JOUR. FAC. SCI., SHINSHU UNIV., Vol. 7, pp. 59–118, December, 1972

Generalized Integral Curves of Generalized Vector Fields

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(Received October 31, 1972)

Contents.

Introduction.

In [4], the authour introduced the notions of $C(S^{n-1})$-differentiable functions and generalized vector fields. On an n -dimensional (paracompact connected) manifold M, they are defined as follows: Let ρ be a metric of M such that if $\rho(x, y) \leqq 2$, then there exists unique the shortest path which joins x and y with respect to ρ. Then, denoting $r_{x,y,t}$ the point on the shortest path which joins x and y such that

$$\rho(x, r_{x,y,t})=t,$$

we call a function f of M is $C(S^{n-1})$-differentiable at x if there exists a continuous function $g=g(x, y)$ on $S_x=\{y \mid \rho(x, y)=1\}$ such that

$$f(r_{x,y,t})=f(x)+g(x, y)t+o(t).$$

As usual, a function f on M is called $C(S^{n-1})$-differentiable on M if f is $C(S^{n-1})$-differentiable at every point of M. Note that, in this case, $g(x, y)$ may not be continuous in x in general.

We denote the space of $C(S^{n-1})$ -differentiable functions on M by $C_{C(S^{n-1})}(M)$. It is a dense subspace of $C(M)$, the space of continuous functions on M with the compact open topology. Then we call a linear operator X from $C_{C(S^{n-1})}(M)$ to $M_{loc.}(M)$, the space of locally bounded functions on M with the compact open topology, to be a generalized vector field or a $C(S^{n-1})$-vector field on M if X satisfies the following (i), (ii), (iii).

(i). *X is a closed operator of $C(M)$.*

(ii). *$X(fg)$ is equal to $(Xf)g+f(Xg)$.*

(iii). *$(Xf)(a)=0$ if $|f(x)-f(a)|=o(\rho(x, a))$.*

It is shown that denoting $C^*(s(M))$ the dual bundle of the $C(S^{n-1})$-bundle associated to $s(M)=\{(x, y)|\rho(x, y)=1, x\in M\}$, the associate S^{n-1}-bundle of the tangent microbundle of M, Xf is written as

$$Xf(x)=<\xi(x), d_\rho f(x)>,$$

where $\xi(x)$ is a (locally bounded) cross-section of $C^*(s(M))$ and denoted by *rep.* X and $d\rho f(x, y)$ is given by

$$d_\rho f(x, y)=\lim_{t\to 0}.\frac{1}{t}(f(r_{x,y,t})-f(x)).$$

The main purpose of this paper is to treat the integral curves of generalized vector fields. For this purpose, first we consider the problem in local, that is, we consider the case $M=\boldsymbol{R}^n$ and ρ is the euclidean metric of $\boldsymbol{R}^n$. In this case, first it is noted that $d_\rho f$ is the Gâteaux-differential of f (*cf.* [8], [9]) and if f is $C(S^{n-1})$-differentiable on $\boldsymbol{R}^n$ and $d_\rho f(x, y)$ is continuous in x, then f is tatally differentiable on $\boldsymbol{R}^n$ (cf. [8], [9], [16]). Since $d_\rho f(x, y)$ is linear in y if f is tatally differentiable, the problem to solve the equation $d_\rho f(x, y)=u(x, y)$ is quite different whether $u(x, y)$ is linear in y or not. In fact, if $u(x, y)$ is $C(S^{n-1})$-differentiable in x and $C(S^{n-2})$-differentiable in y, then to set

$$f(sy, y_1)=\int_0^1 u(tsy, y_1)sdt, \quad ||y||=||y_1||=1,$$

we obtain

$$\begin{aligned}&|u(sy, y_1)\text{-}d\rho f(sy, y_1)|\\ \leqq&\int_0^1 |u(tsy, y)(y, y_1)+d_{\rho,y}u(tsy, \varepsilon_{y,y_1})|\,||y_1-(y, y_1)y||\\ &+d_{\rho,x}u(tsy, y_1, y)ts-(u(tsy, y_1)+d_{\rho,x}u(tsy, y, y_1)ts)|dt,\end{aligned}$$

Here, ε_{y,y_1} means the point of S^{n-1} such that $\rho(y, \varepsilon_{y,y_1})=1$ with y_1-direction, where ρ is the metric on S^{n-1} induced from the euclidean metric and (y, y_1) is the inner product of y and y_1. The right hand side of this inequality is complicated in general. But, since

$$u(x, y_1)=u(x, y)(y, y_1)+d_{\rho,y}u(x, y, \varepsilon_{y,y_1})||y_1-(y, y_1)y||,$$

if $u(x, y)$ is linear in y, the above inequality is reduced to

$$\begin{aligned}&|u(sy, y_1)-d_\rho f(sy, y_1)|\\ &\quad\leqq\int_0^1|d_{\rho,x}u(tsy, y_1, y)-d_{\rho,x}u(tsy, y, y_1)|tsdt.\end{aligned}$$

For this reason, we set the subspace of $C(S^{n-1})$ consisted by linear functions by $l(S^{n-1})$ and decompose $C^*(S^{n-1})$ as follows: *To define a subspace $l^*(S^{n-1})$ of $C^*(S^{n-1})$ by*

$$l^*(S^{n-1})=\left\{\sum_{i=1}^{n}c_i\delta_i\,|\,c_i\in \boldsymbol{R}\right\},$$

where δ_i is the Dirac measure of S^{n-1} concentrated at $(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$, *and set*

$$C^*(S^{n-1})=l^*(S^{n-1})\oplus l(S^{n-1})^{\perp}.$$

In this decomposition, we denote the projections from $C^*(S^{n-1})$ to $l^*(S^{n-1})$ and $l(S^{n-1})^{\perp}$ by p_1 and p_2. Then we define generalized vector fields $D(X)$ and $S(X)$ by

$$\begin{aligned}&D(X)f(x)=\langle p_1(\xi(x)),\ d_\rho f(x)\rangle,\quad rep.\ X=\xi(x),\\ &S(X)f(x)=\langle p_2(\xi(x)),\ d_\rho f(x)\rangle,\end{aligned}$$

for a generalized vector field X on $\boldsymbol{R}^n$. Then we have

(i). *We may consider X to be a usual vector field on $\boldsymbol{R}^n$ if and only if $X=D(X)$.*

(ii). *If $X=S(X)$, then Xf is equal to 0 almost everywhere on $\boldsymbol{R}^n$.*

(ii)′. *If $X=S(X)$ and rep. X is $C(S^{n-1})$-differentiable, then $X(C_{C(S^{n-1})}(\boldsymbol{R}^n))$ contains $l^1_{loc.}(\boldsymbol{R}^n)$. Here, $l^1_{loc.}(\boldsymbol{R}^n)$ is given by*

$$l^1_{loc.}(\boldsymbol{R}^n)=\left\{f\,|\sum_{x\in k}|f(x)|<\infty,\ K\ is\ compact\ in\ \boldsymbol{R}^n\right\}.$$

(iii). *If $\varphi(t)$ is an integral curve of X starts from the origin in the weak sence, that is $\varphi(t)$ satisfies*

$$\lim_{S\to 0}.\frac{1}{S}\lim_{h\to 0}.\int_h^s\frac{f(\varphi(t+r))-f(\varphi(t))}{r}dr=\langle\xi(\varphi(t)),\ d_\rho f(\varphi(t))\rangle,$$

for any $C(S^{n-1})$-differentiable f, then

$$f(\varphi(t))=f(0)+\int_0^t <\xi(\varphi(t)),\ d_\rho f(\varphi(t))>dt.$$

Especially, if $X=S(X)$, then X can not have integral curve although in the weak sence.

But, since $l^*(S^{n-1})\cong \boldsymbol{R}^n$, we may consider $\boldsymbol{R}^n$ to be a subspace of $C^*(S^{n-1})$. Then, since we can extent $\xi(x)(=\text{rep. } X)$ to be a function $\xi^{\#}(x)$: $C^*(S^{n-1})\to C^*(S^{n-1})$ and we can solve the equation

$$\frac{d\varphi(t)}{dt}=\xi^{\#}(\varphi(t)),$$

in $C^*(S^{n-1})$ under suitable assumptions about $\xi^{\#}(x)$, we may consider a generalized vector field X of $\boldsymbol{R}^n$ have an integral curve $\varphi(t)$ starts at any point of $\boldsymbol{R}^n$, in $C^*(S^{n-1})$ under suitable assumptions about X. We call this $\varphi(t)$ to be the generalized integral curve of X. For the generalized integral curves, we have

(i). *If $X=D(X)$, then $\varphi(t)$ is the usual integral curve of X. In general, curve of X. In general, $p_1(\varphi(t))$ is the usual integral curve of $D(X)$.*

(ii). *If $X=S(X)$, then $p_1(\varphi(t))=p_1(\varphi(0))$ for any t.*

(iii). *X has the generalized integral curve starts from x if and only if $D(X)$ has the usual integral curve starts from x.*

Since X has the integral curves, we may consider X generats a 1-parameter local group of transformations $\{T_t\}$, $T_t: \boldsymbol{R}^n\to C^*(S^{n-1})$. Therefore, if we allow to consider the functions from $\boldsymbol{R}^n$ to some space of measures, we can solve the equation

$$Xu=f,$$

for continuous f locally, although $X=S(X)$.

We note although there are many subspaces of $C^*(S^{n-1})$ which can be identified to the dual space of $l(S^{n-1})$ such as

$$l'(S^{n-1})=\{g\omega \mid g\in l(S^{n-1}),\ \omega \textit{ is the standard mesure of } S^{n-1}\}.$$

But, to define the generalized tangent of a curve $\alpha(t)$, $\alpha(0)=x$, to be $\xi(x)\in C^*(S^{n-1})$, where $\xi(x)$ is given by

$$<\xi(x),\ d_\rho f(x)>=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{f(\alpha(t))-f(x)}{t}dt,$$

α is smooth at x if and only if $\alpha(x)=c\delta_y$ where δ_y is the Dirac measure of S^{n+1} concentrated at y. ([4]). Hence no element of $l'(S^{n-1})$ is expressed as a generalized tangent of some smooth curve and it seems to be natural to take $l^*(S^{n-1})$ to be the standard model of the dual space of $l(S^{n-1})$.

To extend the above results to the generalized vector fields on a (connected paracompact n-dimensional) manifold M, first we set

$$l(S_x)=\{d_\rho f(x) \mid d_\rho f \text{ is continuous in } x \text{ on some neighborhood of } x\}$$

Then, it is shown

$$\dim. l(S_x) \leqq n = \dim. M.$$

Hence to set

$$M_{s,\rho,k}=\{x \mid \dim. l(S_x)=n-k\},$$

$$M_{\rho,k}=\bigcup_{m\geqq k} M_{s,\rho,k},$$

we have a decomposition of M as follows:

$$M=\bigcup_{k=0}^{m} M_{s,\rho,k}, \quad M_{s,\rho,i} \cap M_{s,\rho,j}=\emptyset, \quad \text{if } i \neq j.$$

For these $M_{s,\rho,k}$, we can prove

(i). *$M_{s,\rho,k}$ is open in $M_{\rho,k}$.*

(ii). *$\dim. M_{s,\rho,k}$ is at most $n-k$.*

(iii). *$M_{s,\rho}=M_{s,\rho,0}$ allows a differential structure.*

Moreover, we can construct an $(n\text{-}k)$-dimensional subspace $M_{s,\rho,k}{}^{\#}$ which contains $M_{s,\rho,k}$ such that

(i). *$\dim. M_{s,\rho,k}{}^{\#}$ is equal to $n-k$.*

(ii). *$M_{s,\rho,k}{}^{\#}$ allows a differential structure.*

Hence to set

$$M_{s,\rho}{}^{\flat}=M_{s,\rho}-M_{s,\rho} \cap (\bigcup_{m\geqq 1} M_{s,\rho,m}{}^{\#}),$$

$$M_{s,\rho,k}{}^{\flat}=M_{s,\rho,k}{}^{\#}-M_{s,\rho,k}{}^{\#} \cap (\bigcup_{m\geqq k+1} M_{s,\rho,m}{}^{\#}), \quad k\geqq 1,$$

we have a decomposition of M similarly as stratification as follows (cf. [7]):

$$M=M_{s,\rho}{}^{\flat} \cup M_{s,\rho,1}{}^{\flat} \cup \cdots \cup M_{s,\rho,n}{}^{\flat}, \quad M_{s,\rho,i}{}^{\flat} \cap M_{s,\rho,j}{}^{\flat}=\emptyset, \quad i\neq j,$$

$$\dim. M_{s,\rho,k}{}^{\flat}=n-k, \quad M_{s,\rho,0}{}^{\flat}=M_{s,\rho}{}^{\flat}.$$

Moreover, the cotangent bundle of $M_{s,\rho,k}{}^{\flat}$ can be extended to some neighborhood of $M_{s,\rho,k}{}^{\flat}$ in $M-\bigcup_{m\geqq k+1} M_{s,\rho,m}{}^{\flat}$. This cotangent bundle $T(M_{s,\rho,k}{}^{\flat})$ of $M_{s,\rho,k}{}^{\flat}$ is costructed by using $\bigcup_{x\in M_{s,\rho,k}} l(S_x)$. Then to fix the basis $d_\rho f_1, \cdots, d_\rho f_{n-k}$ of $l(S_x)$, we can choose the continuous cross-sections $y_1=y_1(x), \cdots, y_{n-k}=y_{n-k}(x)$ of $s(M)$ such that

$$d_\rho f_i(x, y_j)=\delta_{ij}, \quad i, j=1, \cdots, n\text{-}k.$$

Then to modify the subspace of $C^*(S_x)$ spanned by $\delta_{y_1}, \cdots, \delta_{y_{n-k}}$, we can construct a subspace $l^*(S_x)$ of $C^*(S_x)$ as follows:

(i). *$l^*(S_x)$ is the dual space of $l(S_x)$ as a subspaca of $C^*(S_x)$, if x belongs in $M_{s,\rho,k}$.*

(ii). *$\cup_{x\in M_{s,\rho,k}{}^\flat} l^*(S_x)$ allows the structure of vector bundle and it is the dual bundle $T^*(M_{s,\rho,k}{}^\flat)$ of $T(M_{s,\rho,k}{}^\flat)$ of $T(M_{s,\rho,k}{}^\flat)$.*

Using these we set

$$\tau^{\#*}(M)=\bigcup_{k=0}^{n} T^*(M_{s,\rho,k}{}^\flat),$$

$$\tau^{\#}(M)^{\perp}=\bigcup_{k=0}^{n} T(M_{s,\rho,k}{}^\flat)^{\perp}.$$

Here $T^*(M_{s,\rho,k}{}^\flat)$ and $T(M_{s,\rho,k}{}^\flat)^\perp$ are regarded to be the subspaces of $C^*(s(M))$ whose values are coincide to that of $T^*(M_{s,\rho,k}{}^\flat)$ or $T(M_{s,\rho,k}{}^\flat)^\#$ on $M_{s,\rho,k}{}^\flat$ and vanish on $M\text{-}M_{s,\rho,k}{}^\flat$.

