the* (*right*) *derivative measure of* φ *at* a.

By definition and lemma 6, if E is Lebesgue measurable and $\mathfrak{d}^{+}\varphi(a)$-measurable, then

$$\mathfrak{d}^{+}\varphi(\mathrm{a})(E)=\lim_{\delta\to 0}.\,m(\{t\,|\,\frac{\varphi(a+t)-\varphi(a)}{t}\in E\}\cap(0,\ \delta])/\delta.$$

Especially, by (17)$'$, $\boldsymbol{R}^1$ is always $\mathfrak{d}^{+}\varphi(a)$-measurable, and we have

$$\mathfrak{d}\varphi(a)(\boldsymbol{R}^1)=1. \tag{17}$$

Theorem 4. *If* $\mathscr{F}(\boldsymbol{R}^1)=C_b(\boldsymbol{R}^1)$, *then* $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)$ *exists and only if any Borel set of* $\boldsymbol{R}^1$ *is* $\mathfrak{d}^{+}\varphi(a)$-*measurable and we have*

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^{+}\varphi(a)(f)=\int_{\boldsymbol{R}^1}f(t)d\mathfrak{d}^{+}\varphi(a). \tag{18}$$

Proof. By the definition of $\mathfrak{d}^{+}\varphi(a)$. we have

$$\mathfrak{d}^{+}\varphi(a)(E)=\lim_{s\to 0}.\,\frac{1}{s}\lim_{h\to 0}.\int_h^s \chi_E\Big(\frac{\varphi(a+t)-\varphi(a)}{t}\Big)dt,$$

if E is $\mathfrak{d}^{+}\varphi(a)$-measurable, we have (18) for continuous f by assumption and we get the first assertion.

On the other hand, since there exists a series of functions $\{f_n\}$ in $C_b(\boldsymbol{R}^1)$ such that each $f_n \geqq 0$ and

$$f_n \uparrow \chi_E,$$

if E is a Borel set, we have the second assertion by the same reason as in the proof of lemma 4.

Corollary. *We assume* φ is right $C_b(\boldsymbol{R}^1)$*-derivable at a, then to set*

$$\chi^+\varphi_{(a)}(x) = \mathfrak{d}^+\varphi(a)((-\infty, x]), \tag{19}$$

$$K = \sup_x.\{x \mid \chi^+\varphi_{(a)}(x) = 0\}, \quad L = \inf_x.\{x \mid \chi^+\varphi_{(a)}(x) = 1\},$$

we have

$$car.\, \mathfrak{d}_{C_b(\boldsymbol{R}^1)}{}^+\varphi(a) \subset [K, \ L]. \tag{20}$$

Conversely, if (20) *is hold, then* (19) *is hold.*

6. Although (20) is hold, φ does not satisfy the Lipschitz condition (12)′ at a or the Dini derivates $D_a{}^+\varphi$ and $d_a{}^+\varphi$ can not bound by K, L in general. For example, to set

$$\varphi(t) = 0, \ \frac{1}{2^n} - \frac{1}{3^n} \geqq t \geqq \frac{1}{2^{n+1}}, \ \ or \ t = 0,$$

$$\varphi(t) = \frac{3^n(3^n - 2^{n-1})}{2^{n-2}}\left(t - \frac{1}{2^n}\right), \ \frac{1}{2^n} \geqq t \geqq \frac{1}{2^n} - \frac{1}{2 \cdot 3^n},$$

$$\varphi(t) = -\frac{3^n(3^n - 2^{n-1})}{2^{n-2}}\left(t - \frac{1}{2^n} + \frac{1}{2 \cdot 3^n}\right), \ \frac{1}{2^n} - \frac{1}{2 \cdot 3^n} \geqq t \geqq \frac{1}{2^n} - \frac{1}{3^n},$$

$\varphi(t)$ is continuous at $t = 0$ and we get

$$\mathfrak{d}_{C_b(\boldsymbol{R}^1)}{}^+\varphi(0) = \delta_0, \quad \lim_{t \to +0}.\sup.\frac{\varphi(t)}{t} = \infty.$$

Definition. *We call* φ *to by approximately (right) Dini derivable at a if* $\chi^+\varphi_{(a)}(x)$ *is defined. If* φ *is approximately Dini derivable at a, then we set*

$$AD_a{}^+\varphi = \inf_x.\{x \mid \chi^+\varphi_{(a)}(x) = 1\}, \tag{21}$$

$$Ad_a{}^+\varphi = \sup_x.\{x \mid \chi^+\varphi_{(a)}(x) = 0\}.$$

By definition, we have

Lemma 7. (i). *If* φ *is approximately (right) Dini derivable at a, then*

$$AD_a{}^+\varphi = \inf_x.\{x \mid \ \lim_{\delta \to 0}.\ m(\{t \mid \frac{\varphi(a+t) - \varphi(a)}{t} \leqq x\} \cap (0, \ \delta]/\delta = 1, \tag{21'}$$

$$Ad_a{}^+\varphi = \underset{x}{sup.}\{x \mid \underset{\delta\to 0}{lim.}\, m(\{t \mid \frac{\varphi(a+t)-\varphi(a)}{t} \leqq x\} \cap (0,\ \delta])\delta = 0.$$

(ii). *We have the inequality*

$$d_a{}^+\varphi \leqq Ad_a{}^+\varphi \leqq AD_a{}^+\varphi \leqq D_a{}^+\varphi, \tag{22}$$

and $Ad_a{}^+\varphi = AD_a{}^+\varphi$ *if and only if* φ *is approximately derivable at* a.
(iii). *If* φ *is (right) Lipschitz continuous in the sence of Lévy (cf. [7]) with the bounds* L, K, *then*

$$K \leqq Ad_a{}^+\varphi \leqq AD_a{}^+\varphi \leqq L. \tag{22$'$}$$

Using the approximately Dini derivates, we get

***Theorem* 4$'$.** φ *is (right)* $C_b(\mathbf{R}^1)$*-derivable at* a *if and only if* φ *is approximately (right) Dini derivable at* a *and we have*

$$\mathfrak{d}_{C_b(\mathbf{R}^1)}{}^+\varphi(a)(f) = \int_{-\infty}^{\infty} f(t)\,d\chi^+{}_{\varphi(a)}(t), \tag{18$'$}$$

where the right hand side is the Stieltjes integral. We also have

$$car.\,{}_{C_b(\mathbf{R}^1)}{}^+\varphi(a) \subset [Ad_a{}^+\varphi,\ AD_a{}^+\varphi]. \tag{20$'$}$$

Coversely, we obtain

$$AD_a{}^+\varphi = \underset{x}{inf.}\{x \mid cor.\,\mathfrak{d}_{C_b(\mathbf{R}^1)}{}^+\varphi(a) \cap [x,\ \infty) = \emptyset\},$$

$$Ad_a{}^+\varphi = \underset{x}{sup.}\{x \mid car.\,\mathfrak{d}_{C_b(\mathbf{R}^1)}{}^+\varphi(a) \cap (-\infty,\ x] = \emptyset\}.$$

Or, in other word, $[Ad_a{}^+\varphi,\ AD_a{}^+\varphi]$ *is the shortest interval which containes* $car.\ \mathfrak{d}_{C_b(\mathbf{R}^1)}{}^+(a)$.

We also note that if (20) is hold, then we get for any $\varepsilon > 0$

$$m(\{x \mid K-\varepsilon \leqq \frac{\varphi(a+t)-\varphi(a)}{t} \leqq L+\varepsilon\} \cap (0,\ \delta]) = \delta - o(\delta). \tag{12$''$}$$

7. We assume $\varphi(t)$ is Lipschitz continuous near (the right hand side of) a, that is φ satisfies

$$\mathrm{K} \leqq \frac{\varphi(t_1)-\varphi(t_2)}{t_1-t_2} \leqq L,\quad a \leqq t_1 < t_2 \leqq a+\delta, \tag{23}$$

for some $\delta > 0$. For simple, in the rest, we assume $a=0$ and $\varphi(0)=0$.

***Lemma* 8.** *If* f *is differentiable and* φ *is Lipschitz continuous near the origin and* $\varphi(0)=0$, *then the Radon -Nykodim derivative* $\sigma(f(t\varphi(1/t)))$ *of* $f(t\varphi(1/t))$ *satisfies*

$$\sigma\left(f\left(t\varphi\left(\frac{1}{t}\right)\right)\right) = O\left(\frac{1}{t}\right). \tag{24}$$

Proof. First we remark that since

$$f\left(t\varphi\left(\frac{1}{t}\right)\right)=t\left(\frac{1}{t}f\left(t\varphi\left(\frac{1}{t}\right)\right)\right),$$

and $g(t)=tf(\varphi(t)/t)$ is Lipschitz continuous near the origin and $g(0)=0$, we only need to show the lemma for $t\varphi(1/t)$. But since $|\varphi(t_1)-\varphi(t_2)|\leqq L|t_1-t_2|$, $0\leqq t_1<t_2\leqq\delta$, we get

$$|t_1\varphi\left(\frac{1}{t_1}\right)-t_2\varphi\left(\frac{1}{t_2}\right)|\leqq|t_1-t_2||\varphi\left(\frac{1}{t_1}\right)|+|t_2||\varphi\left(\frac{1}{t_1}\right)-\varphi\left(\frac{1}{t_2}\right)|$$

$$\leqq\frac{L}{|t_1|}|t_1-t_2|+L|t_2||\frac{1}{t_1}-\frac{1}{t_2}|$$

$$=\frac{2L}{|t_1|}|1-\frac{t_2}{t_1}|,\quad t_1,\ t_2\geqq\frac{1}{\delta}.$$

Then, to set $\Psi(t)=t\varphi(1/t)$, we have

$$|\Psi(t_1)-\Psi(t_2)|\leqq\frac{2L}{t_2}|t_2-t_1|,\ \frac{1}{\delta}\leqq t_1\leqq t_2. \tag{25}$$