By the definitions of $\tau^{\#*}(M)$ and $\tau^\#(M)^\perp$, we have,

$$C^*(s(M))=\tau^{\#*}(M)\oplus\tau^\#(M).$$

Then to use this decomposition, we canconstruct the generalized integral curve $\varphi(t)$ of a generalized vector field X on M as a curve in $\tau^\#(M)^\perp$. This $\varphi(t)$ has the following properties.

(i). *If $X=D(X)$, then $\varphi(t)$ is the usual integral curve of X starts from $x=\varphi(0)$. Here M is considered to be the 0-section of $\tau^\#(M)^\perp$.*

(ii). *Denoting the projection of $\tau^\#(M)^\perp$ by $\pi^\#$, $\pi^\#(\varphi(t))$ belongs in $M_{s,\rho,k}{}^\flat$ if $x\in M_{s,\rho,k}{}^\flat$.*

(iii). *If $X=S(X)$, then $\pi^\#(\varphi(t))=x$ for all x.*

(iv). *X has the generalized integral curve starts from x if and only if $D(X)$ has the usual integral curve starts from x.*

We remark that, if M is smooth and ρ is the geodesic distance of a Riemannian metric of M, then $M=M_{s,\rho}$ and $\tau^{\#*}(M)$ is the tangent bundle of M. On the other hand, since $C^*(s(M))$ is the associate $C^*(S^{n-1})$-bundle of the tangent bundle of M, $\tau^\#(M)^\perp$ is also the associate $l(S^{n-1})^\perp$-bundle of the tangent bundle of M. Therefore, $\tau^\#(M)^\perp$ is a Banach manifold modeled by $C^*(S^{n-1})$ ([6], [12]). But, since $C^{**}(S^{n-1})$ is not separable, by the theorem of Restrepo ([6], [14]), $\tau^\#(M)^\perp$ is not C^1-smooth.

On the other hand, if we use the $L^2(S^{n-1})$-differentiable functions and $L^2(S^{n-1})$-vector fields (cf. [4]), then we can construct the above theory using associate $L^2(S^{n-1})$-bundle of the tangent microbundle of M. Hence, if M is smooth, then we obtain the generalized integral curve of an $L^2(S^{n-1})$-vector field of M in the tatal space of the associate $l(S^{n-1})$-bundle of the tangent bundle of M. In this case, the space $\tau^\#(M)^\perp$ is C^∞-smooth ([6]) and by Kuiper's theorem ([11]), we obtain

$$\tau^\#(M)^\perp\cong M\times l(S^{n-1})^\perp\cong M\times H,$$

Where H is the separable Hilbert space. But, since a $L^2(S^{n-1})$-differentiable function at x may not be continuous at x, a smooth curve at x does not have $L^2(S^{n-1})$-tangent at x. For example, in $\boldsymbol{R}^2$, the function f given by

$$f(r,\ \theta)=r\theta^{-1/3},\ \ r>0,\ \ 0<\theta<2\pi,$$

$$f(r,\ 0)=f(0,\ 0)=0,$$

is $L^2(S^1)$-differentiable at the origin by the euclidean metric. But it can not be differentiable although in the weak sence along the line $\alpha(t)=(t,\ t^2)$ at the origin. We remark the above f has the (weak) derivation along the curve $r\theta=1$ at the origin. Therefore, no smooth curve corresponds to the element of $L^2(S^{n-1})$. Moreover, since the generalized tangent of a curve always positive ([4]), no element of $l(S^{n-1})$ corresponds to a curve.

The outline of this paper is as follows: In chapter 1, we state the basic properties of $C(S^{n-1})$-differentiable functions and $C(S^{n-1})$-vector fields. Since the formulae (9)′ and (10) in [4] are false in general, we give the correct form of these formulae in § 2. In § 2, it is also shown that the usual Stokes' theorem can be deduced form the Stokes' theorem of [3] (cf. [5], [7]). The generalized integral curve of a generalized vector field of $\boldsymbol{R}^n$ is defined in chapter 2. It is also shown in § 5 that if $\xi(x)$ is continuous in x and positive as a measure on S^{n-1} for all x, then there exists a continuous family of continuous curves $\varphi_x(t)$ such that

$$\varphi_x(0)=x,$$

$$<\xi(x),\ d_\rho f(x)>=\lim_{s\to 0}.\frac{1}{S}[\lim_{h\to 0}.\int_h^S \frac{f(\varphi_x(t))-f(x)}{t}dt],$$

for any $C(S^{n-1})$-differentiable function f at x for all x. In other word, there exists a 1-parameter family $\{S_t|t\geqq 0\}$ such that S_t is continuous in t, $S_0=I$, the identity map, each S_t is a continuous transformation of $\boldsymbol{R}^n$ and

$$Xf=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^S \frac{S_tf-f}{t}dt.$$

In chapter 3, we define the generalized integral curve of a generalized vector field on a manifold. Since $M_{s,\rho}$ is smooth, it is also shown that if the manifold structure of M is given by $\{(U,\ h_U)|h_U\colon\ U\to\boldsymbol{R}^n\}$ and M has a metric ρ such that ρ satisfies the properties of [3] and each h_U is d_ρ-smooth, that is, $C(S^{n-1})$-differentiable and $d_\rho h_U(x)$ is continuous in X, then M is smooth (Theorem 8). The denseness of the d_ρ-smooth functions of M in $C(M)$ is also proved.

Since the formulae (9)′ and (10) of [4] are not correct in general, the proof of de Rham's theorem in [4] is not correct. But in chapter 3, we give a proof of de Rham's theorem in more refined form as in [4]. It takes the following form.

(i). *The de Rham group of d_ρ-smooth cross-sections of $LAC^p\ (s(M))$ with respect to the differential operator d_ρ is isomorphic to $H^p(M, \boldsymbol{R})$. Here, $AC^p(s(M))$ is the subbundle of the associate $C(\overbrace{S^{n-1}\times\cdots\cdots\times S^{n-1}}^{p})$-bundle of the tangent microbundle of M whose fibre is consisted by those functions $f(x, y_1, \cdots, y_p)$ that*

$$f(x, y_{\sigma(1)}, \cdots, y_{\sigma(p)})=\mathrm{sgn}\,(\sigma)f(x, y_1, \cdots, y_p),\quad \rho\in\mathfrak{S}^p,$$

and $LAC^p(s(M))$ is the subspace of $AC^p(s(M))$ such that $LAC^p(s(M))\,|\,M_{s,\rho,k}{}^p$ is linear for eachk.

(ii). *The element of the homology group of M is represented by those (singular) chain γ that*

$$\gamma=\sum c_i f_i(\sigma),$$
$$\rho(f_i(a_{J+1k}),\ f_i(a_J))\leqq N_i\,|\,a_{j_{k+1}}-a_{j_k}|,\quad \textit{for each } i,$$

where $J=(j_1, \cdots, j_p)$, $J+1_k=(j_1, \cdots, j_{k-1}, j_k+1, j_{k+1}, \cdots, j_p)$ and $a_J=(a_{j_1}, \cdots, a_{j_p})$.

(iii). *Taking the representations γ and φ of the p-th homology group and the p-th de Rham group of M, their duality is given by*

$$<\gamma,\ \varphi>=\int_\gamma \varphi.$$

Chapter 1. Preliminaries.

§ 1. $C(S^{n-1})$-differentiable functions.

1. Let M be a (connected) paracompact n-dimensional manifold with the fixed metric ρ such that the topology of M is given by ρ and satisfies

(i). *If $\rho(x_1, x_2)\leqq 2$, then there exists unique path γ given by $f\colon I\to M$ such that γ joins x_1 and x_2 and*

$$\rho(x_1, x_2)=\int_\gamma \rho=\lim_{|t_i-t_{i+1}|\to 0}\sum_{i=1}^{m}\rho(f(t_i),\ f(t_{i-1})),$$
$$0=t_0<t_1<<t_{m-1}<t_m=1.$$

(ii). *If γ is a curve of M such that*

$$\int_\gamma k_x\delta\rho=0,$$

then there exists a curve γ' of M which contains γ and

$$\int_{\gamma'}\rho=\infty,\quad \int_{\gamma'}k_x\delta\rho=0.$$

Here ρ is regarded to be an Alexander-Spanier 1-cochain of M and x is an *arbitrary point of γ.*

By (i) and (ii), to set

$$S_x=\{y\,|\,\rho(x,\ y)=1\},\quad \overline{B}_x=\{z\,|\,\rho(x,\ z)\leqq 1,$$

there is unique curve $r_{x,y}$ for any $y\in S_x$ which join x and y and

$$\int_{r_{x,y}}\rho=1.$$

Then, for any t, $0\leqq t\leqq 1$, there is unique point $r_{x,y,t}$ of $r_{x,y}$ such that

(1) $$\rho(x,\ r_{x,y,t})=t.$$

Conversely, if $z\in\overline{B}_x$, $z\neq x$, then there is unique $y\in S_x$ such that $z\in r_{x,y}$. We denote this y by $\varepsilon_{x,z}$. By definition, we have

(2) $$r_{x,\varepsilon_{x,z},\rho(x,z)}=z.$$

Definition. *A function f of M at x is called $C(S^{n-1})$-differentiable at x if there exists a continuous function $g(y)$ of S_x such that*

(3) $$f(z)=f(x)+g(\varepsilon_{x,z})\rho(x,\ z)+o(\rho(x,z)),\quad z\in\overline{B}_x.$$

By definition, we have

Lemma 1. *If f is $C(S^{n-1})$-dfferentiable at x, then*

(4)′ $$\lim_{t\to 0}.\frac{1}{t}(f(r_{x,y,t})-f(x))=g(y).\quad y\in S_x.$$

Conversly, if f is continuous at x and the limit of this left hand side exists for all $y\in S_x$ and defines a continuous fuuction on S_x, then f is $C(S^{n-1})$-differentiable at x.

Proof. It only needs to show the converse. Therefore, we set

$$f(z)=f(x)+g(y)\rho(x,\ z)+R(z).$$

Then, if $\lim._{z\to x}R(z)/\rho(x,\ z)\neq 0$, there exists a sequence $\{z_m\}$ such that $\lim._{m\to\infty}z_m=x$ and $|R(z_m)|\geqq c\rho(y,\ z_m)$. But, since S_x is compact, we may assume ε_{x,z_m} converges to $y_0\in S_x$. Then the limit of (4)′ at y_0 must different from $g(y_0)$ and we have the assertion.

Corollary. *In (3), g is determined uniquely by f.*

Definition. *For a function f of M at x, we set*

(4) $$d_\rho f(x,\ y)=\lim_{t\to 0}.\frac{1}{t}(f(r_{x,y,t})-f(x)).$$

Definition. *A function f on M is called $C(S^{n-1})$-differentiable on M if f is $C(S^{n-1})$-differentiable at any point of M.*

By definition, a $C(S^{n-1})$-differentiable function f at x is continuous at x and teh set (of germes) of $C(S^{n-1})$-differentiable functions at x form a ring. Hence a $C(S^{n-1})$-differentiable function M on is continuous on M and the set of $C(S^{n-1})$-differentiable functions on M form a ring.

***Lemma* 2.** *If f is $C(S^{n-1})$-differentiable on M, then to set*

$$||d_\rho f||(x)=\max_{y\in S_x}.\,|d_\rho f(x,\ y)|, \tag{5}$$

$||d_\rho f||(x)$ is locally bounded as a function of x.

Proof. If $||d_\rho f||(x)$ is not locally bounded, then there is a compact set K of M and a series $\{(x_m,\ y_m)|x_m\in K,\ y_m\in S_{x_m}\}$ such that

$$\lim_{m\to\infty}.\,|d_\rho f(x_m,\ y_m)|=\infty.$$

Since K is compact, we may assume $\lim._{m\to\infty}x_m=x$ exists.

For x_m, we set

$$x_m{}'=r_{x_m,\ y_m,\ |d_\rho f(x_m,y_m)|^{-1/2}}$$

Then, since $\lim._{m\to\infty}|d_\rho f(x_m,\ y_m)|=\infty$, we have $\lim._{m\to\infty}x_m{}'=x$ and we also have

$$\lim_{m\to\infty}.\,|d_\rho f(x_m,\ \varepsilon_{x_m,x_m'})\rho(x_m,\ x_m{}'|=\infty.$$

But this is a contradiction. Because f is continuous and we have by (3)

$$\begin{aligned} &d_\rho f(x_m,\ \varepsilon_{x_m,\ x_m'})\rho(x_m,\ x_m{}') \\ &=f(x_m)-f(x_m{}')+o(\rho(x_m,\ x_m{}')). \end{aligned} \tag{3$'$}$$

2. If $M=\boldsymbol{R}^n$, the n-dimensional euclidean space and ρ is the euclidean metric of $\boldsymbol{R}^n$, then a tatally differentiable function f on $\boldsymbol{R}^n$ is $C(S^{n-1})$-differntiable on $\boldsymbol{R}^n$ and we have

$$d_\rho f(x,\ y)=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(x)y_i=(div.f,\ y). \tag{6}$$

$$y=(y_1,\ \cdots,\ y_n),\quad ||y||=\sum_{i=1}^n y_i^2=1.$$

Conversly, if f is $C(S^{n-1})$-differentiable on some neighbourhood of x and $d_\rho f(x,\ y)$ is continuous at x, then f is tatally differentiable at x (cf. [8], [9], [16]).

To show this, first we note that if f is $C(S^{n-1})$-differentiable at x_0 and x_1, then we get

$$d_\rho f(x_0,\ \varepsilon_{x_0,x_1})\rho(x_0,\ x_1)=f(x_1)-f(x_0)+o(\rho(x_0,\ x_1)),$$

$$d\rho f(x_1,\ \varepsilon_{x_1,x_0})\rho(x_1,\ x_0)=f(x_0)-f(x_1)+o(\rho(x_1,\ x^0)).$$

But since we have $\varepsilon_{x_1,x_0}=y$ if $\varepsilon_{x_0,x_1}=\check{y}$, where $\check{y}$ is the antipodal point of y, if $d_\rho f$

is continuous at x, then we get

$$d_\rho f(x\,\check{y})=-d_\rho f(x,\ y). \tag{7}$$

Especially, if $M=\boldsymbol{R}^n$, we get $d_\rho f(x,\ -y)=-d_\rho f(x,\ y)$ if $d_\rho f$ is continuous at x. Hence f is differentiable along any line which pass x. Therefore, fixing a coordinate system $(x_1,\ \cdots,\ x_n)$ of $\boldsymbol{R}^n$, $\partial f/\partial x_i(x)$, $i=1,\ \cdots,\ n$, exists. Then, since $d_\rho f(x,\ y)$ is the derivative of f along the line ty, we obtain

$$d_\rho f(x,\ y)=\sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(x)y_i,$$

if $d_\rho f$ is continuous at x. Hence to set $z=x+ty$, we have by (3),

$$f(z)=f(x)+(div.\,f,\ z-x)+o(||z-x||).$$

Therefore f is tatally differentiable at x.