Then, since $\Psi(t_2)-\Psi(t_1)=\int_{t_1}^{t_2}\sigma(\Psi)(t)dt$, we get

$$|\int_{t_1}^{t}\sigma(\Psi)(s)ds|\leqq 2L\left(1-\frac{t_1}{t}\right),\ t_1\geqq\frac{1}{\delta}.$$

Hence we obtain

(24)′ $\quad |\sigma(\Psi)(t)|<\frac{2Lt_1}{t^2},\ t>t_1\geqq\frac{1}{\delta},$ *and the sign of* $\sigma(\Psi)(t)$ *is definite almost everywhere on* $[t_1,\ t]$.

Therefore we have the lemma.

By theorem 1 and the first remark in the proof of lemma 8, to treat the existence of $\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(0)$ for $\mathscr{F}(R^1)\subset C_b(R^1)\cap C^1(R^1)$, it is sufficient to treat the existence of the Mean value of Ψ. Then, since $\Psi(t)$ is bounded and

$$\int_a^\infty\Psi(t)e^{-xt}dt=-\frac{\Psi(a)}{t}+\frac{1}{t}\int_a^\infty\sigma(\Psi(t))e^{-xt}dt,\ a>0,$$

$\Psi(t)$ has the mean value if $lim._{t\to 0}\int_a^\infty\sigma(\Psi(t))e^{-xt}dt$ exists by Wiener's Tauberian theorem ([12], [13]). For simple, we set $a=0$ in the rest.

Since we get

$$\int_0^\infty \sigma(\Psi(t))e^{-xt}dt=\int_1^\infty \sigma(\Psi(\log. s))s^{-(x+1)}ds,$$

by setting $t=\log. s$, we consider the function $j(\xi, \eta)$ given by

$$j(\xi, \eta)=\int_1^\infty \sigma(\Psi(\log. s))s^{-(\xi+1)}e^{-is\eta}ds. \tag{26}$$

By definition and the theorem of Titchmarsh ([14]), $j(\xi, \eta)$ is defined on $\{(\xi, \eta)\mid \mathrm{Re}.\xi>-1/2,\ \mathrm{Im}.\eta\leqq 0\}$ and holomorphic on its interior. More precisely, we get

***Lemma* 9.** *$j(\xi, \eta)$ has the following properties.*

(i). *$j(\xi, 0)$ is equal to* $\int_1^\infty \sigma(\Psi(\log. s))s^{-(\xi+1)}ds$.

(ii). *To set $j(\xi)(t)=j(\xi, t)$, where ξ is fixed and t is the seal variable, $j(\xi)$ is continuous in t if* $\mathrm{Re}.\xi>0$ *and belongs in* $L^{p/(p-1)}(\boldsymbol{R}^1)$ *if* $p(\mathrm{Re}.\xi+1)>1,\ -1/2<\mathrm{Re}.\xi\leqq 0$.

By lemma 9, (ii), we have

$$j(0)\in \bigcap_{2\geqq p>1} L^p(\boldsymbol{R}^1).$$

$$\gamma_\theta(\xi)=-\left(\frac{1+\sqrt{2e}^{-i\theta/2}\xi^{1/2}}{1-\sqrt{2e}^{-i\theta/2}\xi^{1/2}}\right)^2+1.$$

Then γ_θ maps $D_\theta=\{\xi\mid |\xi|<1/2,\ \arg.\xi\neq\theta\}$ conformally onto $D=\{\eta\mid |\eta+1|<1\}$ and $\lim._{\xi\to 0}\gamma_\theta(\xi)=0$. But, since $j(\xi, \gamma_\theta(\eta))$ is defined and holomorphic on $D_\theta\times\{\eta\mid \mathrm{Re}.\eta<1/2\}$ and since we get for some (positive) constants K and c,

$$\begin{aligned}&|j(\xi, \gamma_\theta(\xi))|\\ &\leqq\int_1^\infty |\sigma(\Psi(\log. s))|\,s^{-(\mathrm{Re}.\xi+1)}e^{s\mathrm{Im}.(\gamma_\theta(\eta))}ds\\ &\leqq K\int_1^\infty s^{-1/2}e^{s\mathrm{Im}.(\gamma_\theta(\eta))}ds<K\Gamma\left(\frac{1}{2}\right)|\mathrm{Im}.\gamma_\theta(\eta)|^{1/2}+c,\end{aligned}$$

$j(\xi, \gamma_\theta(\xi))$ is defined on D_θ for any θ and 0 is either regular point, branching point or pole of $j(\xi, \gamma_\theta(\xi))$. Therefore we obtain

***Lemma* 10.** *The mean value of $\Psi(t)$ either exists and finite or*

$$\lim_{T\to\infty}.\frac{1}{T}\left|\int_a^{a+T}\Psi(t)dt\right|=\infty.$$

***Theorem* 5.** *Denoting $C^1(\boldsymbol{R}^1)$ the space of smooth functions on $\boldsymbol{R}^1$, $\mathfrak{d}_{C^1(\boldsymbol{R}^1)}{}^+\varphi(a)$ exists if φ satisfies* (23) (cf. [2]').

***Proof*.** We may consider $a=0$ and $\varphi(0)=0$. Then by lemma 10, we need only to show that $\lim._{s\to 0}|\mathscr{F}(\sigma)(\Psi))(s)|=\infty$ is impossible. But by the above discussion, if $\lim._{s\to 0}|\mathscr{F}(\sigma(\Psi)))s)|=\infty$, then $|\mathscr{F}(\sigma(\Psi))(s)|$ should be at least $O(\log. s)$ for $s\to 0$.

On the other hand, by the theorem of Titchmarsh, we get

$$\lim_{A\to\infty}\int_0^A \sigma(\Psi(t))e^{-ist}dt=\mathscr{F}(\sigma(\Psi))(s) \text{ in } L^p,\ p\geqq 2,$$

and we also know

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_0^A \sigma(\Psi(t))e^{-ist}\,dtds=\Psi(A)-\Psi(0),$$

for all A. Then, since $\Psi(A)$ is bounded on $\boldsymbol{R}^1$ by the Lipschitz continuity near the origin, $\lim_{s\to 0}|\mathscr{F}(\Psi))(s)|$ should be finite and we have the theorem.

Note. By the same reason as in lemma 3, we have

$$car.\ \mathfrak{d}_{C^1(\boldsymbol{R}^1)}{}^+\varphi(a)\subset[K, L],$$

in this case.

§ 3. Properties of $\mathscr{F}(\boldsymbol{R}^1)$-derivatives.

8. By the definition of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ and (17), we have

***Lemma* 11.** *If 1 can be approximated by the element of $\mathscr{F}(\boldsymbol{R}^1)$ in $L^1_{loc.}(\boldsymbol{R}^1)$, then $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)$ is a probabilistic distribution on $\boldsymbol{R}^1\cup\{\pm\infty\}$. Especially, if φ is $C_b(\boldsymbol{R}^1)$-derivable, then $\mathfrak{d}_{C_b(\boldsymbol{R}^1)}{}^+\varphi(a)$ is a probabilistic distribution on $\boldsymbol{R}^1$.*

***Theorem* 6.** *For any probabilistic distribution ξ on $\boldsymbol{R}^1$ and $a\in\boldsymbol{R}^1$, there exists a (right) continuous function $\varphi(t)$ near (the right hand side of) a such that $\varphi(t)$ is right $C_b(\boldsymbol{R}_1)$-derivable at a and*

$$\mathfrak{d}_{C_b(\boldsymbol{R}^1)}{}^+\varphi(a)=\xi. \tag{27}$$

***Proof*.** For simple, we assume $a=0$ and $\varphi(0)=0$. We denote the distribution function of ξ by $\chi(x)$, that is $\chi(x)=\xi((-\infty, x])$.

For any positive integer n, we define a set of numbers N_n by

$$N_n=\{c\,|\,c=\sum_{i=0}^{n}a_i2^i+\sum_{j=1}^{n}b_j2^{-j},$$
$$a_i,\ b_j \text{ are equal to } 0,\ 1 \text{ or } -1 \text{ and } |c|\leqq\sqrt{n}\}$$

We denote the number of elements of N_n by k_n. By definition, $k_n=O(\log 2^n)$. The m -th number in bigness in N_n is denoted by $c_{n,m}$. Hence, if $n=\sum_{i=-\infty}^{i=\infty}a_i2^i$, then $c_{n,i}=-(\sum_{-n\leqq i\leqq n}a_i2^i)$ *and* $c_{n,k_n}=\sum_{-n\leqq i\leqq n}a_i2^i$. By definition, we also have

$$N_n\subset N_{n+1},\quad \bigcup_{n\geqq 1}N_n \text{ is dense in } \boldsymbol{R}. \tag{28}$$

We also set for sufficiently large n

$$\alpha_n = \min_{\chi(c_{n,m})\neq 0} \{\chi(c_{n,m})\} = \chi(c_{n,\alpha}),$$

$$\beta_n = \max_{\chi(c_{n,m})\neq 1} \{\chi(c_{n,m})\} = (\chi c_{n,\beta}),$$

$$\chi(c_{n,0}) = 0, \quad \chi(c_{n,n_k+1}) = 1, \quad c_{n,n_{k+1}} = c_{n,n_k} + 2^{-(n+1)}.$$