We note that there exists $C(S^{n-1})$-differentiable function f on $\boldsymbol{R}^n$ such that $d_\rho f(x,\ y)$ is discontinuous at any point of $\boldsymbol{R}^n$. For example, taking a countable set of points $\{a_m\}$ of $\boldsymbol{R}^n$ which is dense in $\boldsymbol{R}^n$, to define a function f on $\boldsymbol{R}^n$ by

$$f(x)=\sum_m\frac{1}{2^m}\frac{1}{1+||a_m||}||x-a_m||,$$

we have

$$d_\rho f(x,\ y)=\sum_m\frac{1}{2^m}\frac{1}{1+||a_m||}\frac{(x-a_m,\ y)}{||x-a_m||},\quad x\notin\{a_m,\}$$

$$d_\rho f(a_m,\ y)=\frac{1}{2^m(1+||a_m||)}+\sum_{k\neq m}\frac{1}{2^k}\frac{1}{1+||a_k||}\frac{(x-a_k,\ y)}{||x-a_k||}.$$

Hence $d_\rho f(x,\ y)$ is not continuous at any point of $\boldsymbol{R}^n$. Moreover, since

$$\begin{aligned}&f(z)-d_\rho f(x,\ y)||z-x||\\&=\sum_m\frac{1}{2^m}\frac{1}{1+||a_m||}\frac{||z-a_m||\,||x-a_m||-(z-a_m,\ x-a_m)}{||x-a_m||},\quad x\notin\{a_m\},\\&=\sum_{k\neq m}\frac{1}{2^k}\frac{1}{1+||a_k||}\frac{||z-a_k||\,||x-a_k||-(z-a_k,\ x-a_k)}{||x-a_k||},\quad x=a_m,\end{aligned}$$

f is $C(S^{n-1})$-differentiable at any point of $\boldsymbol{R}^n$. Because since $(||x||\,||a||-(a,\ x))/||a||\,||x-a||$ is bounded on $\boldsymbol{R}^n$ and $lim._{x\to a}(||x||\,||a||-(a,\ x))/||a||\,||x-a||=0$, for any $\varepsilon>0$, there exists an integer m_0 and $\delta>0$ such that

$$\sum_{m\geqq m_0}\frac{1}{2^m}\frac{1}{1+||a_m||}\left|\frac{||z-a_m||\,||x-a_m||-(z-a_m,\ x-a_m)}{||x-a_m||\,||z-x||}\right|<\frac{\varepsilon}{2},$$

$$\left|\frac{||z-a_k||\,||x-a_k||-(z-a_k,\ x-a_k)}{||x-a_k||\,||z-x||}\right|<\frac{\varepsilon}{2},\ \ if\,||z-x||<\delta,$$

$$for\ k=1,\ \cdots,\ m_0-1.$$

Then we get

$$\left|\frac{f(z)-d_\rho f(x,\ y)||z-x||}{||z-x||}\right|<\varepsilon,\ \ if\,||z-x||<\delta.$$

This shows $f(z)-d_\rho f(x,\ y)||z-x||=o(||z-x||)$ and we have our assertion (cf. [10]).

3. ***Definition.*** *A function f on M at x is called $C(S^{n-1})$-analytic at x if there exists a system of continuous functions $\{g_n(y)\}$ of S_x such that*

$$f(z)=f(x)+\sum_{m\geqq 1} g_m(\varepsilon_{x,z})\rho(x,\ z))^m, \tag{8}$$

if $\rho(x,\ z)<\varepsilon$ for some $\varepsilon>0$.

We note that since

$$g_k(y)=\lim_{t\to 0}.\frac{1}{t^k}\left\{f(r_{x,y,t})-(f(x)+\sum_{k-1}^{m=1} g_m(y)t^m)\right\}.$$

in (8), $g_k(y)$ is determined uniquely by f for all k.

Definition. *A function f of M is called $C(S^{n-1})$-analytic on M if it is $C(S^{n-1})$-analytic at any point of M.*

By definition, a $C(S^{n-1})$-analytic function is $C(S^{n-1})$-differentiable and the set of $C(S^{n-1})$-analytic functions on M form a ring.

We set $d_\rho f(x,\ y)=d_{\rho,1}f(x,\ y)$ and define

$$d_{\rho,k}f(x,\ y)=\lim_{t\to 0}.\frac{1}{t^k}\left\{f(r_{x,y,t})-(f(x)+\sum_{m=1}^{k-1} d_{\rho,m}f(x,y)t^m)\right\}, \tag{9}$$

Then, similarly as in lemma 2, we obtain

Lemma 2′. *If f is $C(S^{n-1})$-analytic on M, then to set*

$$||d_{\rho,k}f||(x)=\max_{y\in S_x}.\,|d_{\rho,k}f(x,\ y)|, \tag{5$'$}$$

$||d_{\rho,k}f||(x)$ is locally bounded as a function of x for any k.

If $M=\boldsymbol{R}^n$ and ρ is the euclidean metric of $\boldsymbol{R}^n$, then a real analytic function f of $\boldsymbol{R}^n$ at x is $C(S^{n-1})$-analytic at x and we get

(6)' $$d_{\rho,k}f(x,\ y)=\sum_{i_1+\cdots+i_n=k}\frac{1}{i_1!\,i_n!}\frac{\partial^k f}{\partial x_1^{i_1}\partial x_n^{i_n}}(x)y_1^{i_1}\cdots y_n^{i_n},$$
$$y=(y_1,\ \cdots,\ y_n).$$

On the other hand, although the metric function $f(x)=||x-a||$ is not real analytic at $x=a$, it is $C(S^{n-1})$-analytic at $x=a$ and therefore $f(x)$ is $C(S^{n-1})$-analytic on $\boldsymbol{R}^n$.

Note. If $d_{\rho,k}f(x,\ y)$ is sufficiently regular in x, then $d_{\rho,k}f$ is calculated as follows: Set $d_\rho{}^1 f(x,\ y_1)=d_\rho f(x,\ y_1)$ and define

(10) $$d_\rho{}^k f(x,\ y_1,\ \cdots,\ y_k)$$
$$=\lim_{t\to 0}.\frac{1}{t}\{d_\rho{}^{k-1}f(r_{x,y_k,t},\ \cdots,\ y_{k-1})-d_\rho{}^{k-1}f(x,\ y_1,\ \cdots,\ y_{k-1})\},$$

then we have

(11) $$d_{\rho,k}f(x,\ y)=\frac{1}{k!}d_\rho{}^k f(x,\ y,\ \cdots,\ y).$$

In fact, (11) is true for $k=1$ and assuming (11) is true for $k\leqq m-1$, we get

$$f(r_{x,y,2t})$$
$$=f(x)+\sum_{k=1}^{m}d_{\rho,k}f(x,\ y)+d_{\rho,k}f(r_{x,y,t},\ y)t^k+o(t^m)$$
$$=f(x)+\sum_{k=1}^{m}2^k d_{\rho,k}f(x,\ y)t^k+o(t^m).$$

Hence by inductive assumption, we have

$$2^m d_{\rho,m}f(x,\ y)t^m$$
$$=\left\{\sum_{s=1}^{m-1}d_{\rho,s}(d_{\rho,m-s}f(x,\ y))(y)+2d_{\rho,m}f(x,y)\right\}t^m+o(t^m).$$

Then, since by induction, we obtain

$$d_{\rho,s}(d_{\rho,m-s}f(x,\ y))(y)=\frac{1}{s!(m-s)!}d_\rho{}^m f(x,\ y,\ \cdots,\ y),$$

we have (11) for $k=m$.

We remark that in this proof, to get (11) for $k=m$, we need not the d_ρ-analyticity of f and the continuity of $d_\rho{}^m f$ in x but it needs the continuity in x of $d_\rho{}^k f$ for $k\leqq m-1$. We also note that although f is d_ρ-analytic, $d_\rho{}^k f(x,\ y_1,\ \cdots,\ y_k)$ may not exist for $k\geqq 2$ in general. For example, the metric function $f(x)=||x||$ does not have $d_\rho{}^2 f(x,\ y_1,\ y_2)$ unless $y_1=y_2$.

Lemma 3. *If the metric function* $f(x)=\rho(a, x)$ *is* $C(S^{n-1})$*-analytic for any* $a\in M$, *then the set of* $C(S^{n-1})$*-analytic functions on* M *is dense in* $C(M)$ *by the compact open topology.*

Proof. Since the constant function is $C(S^{n-1})$-analytic and the ring generated by $\{1, \rho(a, x), a\in M\}$ satisfies the assumption of the theorem of Stone-Weirestrass (cf. [18]), we have the lemma.

§ 2. Generalized vector fields.

4. In $M\times M$, *we set*

$$s(M)=\{(x, y)\,|\,x\in M,\ \rho(x, y)=1\}. \tag{12}$$

$s(M)$ is the tatal space of the associate S^{n-1}-bundle of the tangent microbundle of M. We denote the projection from $S(M)$ to M by π. Then we have $\pi^{-1}(x)=S_x$. In general, we set

$$s^p(M)=\{(x, y_1, \cdots, y_p)\,|\,x\in M,\ \rho(x, y_i)=1,\ \mathrm{i}=1, \cdots, \mathrm{p}\}. \tag{12$'$}$$

The associate $C(S^{n-1})$ and $C(\overbrace{S^{n-1}\times\cdots\times S^{n-1}}^{p})$-bundles of $s(M)$ and $s^p(M)$ are denoted by $C(s(M))$ and $C^p(s(M))$. Then by lemma 2, we have

Lemma 4. *If* f *is* $C(S^{n-1})$*-differentiable on* M, *then* $d_\rho f$ *is a locally bounded cross-section of* $C(s(M))$.

Lemma 4′. *If* $d_\rho{}^p f$ *is defined, then* $d_\rho{}^p f$ *is a locally bounded cross-section of* $C^p(s(M))$.

Lemma 5. *If* $f(x, y_1, \cdots, y_p)$ *is a locally bounded cross-section of* $C^p(s(M))$, *then to set*

$$\begin{aligned}&\tilde{f}(x_0, x_1, \cdots, x_p)\\ &=f(x_0, \varepsilon_{x_0,x_1}, \cdots, \varepsilon_{x_0,x_p})\rho(x_0, x_1)\cdots\rho(x_0, x_p),\quad x_i\in\overline{B}_{x_0},\ i=1, \cdots, p,\end{aligned} \tag{13}$$

f defines an Alexander-Spanier p-cochain of M.

By (3), using the above notation, we have

$$\delta\tilde{f}(x_0, x_1)=d_\rho f(x_0, \varepsilon_{x_0,x_1})+o(\rho(x_0, x_1)). \tag{14}$$

Note. If $f(x, y_1, \cdots, y_p)$ is alternative in $y_1, \cdots, y_p$, that is

$$f(x, y_{\sigma(1)}, \cdots, y_{\sigma(p)})=\mathrm{sgn}(\sigma)f(x, y_1, \cdots, y_p),\quad \sigma\in\mathfrak{S}^p,$$

then, to set

$$\begin{aligned}&A\tilde{f}(x_0, x_1, \cdots, x_p)\\ &=\frac{1}{p+1}\sum_{i=0}^{p}(-1)^i f(x_i, \varepsilon_{x_i,x_0}, \cdots, \varepsilon_{x_i,x_{i-1}}, \varepsilon_{x_i,x_{i+1}}, \cdots, \varepsilon_{x_i,x_p}).\end{aligned} \tag{13$'$}$$

$$\rho(x_i,\ x_0)\cdots\rho(x_i,\ x_{i-1})\rho(x_i,\ x_{i+1})\cdots\rho(x_i,\ \mathrm{x}_p),$$

$A\tilde{f}$ is alternative in $x_0,\ x_1,\ \cdots,\ x_p$. On the other hand, if $f(x,\ y_1,\ \cdots,\ y_p)$ is continuous in $(x,\ y_1,\ \cdots,\ y_p)$, alternative in $y_1,\ \cdots,\ y_p$ and for each i, f satisfies

$$f(x,\ y_1,\ \cdots,\ \check{y}_i,\ \cdots,\ \mathrm{y}_p)=-f(x,\ \mathrm{y}_1,\ \cdots,\ y_i,\ \cdots,\ y_p), \tag{15}$$

where $\check{y}_i$ is the unique point of S_x such that $\rho(y_i,\ y_i)=2$, then

$$\begin{aligned} &f(x_{\rho(0)},\ x_{\rho(1)},\ \cdots,\ x_{\rho(p)}) \\ &=\mathrm{sgn}(\sigma)\hat{f}(x_0,\ x_1,\ \cdots,\ x_p)+o(\rho(x_0,\ x_1)\cdots\rho(x_0,\ x_p)),\quad \sigma\in\mathfrak{S}^{p+1}. \end{aligned} \tag{16}$$

We note that, if $f(x,\ y_1,\ \cdots,\ y_p)$ is alternative in $y_1,\ \cdots,\ y_p$ and satisfies

$$\begin{aligned} &f(x,\ y_1,\ \cdots,\ y_{i-1},\ \varepsilon_{x',x''},\ y_{i+1},\ \cdots,\ y_p)\rho(x',\ x'') \\ &=f(x,\ y_1,\ \cdots,\ y_{i-1},\ \varepsilon_{x,x''},\ y_{i+1},\ \cdots,\ y_p)\rho(x,\ x'') \\ &\qquad -f(x,\ y_1,\ \cdots,\ y_{i-1},\ \varepsilon_{x,x'},\ y_{i+1},\ \cdots,\ y_p)\rho(x,\ x')+o(\rho(x',\ x'')), \end{aligned} \tag{17}$$

then, assuming f is $C(S^{n-1})$-differentiable in x, we have

$$\delta\tilde{f}(x_0,\ x_1,\ \cdots,\ x_{p+1})=\widetilde{d_\rho f}(x_0,\ x_1,\ \cdots,\ x_{p+1})+o(\rho(x_0,\ x_1)\cdots\rho(x_0,\ x_{p+1})). \tag{14$'$}$$

In fact, we have

$$\begin{aligned} &\delta\tilde{f}(x_0,\ x_1,\ \cdots,\ x_{p+1}) \\ &=f(x_1,\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1}) \\ &+\sum_{i=1}^{p+1}(-1)^i f(x_0,\ \varepsilon_{x_0,x_1},\ \cdots,\ \varepsilon_{x_0,x_{i-1}},\ \varepsilon_{x_0,x_{i+1}},\ \cdots,\ \varepsilon_{x_0,x_{p+1}}) \\ &\qquad\qquad \rho(x_0,\ x_1)\cdots\rho(x_0,\ x_{i-1})\rho(x_0,\ x_{i+1})\cdots\rho(x_0,\ x_{p+1}). \end{aligned}$$

Then, since f is $C(S^{n-1})$-differentiable in x, we get

$$\begin{aligned} &f(x_1,\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1}) \\ &=f(x_0,\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1}) \\ &+d_\rho f(x_0,\ \varepsilon_{x_0,x_1},\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_0,\ x_1)\rho(x_1,\ x_2) \\ &\cdots\rho(x_1,\ x_{p+1})+o(\rho(x_0,\ x_1)\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1})). \end{aligned}$$

By this and (17), we have

$$\begin{aligned} &f(x_0,\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1}) \\ &=\sum_{i=1}^{p+1}(-1)^{i-l} f(x_0,\ \varepsilon_{x_0,}x_1,\ \cdots,\ \varepsilon_{x_0,x_{i-1}},\ {}_{x_0,x_{i+1}},\ \cdots,\ \varepsilon_{x_0,x_{p+1}})\rho(x_0,\ x_1) \\ &\cdots\rho(x_0,\ x_{i-1})\rho(x_0,\ x_{i+1})\cdots\rho(x_{0,}\ x_{p+1})+o(\rho(x_0,\ x_1)\cdots\rho(x_0,\ x_{p+1})), \\ &d_\rho f(x_0,\ \varepsilon_{x_0,x_1},\ \varepsilon_{x_1,x_2},\ \cdots,\ \varepsilon_{x_1,x_{p+1}})\rho(x_0,\ x_1)\rho(x_1,\ x_2)\cdots\rho(x_1,\ x_{p+1}) \end{aligned}$$

$$=d_\rho f(x_0, \varepsilon_{x_0,x_1}, \varepsilon_{x_0,x_2}, \cdots, \varepsilon_{x_0,x_{p+1}})\rho(x_0, x_1)\rho(x_0, x_2)$$

$$\cdots\rho(x_0, x_{p+1})+o(\rho(x_0, x_1)\cdots\rho(x_0, x_{p+1})).$$

Hence we obtain (14)$'$.