Under these preparations, we define $\varphi(t)$ as follows:

$$\varphi(t) = c_{n,\alpha} t, \quad \frac{1}{n+1} + \frac{n}{2^{n+2}(n+1)(n+2)} \leqq t \leqq$$

$$\leqq \frac{1}{n+1} + \frac{n}{n(n+1)} - \frac{n}{2^{n+1} n(n+1)},$$

$$\varphi(t) = c_{n,m} t, \quad \frac{1}{n+1} + \frac{\chi(c_{n,m-1})}{n(n+1)} + \frac{\chi(c_{n,m-1}) - \chi(c_{n,m-2})}{2^{n+1} n(n+1)} \leqq t \leqq$$

$$\leqq \frac{1}{n+1} + \frac{\chi(c_{n,m})}{n(n+1)} - \frac{\chi(c_{n,m}) - \chi(c_{n,m-1})}{2^{n+1} n(n+1)}, \quad \alpha < m \leqq \beta,$$

$$\varphi(t) = c_{n,\beta+1} t, \quad \frac{1}{n+1} + \frac{\beta_n}{n(n+1)} + \frac{\beta_n - (c_{n,\beta-1})}{2^{n+1} n(n+1)} \leqq t \leqq$$

$$\leqq \frac{1}{n} - \frac{\beta_n}{2^{n+1} n(n+1)},$$

$$\varphi(t) = \frac{2^{n+2}(n+2(c_{n,\alpha} - c_{n+1,\beta+1}) + c_{n,\alpha}\alpha_n + c_{n+1,\beta+1}\beta_{n+1}}{\alpha_n + \beta_{n+1}} \cdot$$

$$\cdot \left(t - \frac{2^{n+2}(n+2) + \alpha_n}{2^{n+2}(n+1)(n+2)}\right) + c_{n,\alpha}\left(\frac{1}{n+1} + \frac{\alpha_n}{2^{n+2}(n+1)(n+2)}\right),$$

$$\frac{1}{n+1} - \frac{\beta_{n+1}}{2^{n+2}(n+1)(n+2)} < t < \frac{1}{n+1} + \frac{\alpha_n}{2^{n+2}(n+1)(n+2)},$$

$$\varphi(t) = \frac{(c_{n,m+1} - c_{n,m}) 2^{n+1}(n + \chi(c_{n,m})) + (c_{n,m+1} + c_{n,m})(\chi(c_{n,m}) - \chi(c_{n,m-1}))}{2(\chi(c_{n,m}) - \chi(c_{n,m-1}))}$$

$$\cdot \left(t - \frac{1}{n+1} - \frac{\chi(c_{n,m})}{n(n+1)} - \frac{\chi(c_{n,m}) - \chi(c_{n,m-1})}{2^{n+1} n(n+1)}\right) +$$

$$+ c_{n,m+1}\left(\frac{1}{n+1} + \frac{\chi(c_{n,m})}{n(n+1)} + \frac{\chi(c_{n,m}) - \chi(c_{n,m-1})}{2^{n+1} n(n+1)}\right),$$

$$\frac{1}{n+1} + \frac{\chi(c_{n,m})}{n(n+1)} - \frac{\chi(c_{n,m}) - \chi(c_{n,m-1})}{2^{n+1} n(n+1)} < t <$$

$$< \frac{1}{n+1} + \frac{\chi(c_{n,m})}{n(n+1)} + \frac{\chi(c_{n,m}) - \chi(c_{n,m-1})}{2^{n+1} n(n+1)}, \quad \alpha < m \leqq \beta,$$

$$\varphi(0) = 0,$$

where $\chi(c)$ means $\lim._{x\to c-0}\chi(x)$.

Since $\varphi(t)=O(\sqrt{t})$ by definition, φ is continuous on $0\leqq t<1/n$ for sufficiently large n and right $C_b(\boldsymbol{R}^1)$-derivable at $t=0$ and we have

$$\mathfrak{d}_{C_b(\boldsymbol{R}^1)}{}^+\varphi(0)=\xi. \tag{27)'}$$

Because by the definition of φ, if $c\in\cup_n N_n$, then $\chi^+{}_{\varphi(0)}(c)$ exists and

$$\chi^+{}_{\varphi(0)}(c)=\chi(c).$$

On the other hand, if $c_1<x<c_2$, c_1, $c_2\in\cup_n N_n$, then since we know

$$\chi^+{}_{\varphi(0)}(c_1)\leqq\lim_{\delta\to0}.\,inf.\,m(\{t\,|\,\frac{\varphi(t)}{t}\leqq x\}\cap(0,\ \delta])/\delta$$

$$\leqq\lim_{\delta\to0}.\,sup.\,m(\{t\,|\,\frac{\varphi(t)}{t}\leqq x\}\cap(0,\ \delta])/\delta$$

$$\leqq\chi^+{}_{\varphi(0)}(c_2),$$

we get $\chi_{\varphi(0)}(x)=\chi(x)$ if φ is continuous at x by (28). But, since φ is continuous almost everywhere on $\boldsymbol{R}_1$, we obtain (27)$'$.