If M is a Riemannian manifold and ρ is the geodesic distance of the Riemannian metric of M, then $s(M)$ is the associate sphere bundle of the tangent bundle of M. In this case, if φ is a differential form of degree p on M with the local expression

$$\varphi(x)=\sum_{i_1,\cdots,i_p} f_{i_1},\cdots,{}_{i_p}(x)dx_{i_1},\cdots, dx_{i_p},$$

Then denoting the coordinate functions corresponding to $dx_1, \cdots, dx_n$ by $y_{,1}, \cdots, y_{,n}$ and set

$$y_{1,i_1}, \cdots, y_{p,i_p}=\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}^p} \mathrm{sgn}\ (\sigma)y_{\sigma(1),i_1}\cdots y_{\sigma(p),i_p},$$

the function

$$\varphi^{\#}(x, y_1, \cdots, y_p)=\sum_{i_1,\cdots,i_p} f_{i_1,\cdots,i_p}(x)y_{1,i_1}, \cdots, y_{p,i_p},$$

$$y_i=(y_{i,1}, \cdots, y_{i,n}),$$

defines a cross-section of $C^p(s(M))$ and alternative in $y_1, \cdots, y_p$. By definition, $\varphi^{\#}$ satisfies (17). On the other hand, we know

$$d_\rho\varphi^{\#}(x, y_0, y_1, \cdots, y_p)$$

$$=\sum_j\sum_{i_1,\cdots,i_p}\frac{\partial f_{i_1},\cdots,{}_{i_p}}{\partial x_j}(x)y_{0,j}(y_{1,i_1}, \cdots, y_{p,i_p}).$$

Hence $d\varphi$ is the *mod.* $(\rho(x_0, x_1)\cdots\rho(x_0, x_{p+1}))$-reduction of $\delta\ddot{\varphi}^{\#}$.

5. we denote the dual bundles of $C(s(M))$ and $C^p(s(M))$ by $C^*(s(M))$ and $C^{*p}(s(M))$. The fibres of $C^*(s(M))$ and $C^{*p}(s(M))$ are $C^*(S^{n-1})$ and $C^*(\overbrace{S^{n-1}\times\cdots\times S^{n-1}}^{p})$.

Definition. *A cross-section f of $C^*(s(M))$ or $C^{*p}(s(M)))$ is called locally bounded if the function $||f||$ defined by*

$$||f||(x)=||f(x)||, \text{ the norm of } f(x) \text{ in } C^*(S_x)(\text{or in } C^*(S_x\times\times S_x)),$$

is locally bounded.

Definition. *Let N be the carrier of some singular chain of M, then a cross-section ξ of $C^*(s(M))$ (or $C^{*p}(s(M)))$ on N is called weakly continuous at x, $x\in N$, if for any cross-section F of $C(s(M))$ (or $C^p(s(M)))$ which is continuous at x, we have*

$$(18)\qquad <\xi(x), F(x)>$$

$$\doteqdot \lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} \frac{\int_{(N\cap N_\delta(x)-N\cap U_\varepsilon(x))} <\xi(t), F(t)> dV}{\int_{(N\cap U_\delta(x)-N\cap U_\varepsilon(x))} dV}, \quad \delta>\varepsilon.$$

Here dV is the volume element of N determined by ρ and $U_\delta(x)$ (or $U_\varepsilon(x)$) means the δ-neighbourhood (or the ε-neighbourhood) of x.

By definition, if ξ is continuous, then ξ is weakly continuous. But there exists ξ which is weakly continuous but not continuous.

Example. Let M *be* $\boldsymbol{R}^3$ and N is a line in $\boldsymbol{R}^3$ parametrized by t, $t\in\boldsymbol{R}$. Since $s(\boldsymbol{R}^3)=\boldsymbol{R}^3\times S^2$, we have $\pi^{-1}(N)=\boldsymbol{R}^1\times S^2$, where π is the projection from $s(M)$ to M. We consider $S^2=\boldsymbol{R}/\boldsymbol{Z}$. Then the map ξ given by

$$\xi(t)=\delta_{(1/t)}, \quad t\neq 0, \quad \xi(0)=d\theta,$$

where $\delta_{(1/t)}$ is the Dirac measure on S^2 concentrated at $1/t$ *mod.* 1 and $d\theta$ is the standard measure of S^2 with the tatal measure 1. Then f is not continuous at $t=0$ but weakly continuous at $t=0$.

We note that if ξ is weakly continuous at x, then

$$(18)' \qquad \|\xi(x)\| \leqq \lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} \frac{\int_{(N\cap U_\delta(x)-N\cap U_\varepsilon(x))} \|\xi(t)\| dV}{\int_{(N\cap U_\delta(x)-N\cap U_\varepsilon(x))} dV}.$$

Definition. *A locally bounded cross-section ξ of $C^*(s(M))$ is called a generalized vector field (or a $C(S^{n-1})$-vector field) on M.*

We denote the spaces of $C(S^{n-1})$ differentiable functions and locally bounded functions on M *by* $C_{C(S^{n-1})}(M)$ and $M_{loc.}(M)$. Then to set

$$(19) \qquad Xf(x)=<\xi(x), d_\rho f(x)>,$$

X is a linear operator from $C_{C(S^{n-1})}(M)$ *into* $M_{loc.}(M)$ by lemma 2. Moreover, X satisfies

(i). *X is a closed operator regarding $C(M)$ and $M_{loc.}(M)$ to be the topological vector spaces by the compact open topology.*

(ii). *$X(fg)$ is equal to $(Xf)g+f(Xg)$.*

(iii). *$(Xf)(a)$ is equal to 0 if $|f(x)-f(a)|=o(\rho(x, a))$.*

Conversely, by the closed graph theorem, if a linear operator X from $C_{C(S^{n-1})}(M)$ into $M_{loc.}(M)$ satisfies the aboue (i), (ii), (iii), then X is written as the form (19) (cf. [4]). Therefore, we may define

Definition. *A linear operator X from $C_{C(S^{n-1})}(M)$ into $M_{loc.}(M)$ which satisfies the above* (i), (ii), (iii) *is called a generalized vector field (or a $C(S^{n-1})$-vector field) on M.*

In (19), we call $\xi(x)$ to be the representation of X and denote

$$\xi(x)=rep. X.$$

Definition. *If rep. X is continuous, or weakly continuous, then we call X is continuous, or weakly continuous.*

Definition. *If rep. X is positive at X, that is, $\xi(x)$ is a positive measure on S_x, then we call X is positive at x. If X is positive at any point of M, then we call X is positive on M.*

Since a measure $\xi(x)$ on S_x is written uniquely as $\xi^+(x)-\xi^-(x)$, where $\xi^+(x)$ and $\xi^-(x)$ are the positive measures on S_x, we have

$$X=X^+-X^-, \quad rep. X^+=\xi^+(x), \quad rep. X^-=\xi^-(x).$$

Definition. *For X, we set*

$$CAR. X=\bigcup_{x\in M} car. \xi(x), \quad car. X=\pi(CAR. X), \quad \xi(x)=rep. X.$$

6. We assume the manifold structure of M is given by $\{(U, h_U)\}$, where h_U is a homeomorphism from U onto $\boldsymbol{R}^n$. Then we know that the transition function of $s(M)$ is given by $\{g_{UV}(x)\}$ where $g_{UV}(x)$ is given by

$$g_{UV}(x)=h_{U,x}h_{V,x}{}^{-1}|S^{n-1}, \quad S^{n-1} \text{ is the unit sphere in } \boldsymbol{R}^n,$$
$$h_{U,x}(x')=h_U(x')-h_U(x), \quad x, x'\in U,$$

(cf. [1], [3], [4]). Then the transition functions of $C(s(M))$ and $C^*(s(M))$ are given by $\{g_{UV}{}^{\#}(x)\}$ and $\{g_{UV}{}^{\#*}(x)\}$. Here $g_{UV}{}^{\#}(x)$ is the induced map of $g_{UV}(x)$ on $C(S^{n-1})$ and $g_{UV}{}^{\#*}(x)$ is the adjoint map of $g_{UV}{}^{\#}(x)$.

If $\xi(x)$ is a generalized vector field on M, then using local coordinates, we may set

$$\xi(x)=\{\xi_U(x)\}, \quad g_{UV}{}^{\#*}(x)\xi_V(x)=\xi_U(x). \tag{20}$$

In (20), if at $x=x_0$, $\xi_U(x)$ satisfies the expression

$$\xi_U(x)=\xi_U(x_0)+\varphi_U(\varepsilon_{x_0,x})\rho(x_0, x)+o(\rho(x_0, x)), \tag{21}$$

where $\varphi_U(\varepsilon_{x_0,x})$ is a bounded map from S^{n-1} to $C^*(S^{n-1})$, then by (20), we have

$$\begin{aligned}\xi_V(x)=\xi_V(x_0)&+(g_{VU}{}^{\#*}(x_0)\varphi_U(\varepsilon_{x_0,x}))\rho(x_0, x)+o(\rho(x_0, x))\\&+(g_{VU}{}^{\#*}(x)-g_{VU}{}^{\#*}(x_0)(\xi_U(x).\end{aligned}$$

But, since we may assume $g_{UV}(x)$ is $C(S^{n-1})$-differentiable, that is, the components of $g_{UV}(x)$ are all $C(S^{n-1})$-differentiable, and $C_{C(S^{n-2})}(S^{n-1})$ is dence in $C(S^{n-1})$, we may set

$$\begin{aligned}&(g_{VU}{}^{\#*}(x)-g_{VU}{}^{\#*}(x_0))\varphi_U(x)\\&\qquad=d_\rho g_{VU}{}^{\#*}(x_0, \varepsilon_{x_0,x})\xi_U(x)\rho(x_0, x)+o(\rho(x_0, x)).\end{aligned}$$

Hence we have

$$\text{(21)}' \qquad \xi_V(x)=\xi_V(x_0)+\{g_{VU}{}^{\#*}(x_0)\varphi_U(\varepsilon_{x_0,x})+d_\rho g_{VU}{}^{\#*}(x_0, \varepsilon_{x_0,x})\xi_U(x)\}\rho(x_0, x)+o(\rho(x_0, x)).$$

Therefore we may define

Definition. *If a generalized vector field $\xi(x)$ of M is given by* (20) *and it satisfies the expression (21) at x_0, then we call ξ is $C(S^{n-1})$-differentiable at x_0,*

If $\xi(x)$ is $C(S^{n-1})$-differentiable at any point of M, then we call $\xi(x)$ is $C(S^{n-1})$-differentiable on M.

In the rest, we denote $\varphi_U=d_\rho\xi_U$ or $d_\rho\xi$ in (21).

If X and Y are generalized vector fields on M such that $rep.X=\xi(x)$, $rep.Y=\eta(x)$ and η is $C(S^{n-1})$-differentiable, then the composition XY is defined for $C(S^{n-1})$-2-differentiable functions. Using local coordinates, XYf is given by

$$\text{(22)}' \qquad XYf(x)=\langle\xi_U(x), \langle d_\rho\eta_U(x, z), d_\rho f(x)\rangle_y\rangle_z + \langle\xi_U(x), \langle\eta_U(x), d_\rho{}^2 f(x, z)\rangle_y\rangle_z.$$

Hence, if ξ and η are both $C(S^{n-1})$-differentiable, then $[X, y]=XY-YX$ is a generalized vector field on M and (by Fubini's theorem) we have

$$\text{(22)} \qquad rep.[X, Y](x)=\langle\xi_U(x), d_\rho\eta_U(x, y)\rangle_z-\langle\eta_U(x), d_\rho\xi_U(x, y)\rangle_z.$$

We note that although XY is only defined for $C(S^{n-1})$-2-differentiable functions, by (22), we may consider $[X, Y]$ is defined for $C(S^{n-1})$-differentiable functions.

Note. If $g_{UV}(x)$ is $C(S^{n-1})$-∞-differentiable, then we may define $C(S^{n-1})$-∞-differentiable generalized vector field on M and the set of all $C(S^{n-1})$-∞-differentiable generalized vector fields on M form a Lie algebra. Similarly, if $g_{UV}(x)$ is $C(S^{n-1})$-analytic for each (U, V), then $C(S^{n-1})$-analytic vector field on M is defined and the set of all $C(S^{n-1})$-analytic generalized vector fields on M form a Lie algebra. In this case, if ξ and η are expressed at x_0 as

$$\xi(x)=\xi(x_0)+\sum_{m\geqq 1}\xi_m(\varepsilon_{x_0,x})(\rho(x_0, x))^n,$$

$$\eta(x)=\eta(x_0)+\sum_{m\geqq 1}\eta_m(\varepsilon_{x_0,x})(\rho(x_0, x))^m,$$

ξ_i, η_i *are bounded functions from S^{n-1} to $C^*(S^{n-1})$, $i\geqq 1$,*

then $rep.[X, Y](x_0)$ is given by

$$rep.[X, Y](x_0)=\langle\xi(x_0), \eta_1\rangle-\langle\eta(x_0), \xi_1\rangle.$$

7. For a curve γ of M given by α: $I\to M$, $I=[0, 1]$, $\alpha(0)=a$, we set

$$\text{(23)} \qquad \mathfrak{X}_\alpha(f)=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s\frac{1}{t}\{f(\alpha(t))-f(a)\}dt,$$

where f is a $C(S^{n-1})$-differentiable function at a. If $\mathfrak{X}_\alpha(f)$ exists for any $C(S^{n-1})$-function of M at a, then there exists an element $\xi(\alpha)$ of $C^*(S_x)$ such that

$$\mathfrak{X}_\alpha(f)=<\xi(\alpha),\ d_\rho f(a)>,$$

for any f. In this case, we call γ is $C(S^{n-1})$-smooth at a and $\xi(\alpha)$ is called the generalized tangent of γ at a and denote $\xi(\alpha)=\tau_a(\alpha)$.