Note. If ξ is a probabilistic distribution on $\boldsymbol{R}^1\cup\{\pm\infty\}$, but not on $\boldsymbol{R}^1$, that is, the distribution function χ of ξ has the property $\lim._{x\to-\infty}\chi(x)\neq0$, or $\lim._{x\to\infty}\chi(x)\neq1$, then we set for $n\geqq2$

$$\chi_n(x)=\chi(x),\quad |x|\leqq\sqrt{n-1},$$

$$\chi_n(x)=0,\quad x<-\sqrt{n-1},$$

$$\chi_n(x)=1,\quad x>\sqrt{n-1},$$

and define $\varphi(t)$ as above, but use $\chi_n(c_{n,m})$, *etc.*, instead of $\chi(c_{n,m})$ *etc.*. Then this $\varphi(t)$ is defined on $0\leqq t<1/(3+n)$ and continuous at $t=0$ and (right) $C(\boldsymbol{R}^1\cup\{\pm\infty\})$-derivable at $t=0$ with $\mathfrak{d}_{C(\boldsymbol{R}^1\cup\{\pm\infty\})}{}^+\varphi(0)=\xi$.

9. ***Lemma* 12.** *We assume $\mathscr{F}(\boldsymbol{R}^1)\subset C(\boldsymbol{R}^1)$, the space of continuous functions on $\boldsymbol{R}^1$, and satisfies the following condition* (*c*):

(*c*). *If $f\in\mathscr{F}(\boldsymbol{R}^1)$ and $a\in\boldsymbol{R}$, then f_a, given by $f_a(x)=f(a+x)$, also belongs in $\mathscr{F}(\boldsymbol{R}^1)$. Then we have*

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+(\varphi_1+\varphi_2)(a)=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi_1(a)*\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi_2(a), \tag{29}$$

if φ_1 and φ_2 are (*right*) *$\mathscr{F}(\boldsymbol{R}^1)$-derivable at a and continuous near* (*of the right hand side*) *of a. Here, $\xi_1*\xi_2$ means the convolution of ξ_1 and ξ_2.*

Proof. First we remark that

$$\lim_{h\to0}.\int_h^s f\left(\frac{\varphi(a+t)-\varphi(a)}{t}\right)dt=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(a)(f)+o(s), \tag{30}$$

if φ is (right) $\mathscr{F}(\boldsymbol{R}^1)$-derivable at a. Then we get

$$\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(0)=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\Big(\frac{1}{c}\rho_c*\varphi\Big)(0), \tag{31}$$

for any $c\neq 0$ by (30), if φ is $\mathscr{F}(\boldsymbol{R}^1)$-derivable at 0 and $\varphi(0)=0$. Because we have

$$\begin{aligned}\int_h^s f\Big(\frac{\varphi(ct)}{ct}\Big)dt&=\frac{1}{c}\int_{ch}^{cs} f\Big(\frac{\varphi(T)}{T}\Big)dT\\ &=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi(0)s+R(h),\end{aligned}$$

where $\lim._{h\to 0} R(h)=0$.

To show (29), we assume $a=0$ *and* $\varphi_1(0)=\varphi_2(0)=0$ for simple. Then we get by the mean value theorem of the integral,

$$\begin{aligned}&\int_h^s f\Big(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(\theta\alpha t)}{\theta\alpha t}\Big)dt \\ &=\frac{1}{t}\int_h^s\int_h^t f\Big(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(\alpha k)}{\alpha k}\Big)dtdk.\end{aligned} \tag{32}$$

But since φ_2 is continuous, we get by the mean value therem of the integral and (30),

$$f\Big(\frac{\varphi_2(t)}{t}\Big)t=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi^2(0)(f)t+o(t)+R(h),\quad \lim_{h\to 0}. R(h)=0.$$

Therefore, by (32), if s is sufficiently small, then for suitable $\alpha>0$, we get

$$\begin{aligned}&\int_h^s f\Big(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(t)}{t}\Big)dt \\ &=\frac{1}{t}\int_h^s\int_h^t f\Big(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(\alpha k)}{\alpha k}\Big)dtdk+o(s)+R(h).\end{aligned} \tag{32)'$$

On the other hand, we obtain by (30) and (31),

$$\begin{aligned}&\lim_{h\to 0}.\frac{1}{t}\int_h^s\int_h^t f\Big(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(\alpha k)}{\alpha k}\Big)dtdk \\ &=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi_1(0)(\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+(\frac{1}{\alpha}\rho_\alpha*\varphi_2)(0)(f(t+k)_k)_t s+o(s) \\ &=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi_1(0)(\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)}{}^+\varphi_2(0)(f(t+k)_k)_t s+o(s).\end{aligned}$$

Hence by (32)$'$, we have

$$\lim_{s\to 0}\frac{1}{s}\lim_{h\to 0}\int_h^s f\left(\frac{\varphi_1(t)}{t}+\frac{\varphi_2(t)}{t}\right)dt$$

$$=\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi_1(0)(\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi_2(0)(f(t+k))_k)_t$$

$$=(\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi_1(0)*\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi_2(0))(f).$$

Therefore, we obtain the lemma.