By [4], if M is a Riemannian manifold, ρ is its geodesic distance and γ is a smooth curve, then γ is $C(S^{n-1})$-smooth at every point and $\tau_a(\alpha)=c(a)\delta_{y(a)}$, where $c(a)$ is a constant and $\delta_{y(a)}$ is the Dirac measure on S_x concentrated at the point $y(a)$. On the other hand, the curve $r\theta=1$ or the graph of $x\sin(1/x)$ with $x>0$ are $C(S^1)$-smooth at the origin.

Similarly, we can define the generalized tangent $\tau_{\alpha(t)}(\alpha)$ of γ at α (t) by

$$(23)' \qquad <\tau_{\alpha(t)}(\alpha),\ d_\rho f(\alpha(t))>$$
$$=\lim_{s\to 0}\frac{1}{s}\lim_{h\to 0}\int_h^s \frac{1}{u}\{f(\alpha(t+u)-f(\alpha(t))\}du,\ \ t<1.$$

By defintion, if γ is $C(S^{n-1})$-smooth at every point, then the map τ_γ defined by

$$\tau_\gamma(\alpha(t))=\tau_{\alpha(t)},$$

gives a cross-section from γ into $C^*(s(M))$. For example, if γ is smooth, then $\tau_\gamma(\alpha(t))=c(t)\delta_{y(t)}$, where $c(t)$ is a continuous function and $y(t)$ is a continuous cross-section from γ to $s(M)$.

***Theorem* 1.** *If τ_γ is defined and the convergence of (23)′ is uniform for $t\leqq s\leqq t+\delta$ for some $\delta>0$, then τ_γ is weakly 1-sided continuous at t, that is, we have at t*

$$(24) \qquad <\tau_{\alpha(t)},(\alpha),\ g>=\lim_{s\to 0}\frac{1}{s}\lim_{h\to 0}\int_h^s <\tau_{\alpha(t+u)}(\alpha),\ g>du,$$
$$s>0,\ \ g\in C(S^{n-1}).$$

***Proof*.** Since the problem is local, we may assume $M=\boldsymbol{R}^n$ and $t=0$, the origin of $\boldsymbol{R}^n$ in (24).

First we note that, if $\alpha(0)=0$ and f is $C(S^{n-1})$-differentiable on some neighborhood of $\alpha(u)$, then

$$(25) \qquad \lim_{r\to 0}\frac{1}{r}\lim_{h\to 0}\int_h^r f(\alpha(t+u)-\alpha(t))dt=f(\alpha(u)),$$

and the convergence is locally uniform in u. Because to set

$$\alpha(t+u)-\alpha(t)=\alpha(u)+\beta(t),$$

we have $\lim_{t\to 0}\beta(t)=0$ and since f is $C(S^{n-1})$-differentiable, we get

$$f(\alpha(t+u)-\alpha(t))$$
$$=f(\alpha(u))+d_p f(\alpha(u),\ \varepsilon_{\alpha(u),\alpha(u)+\beta(t)})||\beta(t)||+o(||\beta(t)||).$$

Then by lemma 2, we obtain for any $\varepsilon>0$,

$$|f(\alpha(t+u)-\alpha(t))-f(\alpha(u))|<\varepsilon,$$

if $t\leqq t_0$ for some $t_0>0$ and this t_0 is independent with u. This shows (25) with its uniformity in u.

To show (24), we assume $g=d_p f(0)$. Then we have

$$<\tau_{\alpha(t)}(\alpha),\ g>$$
$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{1}{u}\{f(\alpha(t+u)-\alpha(t))-f(0)\}du.$$

Therefore for any $\varepsilon_1>0$, we have for some $h_0>0$ and $s_0>0$,

$$|<\tau_{\alpha(t)}(\alpha),\ g>$$
$$-\frac{1}{s}[\int_{h_1}^{s_1}\frac{1}{u}\{f(\alpha(t+u)-\alpha(t))-f(0)\}du]|<\varepsilon_1,\ s_1<s_0,\ h_1<h_0,$$

and we may take these s_0 and h_0 independent with t.

On the other hand, we know that, if $k_1>0$, then

$$\frac{1}{r}\int_{k_1}^{r}\int_{h_1}^{s_1}\frac{1}{u}\{f(\alpha(t+u)-\alpha(t))-f(0)\}dudt$$
$$=\int_{h_1}^{s_1}\frac{1}{u}\left\{\frac{1}{r}\int_{k_1}^{r}\{f(\alpha(t+u)-\alpha(t))-f(0)\}dt\right\}du,$$

and for any $\varepsilon_2>0$, we have by (25),

$$|\{f(\alpha(u))-f(0)\}-\frac{1}{r}\int_{k_1}^{r}\{f(\alpha(t+u)-\alpha(t))-f(0)\}dt|<\varepsilon_2,$$

if $r<r_0$, $k_1<k_0$ for some r_0 and k_0 which are independent with u. Since we may take ε_2 to satisfy $|\varepsilon_2 \log. h_1|<\varepsilon_3$ for any $\varepsilon_3>0$, we have

$$\lim_{r\to 0}.\frac{1}{r}\lim_{k\to 0}.\int_k^s <\tau_{\alpha(t)}(\alpha),\ g>du$$
$$=\lim_{r\to 0}.\frac{1}{r}[\lim_{k\to 0}.\int_k^r \lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{1}{u}\{f(\alpha(t+u)-\alpha(t))-f(0)\}dudt]$$
$$=\lim_{s\to 0}.\frac{1}{s}\lim_{h\to 0}.\int_h^s \frac{1}{u}\{f(\alpha(u))-f(0)\}du$$

$$=\langle \tau_{\alpha(0)},\ g\rangle.$$

This shows the theorem.

§ 3. Some ineqalities.

8. ***Definition.*** *If f is a locally bounded cross-section of $C^p(s(M))$ and γ a singular p-chain of M, then we define the integral $\int_\gamma f$ of f on γ by*

$$\int_\gamma f=\int_\gamma \tilde{f}, \tag{26}$$

where $\int_\gamma \tilde{f}$ is the integral of the Alexander-Spanier cochain $\tilde{f}$ on γ (*cf.* [3]).

By definition, if γ is given by φ: $I^p\to M$, $I^p=\{(a_1, \cdots, a_p)|0\leqq a_i\leqq 1,\ i=1, \cdots, p\}$, then $\int_\gamma f$ is given by

$$\int_\gamma f=\lim_{|a_{J+1_i}-a_J|\to 0}\sum_J f(\varphi(a_J),\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_1})},\ \cdots,\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_p})})$$
$$\rho(\varphi(a_J),\ \varphi(a_{J+1_1}))\cdots\rho(\varphi(a_J),\ \varphi(a_{J+1_p})),$$

$J=(j_1, \cdots, j_p)$, *j_i are the integers and* $0\leqq j_i\leqq m_i$,

$J+1_i=(j_1, \cdots, j_{i-1},\ j_i+1,\ j_{i+1}, \cdots, j_p)$,

$a_J=(a_{j_1}, \cdots, a_{j_p})$, $0=a_{0_i}<a_{1_i}<<a_{m_i-1}<a_{m_i}=1$.

In this case, if φ satisfies

$$\rho(\varphi(a_{J+1_i}),\ \varphi(a_J))\leqq N|a_{j_i+1}-a_{j_i}|, \tag{27}$$

for some $N>0$, then

$$\overline{\int_\gamma}|f|$$
$$=\overline{\lim_{|a_{J+1_i}-a_J|\to 0}}\sum_J |f(\varphi(a_J),\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_1})},\ \cdots,\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_p})})|$$
$$\rho(\varphi(a_J),\ \varphi(a_{J+1_1}))\cdots\rho(\varphi(a_J),\ \varphi(a_{J+1_p})),$$

$$\underline{\int_\gamma}|f|$$
$$=\underline{\lim_{|a_{J+1_i}-a_J|\to 0}}\sum_J |f(\varphi(a_J),\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_1})},\ \cdots,\ \varepsilon_{\varphi(a_J),\ \varphi(a_{J+1_p})})|$$
$$\rho(\varphi(a_J),\ \varphi(a_{J+1_1}))\cdots\rho(\varphi(a_J),\ \varphi(a_{J+1_p})),$$

both exist and if f is continuous, then $\overline{\int_\gamma}|f|=\underline{\int_\gamma}|f|$. Hence $\int_\gamma f$ exists if f is continuous and γ satisfies (27).

We note that, by definition, we have

$$|\int_{\varphi(I^p)} f| \leqq \int_{\varphi(I^p)} |f|. \tag{28}$$

Example. *If* $M=R^n$ *and* ρ *is the euclidean metric,* φ *a smooth map given by* $(\varphi_1, \cdots, \varphi_n)$, then

$$\varepsilon_{\varphi(a_J),\, \varphi(a_{J+1_i})}$$

$$=\frac{1}{\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial x_i}(a_J)\right)^2}}\left(\frac{\partial\varphi_1}{\partial x_i}(a_J),\ \cdots, \frac{\partial\varphi_n}{\partial x_i}(a_J)\right)+o(||\varphi(a_{J+1_i})-\varphi(a_J)||),$$

$$\rho(\varphi(a_J),\ \varphi(a_{J+1_i}))=||\varphi(a_{J+1_i})-\varphi(a_J)||$$

$$=\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial x_i}(a_J)\right)^2}|a_{j_{i+1}}-a_{j_i}|+o(||\varphi(a_{J+1_i})-\varphi(a_J)||).$$

Hence we have

$$\int_{\varphi(I^p)} f \tag{29}$$

$$=\int_{I^p} f(\varphi(t), \frac{1}{\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial t_p}(t)\right)^2}}\left(\frac{\varphi\partial_1}{\partial t_1}(t),\ \cdots,\ \frac{\partial\varphi_n}{\partial t_1}(\mathrm{t})\right), \cdots,$$

$$\frac{1}{\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial t_p}(t)\right)^2}}\left(\frac{\partial\varphi_1}{\partial t_p}(t),\ \cdots, \frac{\partial\varphi_n}{\partial t_p}(t)\right)\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial t_1}(t)\right)^2}$$

$$\sqrt{\sum_k\left(\frac{\partial\varphi_k}{\partial t_p}(t)\right)^2}dt_1\cdots dt_p,$$

where $t=(t_1, \cdots, t_p)$.

Lemma 6. *If a cross-section* f *of* $C^2(s(R^n))=R^n\times S^{n-1}\times S^{n-1}$ *satisfies*

$$|f(x,\ y_1,\ y_2)|\leqq L||x||^k,\quad ||x|| \text{ is the euclidean norm of } x,$$

$$k\geqq-1,$$

for some $L>0$ *and* f *is integrable on* $\varphi(I^2)=\Delta_{x_1,x_1+hx_2}$ *for any* $h>0$, *where* Δ_{x_1,x_1+hx_2} *is the triangle with the vertexes are* 0, x_1 *and* x_1+hx_2, *then we have*

$$\lim_{h\to 0}.\ |\int_{\Delta_{x_1,x_1+hx_2}} f|\leqq\frac{L}{k+2}||x_1||^{k+2}||x_2||. \tag{30}$$

Proof. By the assumption, (28) and (29), we obtain

$$|\int_{\Delta_{x_1, x_1+hx_2}} f| \leqq \int_{\Delta_{x_1, x_1+hx_2}} |f| \leqq$$

$$\leqq \int_0^\theta \int_0^{x_1(1+h||x_2||/||x_1||)} Lr^{k+1} dr d\theta = \frac{1}{k+2}(||x_1||+h||x_2||)^{k+2}\theta,$$

where $\tan(\theta/2)=h/2||x_2||$. Hence we have (30).

9. On M and on $s^p(M)$, we can define the standard measure $m=m(\rho)$ from the metric ρ (cf. [3], [4]).

Definition. *A measurable function f on $s^p(M)$ is called a measurable cross-section of $C^p(s(M))$.*

We note that since f is defined almost everywhere on $s^p(M)$ and $m(\rho)$ is the normal measure, we have

$$m(\rho)(x|m(\rho)((y_1, \cdots, y_p)|f(x, y_1, \cdots, y_p) \text{ is not defined}) \neq 0)=0, \tag{31}$$

if f is measurable.

Lemma 7. *If f is an Alexander-Spanier p-cochain of M such that if $\varphi(I^p)$ is a singular p-simplex of M which satisfies (27), then f is absolutely and uniformly integrable on φ (I^p), then to set*

$$d_\rho{}^\flat f(x, y_1, \cdots, x_p) \tag{32}$$

$$= \lim_{t_1\to 0, \cdots, t_p \to 0} \frac{1}{t_1 \cdots t_p} f(x, r_{x,y_1,t_1}, \cdots, r_{x,y_p,t_p}),$$

$d_\rho{}^\flat f$ is a measurable cross-section of $C^p(s(M))$.

Proof. *By the definition of the integral and (27), we have*

$$|f(x, r_{x,y_1,t_1}, \cdots, r_{x,y_p,t_p})| = o(t_1 \cdots t_p),$$

almost everywhers on $s^p(M)$ and the limit of the right hand side of (32) should be exists almost everywhere no $s^p(M)$. Hence we get the lemma.

Corollary 1. *Under the same assumptions, we have*

$$\int_\gamma f = \int_\gamma d_\rho{}^\flat f, \tag{33$'$}$$

where $\gamma=\sum_i a_i\varphi_i(I^p)$ and each φ_i satisfies (27).

Corollary 2. *If f is a continuous cross-section of $C^p(s(M))$ and δf satisfies the assumption of lemma 7, then*

$$\int_{\partial\gamma} f = \int_\gamma d_\rho{}^\flat(\delta \tilde{f}), \tag{33$''$}$$

where $\gamma=\sum_i a_i\varphi_i(I^{p+1})$ and each φ_i satisfies (27).

In the rest, we set

$$d_\rho\# f = d_\rho{}^\flat(\delta f).$$

Then $(33)''$ is rewritten as

$$(33)\qquad \int_{\partial\gamma} f = \int_\gamma d_\rho\# f.$$

Note. If f is a function, or f is alternative in $y_1, \cdots, y_p$ and satisfies (17), then by $(14)'$, we have

$$d_\rho\# f = d_\rho f.$$

Hence for those f, we get

$$(34)\qquad \int_{\partial\gamma} f = \int_\gamma d_\rho f.$$

Especially, usual Stokes' theorem follows from the Stokes, theorem for the integration of Alexander-Spanier cochains (Theorem 4 of [3], cf. [5], [7]).

On the other hand, although f is bounded, $d_\rho\# f$ does not exist in general. For example, in $\boldsymbol{R}^n$ with the euclidean metric, the constant cross-section c of $C^p(s(R^n))$ defined by $c(x, y_1, \cdots, y_p) = c$, a constant, does not have bounded $d_\rho\# c$.

10. By lemma 6 and corollary 2 of lemma 7, if $M = \boldsymbol{R}^n$, ρ is the euclidean metric of $\boldsymbol{R}^n$ and f is a continuous cross-section of $C^2(s(\boldsymbol{R}^n))$ such that $d_\rho\# f$ exists and

$$(35)'\qquad |d_\rho\# f(x, y)| \leqq L\,||x||^k,\quad k \geqq -1,\quad x \in U(r_{sy}),$$

where r_{sy} means $r_{0,y,s}$, $||y|| = 1$, then we get

$$(36)'\qquad \Big|f(sy, y_1)h - \Big(\int_{r_{sy+hy_1}} f - \int_{r_{sy}} f\Big)\Big|$$

$$\leqq \frac{L}{k+2} s^{k+2} h + o(h),\quad ||y_1|| = 1.$$

Because we have

$$\int_{r_{sy+hy_1}} f - \int_{r_{sy}} f = \int_{\partial\Delta_{sy+hy_1,\, sy}} f - \int_{r_{sy,\, y_1,\, h}} f$$

$$= \int_{sy+hy_1,\, sy} d_\rho\# f - \int_{r_{sy,\, y_1,\, h}} f,$$

and by the definition of the integral and the continuity of f, we also obtain

$$\int_{r_{sy, y_1, h}} f = f(sy, y_1)h + o(h).$$

For general f, first we remark that, by the definition of the integral, we have

$$\int_{r_{sy}} f = \int_0^1 f(tsy, y)sdt,$$

where the right hand side is the usual (Riemannian) integral. Hence, if f is $C(S^{n-1})$-differentiable in x and $C(S^{n-2})$-differentiable in y, then

$$\int_{r_{sy+hy_1}} f - \int_{r_{sy}} f \tag{37}$$

$$= h\int_0^1 [f(tsy, y)(y, y_1) + d_{\rho,y} f(tsy, y, \varepsilon_{y,y_1}) ||y_1 - (y, y_1)y||$$

$$+ d_{\rho,x} f(tsy, y_1, y)\ ts]dt + o(h),$$

where (y, y_1) is the inner product of y and y_1. Because we know

$$||sy + hy_1|| = s + h(y, y_1) + o(h),$$

$$\frac{sy + hy_1}{||sy + hy_1||} = y + \frac{h}{s}(y_1 - (y, y_1)y) + (h).$$

On the other hand, since

$$\lim_{h\to 0}. \frac{1}{h}\int_1^{1+h} f(tsy, y_1)dt = f(sy, y_1),$$

we have

$$f(sy, y_1) = \int_0^1 \{f(tsy, y_1) + d_{\rho,x} f(tsy, y, y_1)ts\}dt, \tag{38}$$

if f is $C(S^{n-1})$-differentiable in x.

Combinning (37) and (38), we obtain

***Lemma* 8.** *If $M = \boldsymbol{R}^n$, ρ is the euclidean metric of $\boldsymbol{R}^n$ and f is a continuous cross-section of $C^2(s(\boldsymbol{R}^n))$ such that $d_\rho \# f$ exists and satisfies (35)′ or f is $C(S^{n-1})$-differentiable in x, $C(S^{n-2})$-differentiable in y and satisfies*

$$|f(tsy, y)(y, y_1) + d_{\rho,y} f(tsy, y, \varepsilon_{y,y_1})||y_1 - (y, y_1)y|| \tag{35''}$$

$$+ d_{\rho,x} f(tsy, y_1, y)ts - (f(tsy, y_1) + d_{\rho,x} f(tsy, y, y_1)ts)|$$

$$\leqq L|ts|^k,\ k \geqq 0,\ 0 \leqq t \leqq 1.$$

Then we have (36)′ if $d_\rho \# f$ satisfies (35)′ and

(36) $$|f(sy,\ y_1)h-\left(\int_{r_{sy+hy_1}} f-\int_{r_{sy}} f\right)|\leqq\frac{L}{k+1}s^k h+o(h),\ \ k\geqq 0,$$

if f satisfies (35)″.

Note 1. There exists f which does not have bounded $d_\rho\#f$ but (35)″ holds. For example, if f is a constant c, then $d_\rho\#f$ does not exist but the left hand side of (35)″ bounds by $2|c|$.

Note 2. If $f(x,\ y)$ is linear in y, then

$$f(x,\ y)(y,\ y_1)+d_{\rho,y}f(x,\ y,\ e_{y,y_1})||y_1-(y,\ y_1)y||=f(x,\ y_1).$$

Therefore (35)″ is rewritten as

$$|d_{\rho,x}f(tsy,\ y_1,\ y)-d_{\rho,x}f(tsy,\ y_1)|\leqq L|ts|^k,\ \ k\geqq 0,\ \ 0\leqq t\leqq 1.$$

Especially, if $f(x,\ y)$ is linear in y and

$$d_{\rho,x}f(x,\ y_1,\ y_2)=d_{\rho,x}f(x,\ y_2,\ y_1),$$

then to set

$$g(x)=\int_{r_{sy}} f,\ \ x=sy,$$

we have $d_\rho g(x,\ y)=f(x,\ y)$.

Note 3. *The right hand side of (35)′ or (35)″ may be replaced by a positive coefficients polynomial* $p(||x||)$ *or* $P(ts)$. *Then (36) takes the form*

(36)″ $$|f(sy,\ y_1)h-\left(\int_{r_{sy+hy_1}} f-\int_{r_{sy}} f\right)|\leqq\int_0^s p(t)dt.$$

By lemma 8, we obtain

Lemma 9. *Let U be a neighborhood of 0, the origin of $\boldsymbol{R}^n$, such that if $sy\in U$, then $r_{sy}\subset U(||y||=1)$, and e a cross-section of $C^2(s(\boldsymbol{R}^n))$ on U, then to set*

(39) $$f(x)=\int_{r_{sy}} e,\ \ x=sy,$$

we have

(i). *If $e(x,\ y)$ is continuous in x, then*

$$d_\rho f(sy,\ y)=e(sy,\ y).$$

(ii). *If $e(x,\ y)$ satisfies either (35)′ or (35)″, then $f(x)$ is $C(S^{n-1})$-differentiable and*

(40) $$|e(x,\ y)-d_\rho f(x,\ y)|\leqq\frac{L}{k+2}||x||^{k+2},\ \ \text{if } e \text{ satisfies (35)}',$$

$$|e(x, y)-d_\rho f(x, y)|\leqq\frac{L}{k+2}||x||^k, \quad \textit{if } e \textit{ satisfies } (35)''.$$

Chapter 2. Local analytic properties of generalized vector fields.

§ 4. Local integration of the equation $Xu=f$.

11. In this §, we consider the equation

$$Xu=f, \tag{41}$$

in local. Here X is a $C(S^{n-1})$-vector field on M. But, since the problem is local, we assume M is $\boldsymbol{R}^n$ and ρ is the euclidean metric of $\boldsymbol{R}^n$. Then, since $s(\boldsymbol{R}^n)=\boldsymbol{R}^n\times S^{n-1}$, we may set

$$Xu(x)=\int_{S^{n-1}}d_\rho u(x, y)d\xi(x)=<\xi(x), d_\rho u(x)>,$$

where $\xi(x)$ is a Radon measure on S^{n-1}.

We regard S^{n-1} to be the unit sphere of $\boldsymbol{R}^n$ and denote the Dirac mesure on S^{n-1} concentrated at $e_i=(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ by δ_i. We note that δ_i is the $C(S^{n-1})$-tangent of the line te_i. The subspace of $C^*(S^{n-1})$ spanned by $\delta_1, \cdots, \delta_n$ is denoted by $i^*(S^{n-1})$. Then $l^*(S^{n-1})$ is considered to be the dual space of $l(S^{n-1})$, the subspace of $C(S^{n-1})$ consisted by the linear functions on S^{n-1}. Then, denoting the annihilator of $l(S^{n-1})$ in $C^*(S^{n-1})$ by $l(S^{n-1})^\perp$, we have

$$C^*(S^{n-1})=l^*(S^{n-1})\oplus l(S^{n-1})^\perp. \tag{42}$$

In (42), we denote the projections from $C^*(S^{n-1})$ to $l^*(S^{n-1})$ and to $l(S^{n-1})^\perp$ by p_1 and p_2.

Definition. *Denoting rep. $X=\xi(x)$, we define the generalized vector fields $D(X)$ and $S(X)$ for X by*

$$rep. D(X)=p_1\xi(x)), \quad rep. S(X)=p_2(\xi(x)).$$

By definition, we have

$$X=D(X)+S(X).$$

***Theorem* 2.** *X is a usual vector field if and only if $X=D(X)$. On the other hand, $X=S(X)$ if and only if $Xu=0$ if u is smooth.*

Proof. We note that if u is smooth, then $d_\rho u(x)$ belongs in $l(S^{n-1})$ for any x. In fact, we get

$$d_\rho u(x, y)=\sum_{i=1}^n\frac{\partial u}{\partial x_i}(x)y_i, \quad y=(y_1, \cdots, y_n).$$

Then, since we may set

$$rep.\,X(x)=\sum_{i=1}^{n}a_i(x)\delta_i,$$

if $X=D(X)$, we have

$$Xu(x)=\sum_{i=1}^{n}a_i(x)\frac{\partial u}{\partial x_i}(x),$$

if $X=D(X)$. On the other hand, if X' is a usual vector field on $\boldsymbol{R}^n$, then we may set $X'=\sum_{i=1}^{n}c_i(x)\partial/\partial x_i$. Hence to correspond X' the generalized vector field $X=\sum_{i=1}^{n}c_i(x)\delta_i$, we have the first assertion. The second assertion follows from the definition.

Corollary. *If u is smooth, then we have*

$$Xu=D(X)u. \tag{43}$$

By (43), if $D(X)\neq 0$, then for continuous f, the equation (41) is reduced to the equation

$$D(X)u=f, \tag*{$(41)_D'$}$$

or, to set

$$<\xi(x),\ y_i>=c_i(x),\ \ \xi(x)=rep.\,X,\ \ y_i=x_i|S^{n-1},\ \ i=1,\ \cdots,\ n,$$

to the equation

$$\sum_{i=1}^{n}c_i(x)\frac{\partial u}{\partial x_i}(x)=f(x). \tag*{$(41)_D$}$$

We note that $(41)_D$ has a solution locally if the vector $(c_1(x),\ \cdots,\ c_n(x))\neq 0$ (cf. note 2 of n°10).

12. ***Lemma 10.*** *If $X=S(X)$ and $\xi(0)\neq 0$, $\xi=rep.\,X$, then there is a $C(S^{n-1})$-differentiable function u of $\boldsymbol{R}^n$ such that*

$$Xu(0)=1,\ \ Xu(x)=0,\ \ x\neq 0. \tag{44}$$

Proof. Since $C^1(S^{n-1})$ is dense in $C(S^{n-1})$, there exists a differentiable fuction $g(y)$ of S^{n-1} such that

$$<\xi(0),\ g>=\int_{S^{n-1}}g(y)d\xi(0)=1.$$

Then to set

$$u(x)=g\left(\frac{x}{||x||}\right)||x||,\ \ x\neq 0,\ \ u(0)=0,$$

$u(x)$ is $C(S^{n-1})$-differentiable and satisfies (44). Because by definition, $u(x)$ is $C(S^{n-1})$-differentiable at $x=0$ and $d_\rho u(0)=g(x)$. On the other hand, since

$$||x+ty||=||x||(1+t\frac{(x,\ y)}{||x||^2}+o(t)),$$

$$\frac{x+ty}{||x+ty||}=\frac{x}{||x||}+\frac{t}{||x||^3}(y||x||^2-x(x,\ y))+o(t),$$

to set $z=(y||x||^2-x(x,\ y))/||y||x||^2-x(x,\ y)||$, we get

$$\begin{aligned}&u(x+ty)\\&=u(x)+\left\{g\left(\frac{x}{||x||}\right)\frac{(x,\ y)}{||x||}+d_\rho g\left(\frac{x}{||x||},\ \varepsilon_{\left(\frac{x}{||x||},z\right)}\right)\cdot\right.\\&\qquad\left.\sqrt{||x||^2-(x,\ y)^2}\right\}t+o(t),\end{aligned}$$

foe $x\neq 0$, because g is $C(S^{n-2})$-differentiable. Hence u is $C(S^{n-2})$-differentiable and we have

$$\begin{aligned}&d_\rho u(x,\ y)\\&=g\left(\frac{x}{||x||}\right)\frac{(x,\ y)}{||x||}+d_\rho g\left(\frac{x}{||x||},\ \varepsilon_{\left(\frac{x}{||x||},z\right)}\right)\sqrt{||x||^2-(x,\ y)^2},\quad x\neq 0.\end{aligned}\tag{45}$$

Then, since g is differentiable, $d_\rho g$ is continuous in $x/||x||$ and therefore $d_\rho u$ is continuous in x if $x\neq 0$. Hence we have (44) by theorem 2, because $X=S(X)$.

Note. If $u_1(x)$ is given by

$$u_1(x)=u(x)v(x),\ v \textit{ is smooth and } v(0)=1,\tag{46}$$

then u_1 also satisfies (44).

***Theorem* 3.** *Let X be $S(X)$ on U, an open set of $\boldsymbol{R}^n$, and set rep. $X=\xi(x)$, $\{x_i\}$ a dense subset of countable points of U, $g_i(y)$ the smooth functions on S^{n-1} such that*

$$<\xi(x_i),\ g_i>=1,\quad |g_i(y)|\leqq A_i,\quad |d_\rho g_i(y,\ z)|\leqq B_i,\tag{47}$$

and $\{c_i\}$ a series of non-zero positive numbers such that

$$\sum_{i=1}^\infty c_i<\infty,\quad \sum_{i=1}^\infty c_i\frac{B_i}{A_i}<\infty.\tag{48$'$}$$

Then, if $f(X)$ is bounded on U and satisfies

$$\sum_{i=1}^\infty A_i|f(x_i)|<\infty,\tag{48}$$

there exists a $C(S^{n-1})$-differentiable function $u(x)$ on U such that

(49) $$Xu(x_i)=f(x_i),\ \ Xu(x)=0,\ \ x\notin\{x_i\}.$$

Proof. We set

$$u_i(x)=g_i\Big(\frac{x-x_i}{||x-x_i||}\Big)||x-x_i||,\ \ x\neq x_i,\ \ u_i(x_i)=0.$$

Then we have by (45) and (47),

(47)′ $$|u_i(x)|\leqq A_i||x-x_i||,\ \ |d_\rho u_i(x,\ y)|\leqq A_i+B_i||x||.$$

Next we set

$$v_i(x)=\frac{c_i}{A_i}\frac{1}{||x-x_i||},\ \ ||x-x_i||\geqq\frac{3}{2}\frac{c_i}{A_i},$$

$$=-\frac{4A_i^2}{27c_i^2}||x-x_i||^2+1,\ \ ||x-x_i||<\frac{3}{2}\frac{c_i}{A_i}.$$

Then $lim._{m\to\infty}\sum_{i=1}^{m}u_i(x)v_i(x)f(x_i)$ exists and to set

$$u(x)=\sum_{i=1}^{\infty}u_i(x)v_i(x)f(x_i),$$

$u(x)$ is $C(S^{n-1})$-differentiable and satisfies (49). Because, we have by the definition of v_i and (47), (48)′,

$$|u_i(x)v_i(x)|\leqq c_i,\ \ x\in \boldsymbol{R}^n,$$

$$|d_\rho(u_iv_i)(x,\ y)|\leqq A_i+c_i\frac{B_i}{A_i},\ \ (x,\ y)\in\boldsymbol{R}^n\times S^{n-1}.$$

Therefore by (48)′ and (48), $u(x)$ and $d_\rho u(x,\ y)$ both exist. Hence by lemma 1, $u(x)$ is $C(S^{n-1})$-differentiable on U and by lemma 10, we get (49).