On the other hand, by the definition of $\mathscr{F}(\boldsymbol{R}^1)$-derivatives, we obtain

***Lemma* 13.** *If φ is $\mathscr{F}(\boldsymbol{R}^1)$-derivable at a, then for any $c\in\boldsymbol{R}$, $c\varphi$ is $\mathscr{F}(\boldsymbol{R}^1)$-derivable at a and we have*

$$\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}(c\varphi)(a)=\rho_c{}^*(\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)),\tag{33}$$

where $\rho_c{}^*$ means the adjoint operator or ρ_c.

By lemma 12 and lemma 13, we have

***Lemma* 14.** *If φ_1 and φ_2 are $C_b(\boldsymbol{R}_1)$-derivable at a, then $\varphi_1\varphi_2$ is also $C_b(\boldsymbol{R}_1)$-derivable at a and we have*

$$\begin{aligned}&\mathfrak{d}_{C_b(R^1)}{}^{+}\varphi_1\varphi_2)(a)\\&=(\rho_{\varphi_1(a)}{}^*(\mathfrak{d}_{C_b(R^1)}{}^{+}\varphi_2(a)))*(\rho_{\varphi_2(a)}{}^*(\mathfrak{d}_{C_b(R^1)}{}^{+}\varphi_1(a))).\end{aligned}\tag{34}$$

***Proof*.** By the usual calculation, it only needs to show

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{s}\lim_{h\to 0}\int_h^s f\left(\frac{\varphi_1(a+t)(\varphi_2(a+t)-\varphi_2(a))}{t}\right)dt\\&=\rho_{\varphi_1(a)}{}^*(\mathfrak{d}_{C_b(R^1)}{}^{+}\varphi_2(a))(f).\end{aligned}\tag{35}$$

To show (35), we assume $a=0$ and $\varphi_1(0)=\varphi_2(0)=0$ for simple. Then by lemma 11, for any $\varepsilon>0$, there exists $K>0$ and $L>0$ such that if $\delta<L$, then

$$m(\{t\mid |\frac{\varphi_i(t)}{t}|\leqq K\}\cap(0,\ \delta])>(1-\frac{\varepsilon}{2})\delta,\ \ i=1,2.$$

Hence we get for these L and K,

$$m(\{t\mid |\varphi_1(t)\varphi_2(t)|\leqq K^2t^2\}\cap(0,\ \delta])>(1-\varepsilon)\delta.$$

Hence for any $\varepsilon>0$ and $\alpha>0$, there exists $M>0$ such that if $\gamma<M$, then

$$m(\{t\mid |\frac{\varphi_1(t)\varphi_2(t)}{t}|\leqq\alpha\}\cap(0,\ \gamma])>(1-\varepsilon)\gamma.$$

Therefore, since we censider the case $\mathscr{F}(\boldsymbol{R}_1)=C_b(\boldsymbol{R}^1)$, we obtain (35).

10. Since $\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)$ is a probabilistic distribution if $\mathscr{F}(\boldsymbol{R}^1)$ containes

$C_0(\boldsymbol{R}^2)$, we get

$$\lim_{c\to 0}.\ \mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}(c\varphi)(a)=\delta\ (=\delta_0), \tag{36}$$

if $C_0(\boldsymbol{R}^1)\subset\mathscr{F}(\boldsymbol{R}^1)\subset C_b(\boldsymbol{R}^1)$ by the Lebesgue's convergence theorem.

Definition. *If $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)$ is a probabilistic distribution and its characteristic function $\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)](t)=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)(e^{-2\pi\sqrt{-1}st})_s$ is positive on $\boldsymbol{R}^1$, then we define $d_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)$ by*

$$d_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)=\frac{-1}{2\pi\sqrt{-1}}\log.\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)]. \tag{37}$$

Since we have by lemma 12,

$$\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}(\varphi_1+\varphi_2)(a)]=\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi_1(a)].\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi_2(a)], \tag{29'}$$

we obtain

$$d_{\mathscr{F}(\boldsymbol{R}^1)^+}(\varphi_1+\varphi_2)(a)=d_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi_1(a)+d_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi_2(a), \tag{38}$$

if $d_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi_i(a)$, $i=1, 2$, exist.

Similarly, by (36), we get

$$\lim_{c\to 0}.\ d_{\mathscr{F}(\boldsymbol{R}^1)^+}(c\varphi)(a)=0. \tag{36'}$$

Lemma 15. *If the expectation of $\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)$ exists, that is φ is φ is $C_{|x|}(\boldsymbol{R}^1)$-derivable at a, then*

$$d_{\mathscr{F}(\boldsymbol{R}^1)^+}(c\phi)(a)=ct\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)(s)+o(c), \tag{39}$$

for $c\neq 0$.