13. ***Theorem 4.*** *If $X=S(X)$, then Xf is equal to* 0 *almost everywhere on $\boldsymbol{R}^n$ (with respect to the Lebesgue measure).*

Proof. First we note that if u is $C(S^{n-1})$-differentiable and *car.* u is compact, then (considering the integral along the line ty)

(50)′ $$\int_{R^n}d_\rho u(x,\ y)dx=0.$$

Hence for an element T of $C_0(\boldsymbol{R}^n)^*$, the dual space of the space of continuous functions with compact carrier, we can define XT, X is a $C(S^{n-1})$-vector field on $\boldsymbol{R}^n$, to be an element of $C^1{}_0(\boldsymbol{R}^n)^*$, the dual space of the space of C^1-class functions with compact carrier, by

(51) $$XT[u]=-T[Xu],$$

and if $T=T_f$ is given by $T_f[u]=\int_{R^n} f(x)u(x)dx$, then we get by (50)′ and (51),

(50) $$XT_f=T_{Xf}.$$

Then by definition, if $X=S(X)$, XT is equal to 0 as an element of $C^1{}_0(\boldsymbol{R}^n)^*$. But, since $C^1{}_0(\boldsymbol{R}^n)$ is dense in $C_0(\boldsymbol{R}^n)$, we have the theorem by (50).

Corollary 1. *To set*

$$N(\boldsymbol{R}^n)=\{f \mid f \textit{ is locally bounded and } m\ (car.\, f)=0\},$$
$$m \textit{ is the Lebesgue measure,}$$

we have

(52) $$X(C_{C(S^{n+1})}(\boldsymbol{R}^n))\subset N(\boldsymbol{R}^n), \textit{ if } X=S(X).$$

Corollary 2. *If $f(x)$ is $C(S^{n-1})$-differentiable on U, an open set of $\boldsymbol{R}^n$, then $d_\rho f(x)$ belongs in $l(S^{n-1})$ almost everywhere on U.*

We note that by theorem 3, we also obtain

Corollary 1′. *If $X=S(X)$ and to set rep. $X=\xi(x)$, if there exists a function $e(x, y)$ on $U\times S^{n-1}$ such that*

$$<\xi(x),\ e(x)>=1,\ x\in U,$$
$$|e(x, y)|\leqq A,\ |d_{\rho,y}e(x, y, z)|\leqq B,\ x\in U,\ Y\in S^{n-1},\ z\in S^{n-2},$$

then

(52)′ $$X(C_{C(S^{n-1})}(U))\supset l^1{}_{loc.}U).$$

Here $l^1{}_{loc.}(U)$ is given by

$$l^1{}_{loc.}(U)=\{f \mid \sum_{x\in K}|f(x)|<\infty \textit{ for any compact subset } K \textit{ of } U\}.$$

Since $\{x \mid f(x)\neq 0\}\cap K$ is a countable set for any compact subset K of U if $f\in l^1{}_{loc.}(U)$, we have

$$C_{C(S^{n-1})}(U)\cap l^1{}_{loc.}(U)=\{0\}.$$

Hence we can extend X to $l^1{}_{loc.}\ (U)$ to be the 0-map. Then to set

$$C_X(U)=X^{-1}(l^1{}_{loc.}(U))\oplus l^1{}_{loc.}(U),$$

we may consider X is defined on $C_X(U)$. The subspace of $C_X(U)$ constructed by the compact carrier functions is denoted by $C_{X.0}(U)$.

Taking

$$U(f,\ k,\ \varepsilon;\ \varepsilon')$$
$$=\{g \mid |f(x)-g(x)|<\varepsilon,\ x\in K,\ \sum_{x\in K}|X(f-g)(x)|<\varepsilon',$$

K is a compact subset of $U\}$,

to be the neighborhood basis of f, we give a topology of $C_X(U)$ (or $C_{X,0}(U)$). The dual space of $C_{X,0}(U)$ (under this topology) is denoted by $C_{X,0}(U)^*$. Then we define the inclusion map ι from $C_{X,0}(U)^*$ to $L(C_{X,0}(U),\ C_{X,0}(U)^*)$, the space of continuous homomorphisms from $C_{X,0}(U)$ to $C_{X,0}(U)^*$, by

$$\iota(T[f])=T_{T[f]},\ T_g[h]=\int_u g(x)h(x)dx.$$

For an element T of $L(C_{X,}(U),\ C_{X,0}(U)^*)$, we define XT, X is a $C(S^{n-1})$-vector field, by

$$\begin{aligned}((XT)[f])[g]&=(X(T[f]))[g]-(T[Xf])[g]\\&=-(T[f])[Xg]-(T[Xf])[g].\end{aligned} \tag{53}$$

Lemma 11. *The equation*

$$XT=\delta,\quad \delta \text{ is the Dirac measure,} \tag{54}$$

has a solution in $L(C_{X,0}(U),\ C_{X,0}(U)^*)$.

Proof. We define an element δ^2 of $L(C_{X,0}(U),\ C_{X,0}(U)^*)$ by

$$\delta^2\ [f]=f(0)\delta,\ \delta \text{ is the Dirac measure.}$$

Then by (53), taking the function $u(x)$ defined for X by lemma 10, we get (54) to set

$$T=-e^{ux}\delta^2.$$

Here $f(x)T[g]$ is given by $T[fg]$.

By lemma 11, if *car.* f is compact and $X=S(X)$ on $\boldsymbol{R}^n$ such that *rep.* $X(x)\neq 0$ for all x, then we may consider the equation (41) has a solution in $F(R^n,\ C_{X,0}(\boldsymbol{R}^n)^*)$, the space of functions from $\boldsymbol{R}^n$ to $C_{X,0}(\boldsymbol{R}^n)^*$. In fact, by assumption, there exists a function $e(x,\ y)$ on $\boldsymbol{R}^n\times S^{n-1}$ such that $e(x,\ y)$ is smooth in y and $\langle rep.X(x),\ e(x)\rangle =1$ for all x. Then to set

$$U(x,\ \xi)=e\left(x,\frac{\xi}{||\xi||}\right)||\xi||,$$

the solution $u(x)$ of (41) is given by

$$u(x)=e^{U(x,\ \xi)}\delta^2(\xi)[f(x-\xi)].$$

Note. In the next §, we show that a $C(S^{n-1})$-vector field of $\boldsymbol{R}^n$ generates a local 1-parameter group of transformation of $\boldsymbol{R}^n$ into $C^*(S^{n-1})$. Therefore we may consider the equation (41) can be solved in $F(\boldsymbol{R}^n,\ C^*S^{n-1}))$, locally.

§ 5. Generalized integral curves of generalized vector fields on R^n.

14. If $X=S(X)$, we define a transformation T_t of $C_X(R^n)$ by

$$T_t f=f+tXf, \quad t\geqq 0.$$

Then, since $X^2=0$ on $l^1_{loc.}(R^n)$, T_t is a 1-parameter (semi-) group of linear transformations of $C_X(R^n)$ and it is differentiable in t and we have

$$\frac{d}{dt}(T_t f)=XT_t f.$$

But, since T_t is the identity map on $C^1(R^n,)$ T_t does not induce any (non-trivial) transformation of R^n.

On the other hand, if X is positive, then we can construct a family of (continuous) curves $\varphi(t, x)$ of R^n with the parameter x such that

(i). $\varphi(0, x)=x, \quad x\in R^n$.

(ii). *The generalized tangent of* $\varphi(t, x)$ *at* $t=0$ *is rep.* $X(x), \quad x\in R^n$.

(iii). *If* X *is continuous, then* $\varphi(t, x)$ *is continuous in* x.

To construct such family of curves, first we fix a countable set of points $\{y_p\}$ of S^{n-1} such that

(a). $\{y_p\}$ *is dense in* S^{n-1}.

(b). $y_p\neq\pm y_q$ *if* $p\neq q$.

For this $\{y_p\}$, we fix a family of Borel sets $\{E_p{}^q\}$ of S^{n-1} such that

$$y_p\in E_p{}^q, \quad \lim_{q\to\infty} dia.(E_p{}^q)=0,$$

$$S^{n-1}=\bigcup_{p\leqq q} E_p{}^q \text{ for any fixed } q, \quad E_{p'}{}^q\cap E_{p''}{}^q=\phi, \text{ if } p'\neq p''.$$

For these $\{E_p{}^q\}$, we define a series of positive real numbers $\{t_{q,p}(x)\}$, x is the parameter, as follows: First we set *rep.* $X=\xi(x)$ and set

$$\xi(x)(S^{n-1})=v(x).$$

By assumption, $v(x)\geqq 0$ and if X is continuous, then $v(x)$ is continuous. Using $\xi(x)$ and $v(x)$, we set

$$t_{q,1}(x)=\frac{v(x)}{q!},$$

$$t_{q,p}(x)=\frac{v(x)}{(q+1)!}+\frac{q}{(q+1)!}\sum_{r\geqq p}\xi(x)(E_r{}^q), \quad p\leqq q.$$

Since $\sum_{p\in q}\xi(x)(E_p{}^q)=v(x)$, these $t_{q,p}(x)$ are well defined and since $\xi(x)$ is a positive measure, we obtain

$$t_{q,p}(x)\geqq t_{q,p+1}(x), \text{ if } p+1\leqq q, \quad t_{q,q}(x)\geqq t_{q+1,1}(x),$$

$$\lim_{q\to\infty} t_{q,p}(x)=0,\ \lim_{q\to\infty}\frac{t_{q+1,1}(x)}{t_{q,1}(x)}=0,\ \textit{if}\ v(x)\neq 0,$$

$$t_{q,p}(x) \textit{ is continuous in } x \textit{ if } X \textit{ is continuous in } x.$$

Then we define a function $\Psi(t, x)$: $\boldsymbol{R}^+\to\boldsymbol{R}^n$, $\boldsymbol{R}^+=\{t|\ t\geqq 0\}$, with the parameter x, $x\in\boldsymbol{R}^n$, by

$$\Psi(t, x)=0,\ \textit{if}\ v(x)=0,\ \Psi(0, x)=0.$$

$$\Psi(t_{q,p}(x), x)=t_{p,p}(x)y_p,\ \textit{if}\ t_{q,p}(x)\neq t_{q,p+1}(x).$$

$$\Psi(t_{q,q}(x), x)=t_{q,q}(x)y_q,\ \textit{if}\ t_{q,q}(x)\neq t_{q+1,1}(x).$$

$$\Psi(t, x)=\frac{t_{q,p}(x)-t}{t_{q,p}(x)-t_{q,p+r}(x)}\Psi(t_{q,p+r}(x), x)+\frac{t-t_{q,p+r}(x)}{t_{q,p}(x)-t_{q,p+r}(x)}\Psi(_{q,p}(x), x),$$

$$\textit{if}\ t_{q,p}(x)=t_{q,p+1}(x)=\cdots=t_{q,p+r-1}(x),\ t_{q,p}(x)>t>t_{q,p+r}(x).$$

In this last formula, we consider $t_{q,p+r}(x)=t_{q+1,\ p+r-q}(x)$ if $p+r>q$.

By definition, if X is continuous in x, then $\Psi(t, x)$ is continuous in x and we have

$$||\Psi(t, x)||\leqq v(x)|t|,$$

$$\Psi(t, x)\neq 0\ \textit{if}\ v(x)\neq 0\ \textit{and}\ t\neq 0.$$

Using this $\Psi(t x)$, we define $\varphi(t, x)$ by

$$\varphi(t, x)=x+v(x)\frac{\Psi(t, x)}{||\Psi(t, x)||},\ t\neq 0,\ v(x)\neq 0,$$

$$\Psi(t, x)=x,\ \textit{for all } t \textit{ it } v(x)=0,$$

$$\Psi(0, x)=x.$$

Then by definition, $\Psi(t, x)$ is continuous in t and it satisfies (i), (ii), (iii) (cf. [4]).

By (ii), to set

$$S_t(x)=\varphi(t, x), \tag{55}$$

S_t maps $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ and we have

$$\lim_{s\to 0}\frac{1}{s}\lim_{h\to 0}\int_h^s\frac{S_t{}^*f(x)-f(x)}{t}dt=Xf(x), \tag{56}$$

if f is $C(S^{n-1})$-differentiable on $\boldsymbol{R}^n$.

By (i), S_0 is the identity map of $\boldsymbol{R}^n$ and by (iii), if X is continuous, then S_t is a continuous map. Therefore, if X is continuous, then $S_t{}^*$ maps $C(\boldsymbol{R}^n)$ into $C(\boldsymbol{R}^n)$ and it is a bounded operator as a map from $C_b(\boldsymbol{R}^n)$, the Banach space of bounded continuous functions on $\boldsymbol{R}^n$, into $C_b(\boldsymbol{R}^n)$.

15. We call $\varphi(t)$ to be the integral curve of X starts from the origin of $\boldsymbol{R}^n$ in the weak sense if it satisfies

$$\lim_{t\to 0} \frac{1}{s} \lim_{h\to 0} \int_h^s \frac{f(\varphi(t+r))-f(\varphi(t))}{r} dr = <\xi(\varphi(t)),\ d_p f(\varphi(t))>,$$

for any $C(S^{n-1})$-differentiable function f of $\boldsymbol{R}^n$, where $\xi(x)$ is *rep.* X. We note that, if $\lim_{r\to 0}(\varphi(t+r)-\varphi(t))/r$ exists and f is differentiable, then the above formula reduces to

$$\frac{d(f(\varphi(t))}{dt} = Xf(\varphi(t)).$$

If $\varphi(t)$ is the integral curve of X in the weak sence starts from the origin, then we have

$$\int_0^t \frac{1}{s} \int_h^s \frac{f(\varphi(k+r))-f(\varphi(k))}{r} drdk$$

$$= \frac{1}{s} \int_h^s \int_0^t \frac{f(\varphi k+r))-f(\varphi(k))}{r} dkdr$$

$$= \frac{1}{s} \int_h^s [f(\varphi(t+\theta r))-f(\varphi(\theta' r))] dr$$

$$= \frac{s-h}{s} [f(\varphi(t+\mu s))-f(\varphi(\mu' s))],\quad 0<\theta, \theta'<1,\quad 0<\mu, \mu'<1,$$

by the mean value theorem. Hence we get

$$f(\varphi(t)) = f(0) + \int_0^t <\xi(\varphi(\mathrm{t})),\ d_p f(\varphi(t))> dt, \tag{57}$$

if φ is an integral curve of X in the weak sence starts from the origin.