Proof. By assumption, we can set

$$\begin{aligned}&\mathscr{F}[\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}(c\varphi)(a)](t)\\&=\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)(e^{-2\pi\sqrt{-1}cts})_s\\&+\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)(1-2\pi\sqrt{-1}cts)_s+\mathfrak{d}_{\mathscr{F}(\boldsymbol{R}^1)^+}\varphi(a)(e^{-2\pi\sqrt{-1}cts}-1+2\pi\sqrt{-1}cts)_s.\end{aligned}$$

Then, since we get by mean value theorem,

$$\begin{aligned}&\left|\frac{1}{c}(e^{-2\pi\sqrt{-1}cts}-1+2\pi\sqrt{-1}cts)\right|\\&=\left|2\pi\sqrt{-1}ts+\frac{1}{c}(e^{-2\pi\sqrt{-1}cts}-1)\right|\end{aligned}$$

$$\leqq |2\pi\sqrt{-1}ts - 2\pi\theta_1 ts \sin 2\pi\theta_1 cts - 2\pi\sqrt{-1}\theta_2 ts \cos\theta_2 ts|$$

$$\leqq 6\pi|ts|,$$

we have by Lebesgue's convergence theorem,

$$\lim_{c\to 0}.\frac{1}{c}\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(e^{-2\pi\sqrt{-1}cts} - 1 + 2\pi\sqrt{-1}cts) = 0.$$

Hence we may set

$$\mathscr{F}[\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}(c\varphi)(a)]$$

$$= \mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(1) - 2\pi\sqrt{-1}ct\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(s) + o(c).$$

Then, since $\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}(a)(1) = 1$, we obtain the lemma.

We note that to denote $B_r\varphi(a)$ the right Borel derivate of φ at a, then we get

$$\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(s) = B_r\varphi(a).$$

Note. By the same reason, if k-th moment of $\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)$ exists, then

$$d_{\mathscr{F}(R^1)}{}^{+}(c\varphi)(a) = \frac{-1}{2\pi\sqrt{-1}}\sum_{m\leqq k}\frac{1}{m!}(-2\pi\sqrt{-1}ct)^m\{\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(s^m) - $$
$$-\frac{n!}{1}\sum_{i\leqq[n/2]}\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(s^i)\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(^{m-i}) + \cdots\} + o(c^k). \tag{39$'$}$$

Especially, if $\mathfrak{d}_{\mathscr{F}(R^1)}\varphi(a)(s^m) = \{\mathfrak{d}_{\mathscr{F}(R^1)}\varphi(a)(s)\}^m$, then $d_{\mathscr{F}(R^1)}{}^{+}(c\varphi)(a) = ct\,\mathfrak{d}_{\mathscr{F}(R^1)}{}^{+}\varphi(a)(s) + o(c^k)$.

Summarising these, we have

Theorem 7. *$d_{\mathscr{F}(R^1)}{}^{+}$ is additive and if $d_{\mathscr{F}(R^1)}{}^{+}\varphi(a)$ is differentiable, then*

$$\lim_{c\to 0}.\frac{1}{c}(d_{\mathscr{F}(R^1)}{}^{+}(c\varphi)(a)) = B_r\varphi(a)t. \tag{40}$$

Moreover, we get $d_{\mathscr{F}(R^1)}{}^{+}\varphi(a) = B_r\varphi(a)t$ if and only if φ is (right) approximately derivable at a.

Bibliography

[1]. Asada, A. : Generalized tangents of curves and generalized vector fields, J. Fac. Sci. Shinshu Univ. 6(1971), 45-75.

[2]. Asada, A. : Generalized integral curves of generalized vector fields, J. Fac. Sci. Shinshu Univ. 7(1972), 59-118.

[2]'. Asada, A. : Generalized vector field and its local integration, Proc. of Jap. Acad. 49 (1973), 73-76.

[3]. Besicovic, A. S. : *Almost periodic functions*, Cambridge, 1932.
[4]. Burkil, J. C. : Integrals and trigonometric series, Proc. London Math. Soc. (3) 1(1951), 46-57.
[5]. James, R. D. : Integrals and summable trigonometric series, Bull. Amer Math. Soc. 61 (1953), 1-15.
[6]. Khintchin, A. : Recherches sur la structure des fonctions mesurable, Fund. Math. 9 (1927), 212-279.
[7]. Lévy, P. : *Théorie de l'addition des variables aléatoires*, Paris, 1937.
[8]. Linnik, Yu. V. : *Decomposition of probability distributions*, Mosccow, 1960 (English Translation, London, 1964).
[9]. Marcinkiewicz, J. -Zygmund, A. : On the differentiability and summability of trigonometrical series, Fund. Math. 26(1936), 1-43.
[10]. Saks, S. : *Theory of the integral*. Warszaw, 1937.
[11]. Sargent, : The Borel derivative of a function, Proc. London Math. Soc. 38(1934), 180-196.
[12]. Widder, D. V. : *The Laplace transform*, Princeton, 1946.
[13]. Wiener, N, : Tauberian theorems, Ann. of Math. 33(1932), 1-100.
[14]. Zygmund, A. : *Trigonometric series*, I, II, Cambridge, 1959.