By (57), if $X=S(X)$, then for any smooth f, we have $f(\varphi(t))=f(0)$. But, since $C^1(\boldsymbol{R}^n)$ is dense in $C(\boldsymbol{R}^n)$, it occurs only the case $\varphi(t)=0$ for all t. Therefore we obtain

Theorem 5. *If $X=S(X)$, then X has no integral curve although in weak sense.*

Note. In $\boldsymbol{R}^2$, the generalized vector field X given by

$$X(x,\ y)$$

$$= \left(\frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\right)\delta_1 + \left(\frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\right)\delta_2,\quad (x,\ y) \neq (0,\ 0),$$

$$X(0,\ 0) = \frac{1}{2\pi} d\theta,$$

where δ_1 and δ_2 are the Dirac measures concentrated at (1, 0) and (0, 1) and $d\theta$ is the standard measure of S^1, is weakly continuous on $\boldsymbol{R}^2$ but $X(0, 0)=S(X)(0, 0)$. This X has the integral curve starts from $(0, 0)$ which is given by

$$x(t)=t\cos\left(\frac{1}{t}+c\right),\quad y(t)=t\sin\left(\frac{1}{t}+c\right),\quad t>0,$$

$$x(0)=y(0)=0,\quad 0\leqq c<2\pi.$$

16. Since $l^*(S^{n-1})=\boldsymbol{R}^n$, we consider $\boldsymbol{R}^n$ to be a subspace of $C^*(S^{n-1})$ in this manner. Then we can extend a function f or a generalized vector field X of $\boldsymbol{R}^n$ to a function $f^\#$ of $C^*(S^{n-1})$ or a map $X^\#$: $C^*(S^{n-1})\longrightarrow C^*(S^{n-1})$ by

$$f^\#(\xi)=f(p_1(\xi)),\quad X^\#(\xi)=(rep.\,X)(p_1(\xi)). \tag{58}$$

We note that since we get

$$p_1(\xi)=\sum_{i=1}^{n}\xi(y_i)\delta_i,\quad y_i=x_i\,|\,S^{n-1},$$

we obtain

$$||p_1(\xi)||\leqq n||\xi||.$$

Therefore, if f is $C(S^{n-1})$-differentiable, then we get

$$f^\#(\xi+t\eta)=f^\#(\xi)+d_p f\Big(p_1(\xi),\frac{p_1(\eta)}{||p_1(\eta)||}\Big)||p_1(\eta)||t+o(t). \tag{59}$$

Hence $f^\#$ is (1-sidede) Gâteaux differentiable (with respect to the real numbers) and it is Fréchet differentiable if and only if f is tataly differentiable. Because we may consider $C(S^{n-1})\subset C^{**}(S^{n-1})$ and $\boldsymbol{R}^n\cong l(S^{n-1})\subset C(S^{n-1})$.

On $C^*(S^{n-1})$, similarly as usual ordinary differential equation, we have

***Lemma* 12.** *If Y is a map from $C^*(S^{n-1})$ to $C^*(S^{n-1})$ such that*

$$||Y(\xi)-Y(\eta)||\leqq L||\xi-\eta||, \tag{60}$$

for some positive number L, then for any $\xi\in C^*(S^{n-1})$, there exists unique function $\varphi(t)$, $t\in\boldsymbol{R}$ with values in $C^*(S^{n-1})$ such that

$$\frac{d\varphi(t)}{dt}=Y(\varphi(t)),\quad \varphi(0)=\xi. \tag{61}$$

Moreover, if the value of Y all belongs in $l^*(S^{n-1})$ and $\varphi(0)$ also belongs in $l^*(S^{n-1})$, then the value of $\varphi(t)$ also belongs in $l^*(S^{n-1})$.

Note. If a $C(S^{n-1})$-vector field X satisfies the Lipschitz condition

$$||\xi(x)-\xi(x')||\leqq L'||x-x'||,\quad \xi(x)=rep.\,X, \tag{62$'$}$$

then we have

$$||X^{\#}(\eta)-X^{\#}(\xi)||\leqq n^2L'||\eta-\xi||.$$

If $\varphi(t)$ is a solution of the equation

$$\frac{d\varphi(t)}{dt}=X^{\#}(\varphi(t)),$$

and f is a $C(S^{n-1})$-differentiable function of $\boldsymbol{R}^n$, then by (59), we get

$$\frac{d}{dt}(f^{\#}(\varphi(t))=<X^{\#}(t)),\quad (d_\rho f)^{\#}(\varphi(t))>. \tag{63)'$$

Especially, if f is smooth, then

$$\frac{d}{dt}(f^{\#}(\varphi(t))=<X^{\#}(t)),\quad d(f^{\#})(\varphi(t))>, \tag{63}$$

where $d(f^{\#})$ is the Fréchet differential of $f^{\#}$ and it is considered to be an element of $C^{**}(S^{n-1})$.

By (63) and (63)', we may define

Definition. *A curve $\varphi(t)$ in $C^*(S^{n-1})$ is called the generalized integral curve of a generalized vector field X starts from x if it satisfies*

$$\frac{d\varphi(t)}{dt}=X^{\#}(\varphi(t)),\quad \varphi(0)=\iota(x),\quad x\in\boldsymbol{R}^n. \tag{64}$$

Here $\iota(x)$ means

$$\iota(x)=\sum_{i=1}^{n}x_i\delta_i,\quad x=(x_1,\ \cdots,\ x_n).$$

Then by lemma 12, we have

Theorem 6. *If X is a generalized vector field on $\boldsymbol{R}^n$ such that rep. $X=\xi(x)$ satisfies the Lipschitz condition (62)' for some positive L' on $\boldsymbol{R}^n$, then X has a generalized integral curve starts at any point of $\boldsymbol{R}^n$. Moreover, if $X=D(X)$, then we may consider the generalized integral curve of X to be the usual integral curve of X.*

Corollary. *Under the same assumptions, X generates a local 1-parameter group $\{T_t\}$ of transformations of $C^*(S^{n-1})$. If $X=D(X)$, then this group is identified the usual local 1-parameter group of transformations of $\boldsymbol{R}^n$ generated by X.*

Note 1. In general, if $\varphi(t)$ is the generalized integral curve of X starts from x, then to set

$$\Psi(t)=\iota^{-1}(p_1(\varphi t))),$$

$\Psi(t)$ is the usual integral curve of $D(X)$ starts from x. Especially, if $X=S(X)$, then $\Psi(t)=x$ for all x.

Note 2. Denoting the integral curve of $D(X)$ (*in* $C^*(S^{n-1})$) by $\varphi_1(t)$, the solution $\varphi(t)$ of the equation (64) takes the form

$$\varphi(t)=(\varphi_1(t), \int_0^t p_2(\xi)(\varphi_1(s)ds). \tag{65}$$

By (65), if $X=S(X)$, then the generalized integral curve of X starts from x is given by

$$\varphi(t)=(x, \xi(x)t). \tag{65$'$}$$

By (65), we obtain

Theorem 6'. *A $C(S^{n-1})$-vector field X on U, a neighborhood of x, has the generalized integral curve starts from x if $D(X)$ has the integral curve (in the usual sence) starts from x and $S(X)$ is integrable.*

By (65)$'$, if $X=S(X)$, then the 1-parameter group $\{T_t\}$ generated by X is given by

$$T_t(x)=(x, \xi(x)t), \quad rep. \, X=\xi(x).$$

On the other hand, if $X=D(X)$, then the generalized integral curve of X is given by

$$\varphi(t)=(\varphi_1(t), p_2(\eta)), \tag{65$''$}$$

if it starts from $\eta\in C^*(S^{n-1})$. Here $\varphi_1(t)$ is the ι-image of the usual integral curve of $D(X)$ starts from $\iota^{-1}(p_1(\eta))$. By (65)$''$, the 1-parameter group T_t of transformations of $C^*(S^{n-1})$ generated by X takes the form

$$T_t(\eta)=(T_t'(p_1(\eta)), p_2(\eta)),$$

where T_t' is the (ι-image of) the usual 1-parameter group generated by X ($=D(X)$).

Chapter 3. Generalized vector fields on manifolds.

§ 6. $C(S^{n-1})$-smooth functions on manifolds.

17. ***Definition.*** *If f is $C(S^{n-1})$-differentiable on M and $d_\rho f$ defines a continuous cross-section of $C(s(M))$, then we call f to be $C(S^{n-1})$–1-smooth on M.*

Similarly, if f is $C(S^{n-1})$-differentiable on some neighborhood of $x\in M$ and $d_\rho f$ is continuous at x, then we call f is $C(S^{n-1})$–1-smooth at x.

Definition. *If $d_\rho{}^p f$ is defined on M and it defines a continuous cross-section of $C^p(s(M))$, then we call f to be $C(S^{n-1})$-p-smooth. If f is $C(S^{n-1})$-p-smooth for all p, then we call f to be $C(S^{n-1})$-∞-smooth or $C(S^{n-1})$-smooth.*

$C(S^{n-1})$-p-smooth at x or $C(S^{n-1})$-smooth at x are also defined similarly.

In $C(S_x)$, we set

(66) $$l(S_x)=\{d_\rho f(x)\,|\, f \text{ is } C(S^{n-1})\text{-smooth on some neighborhood of } x\}.$$

Note. Similarly, we may set

(66) $$l_k(S_x)=\{d_\rho f(x)\,|\, f \text{ is } C(S^{n-1})\text{-}k\text{-smooth on some neighborhood of } x\},$$

for each k. Starting from these $l_k(S_x)$, we have same results as in this chapter.

***Lemma* 13.** *If $dim. M=n$, then*

(67) $$dim.\, l(S_x)\leqq n.$$

Proof. If f is $C(S^{n-1})$-smooth at x, then to define φ_y: $(-1, 1)\to M$ by

$$\varphi_y(t)=r_{x,y,t},\ \ t\geqq 0,\ \ \varphi_y(t)=r_{x,\check{y},-t},\ \ t<0,$$

$f(\varphi_y(t))$ is differentiable at $t=0$ and the generalized tangent of $\varphi_y(t)$ at $t=0$ is δ_y. But, since $\pi^{-1}(U)=S^{n-1}\times U$ and a paracompact topological manifold always has a topological connection and a topological connection can be considered to be a local parallel displacement ([1]), we may set

$$f(\varphi_y(t))=f(\Psi_1(\varphi_{y_1}(t)), \cdots, \Psi_n(\varphi_{y_n}(t))),\ \ -\varepsilon<t<\varepsilon,$$

where $y_1, \cdots, y_n$ are suitable points of S_x, because $dim. M=n$. Then, since $\varphi_y(t)$ and $\varphi_{y_1}(t), \cdots, \varphi_{y_n}(t)$ are smooth in t, we have

(68) $$d_\rho f(x,\ y)=\sum_{i=1}^{n}\left(\frac{d\Psi_i}{dt}(0)\right)^{-1} d_\rho f(x,\ y_i).$$

Hence we have the lemma.

Note. This lemma is hold for $l'(S_x)=\{d_\rho f(x)\,|\ f \text{ is } C(S^{n-1})\text{-smooth at } x\}$.

Lemma 14. *For any $x\in M$, there exists $\varepsilon<0$ such that*

(69) $$dim.\, l(S_{x_1})\geqq dim.\, l(S_x),\ \ if\ \rho(x_1,\ x)<\varepsilon.$$

Proof. Since $dim.\, l(S_x)<\infty$ by lemma 13, we may take $C(S^{n-1})$-smooth functions $f_1, \cdots, f_m$ on some neighborhood of x such that $d_\rho f_1(x), \cdots, d_\rho f_m(x)$ form the basis of $l(S_x)$. Then, since $d_\rho f_1, \cdots, d_\rho f_m$ are continuous in x on some neighborhood of x, if $\rho(x, x_1)<\varepsilon$, then $d_\rho f_1(x_1), \cdots, d_\rho f_m(x_1)$ are linear independent in $C(S_{x_1})$ for some $\varepsilon>0$. Since $d_\rho f_1(x_1), \cdots, d_\rho f_m(x_1)$ belongs in $l(S_{x_1})$, this means

$$dim.\, l(S_{x_1})\geqq m=dim.\, l(S_x).$$

Hence we have the lemma.

Corollary. *If $dim.\, l(S_x)=n(=dim. M)$ and $d_\rho f_1(x), \cdots, d_\rho f_n(x)$ form the basis of $l(S_x)$, then for some $\varepsilon>0$, if $\rho(x, x_1)<\varepsilon$,* then *$d_\rho f_1(x_1), \cdots, d_\rho f_m(x_1)$ form the basis of $l(S_x)$.*

Proof. By the above proof, $d_\rho f_1(x_1), \cdots, d_\rho f_m(x_1)$ are linear independent in $l(S_{x_1})$ if $\rho(x, x_1)<\varepsilon$ for sufficiently small ε. But since $dim.\, l(S_x)\leqq n$ by lemma 13,

$d_\rho f_1(x_1), \cdots, d_\rho f_n(x_1)$ should be the basis of $l(S_x)$.

In the rest, we set

$$dim.\, l(S_x)=l_x,$$
$$\{x \mid l_x=n\}=M_{s,p}.$$

By the corollary of lemma 14, $M_{s,\rho}$ is an openset of M.

18. ***Lemma 15.*** *To set*

$$T(M_{s,\rho})=\bigcup_{x\in M_{s,\rho}} l(S_x),$$

$T(M_{s,\rho})$ is a (tatal space of a) vector bundle over $M_{s,\rho}$.

Proof. By the corollary of lemma 14, to define a map: $T(M_{s,\rho})\to M_{s,\rho}$ by

$$p(g)=x, \quad g\in l(S_x),$$

we have

$$p^{-1}(U(x, \varepsilon))=U(x, \varepsilon)\times l(S_x),$$

for some $\varepsilon>0$. Moreover, if $d_\rho f_1(x_1), \cdots, d_\rho f_n(x_1)$ and $d_\rho f_1'(x_1), \cdots, d_\rho f_n'(x_1)$ both form the basis of $l(S_{x_1})$, then it should be

$$d_\rho f_i'(x_1)=\sum_{j=1}^n a_{ij}(x_1) d_\rho f_j(x_1), \quad (a_{ij}(x_1))\in GL(n, \boldsymbol{R}.) \tag{70}$$

Hence we have the lemma.

Theorem 7. *$M_{s,\rho}$ allows the structure of differentiable manifold and its cotangent bundle is equivalent to $T(M_{s,\rho})$.*

Proof. By (67), if $\{d_\rho f_1(x), \cdots, d_\rho f_n(x)\}$ form a basis of $l(S_x)$, then there exist continuous cross-sections $y_1=y_1(x), \cdots, y_n=y_n(x)$ from $U(x, \varepsilon)$ to $s(M)$ such that

$$d_\rho f_i(x', x_j(x'))\in GL(n, \boldsymbol{R}), \quad x'\in U(x, \varepsilon).$$

For simple (if necessary, to change $f_1, \cdots, f_n$ linearly), we assume

$$d_\rho f_i(x', y_j(x'))=\delta_{ij}, \quad i, j=1, \cdots, n, \quad x'\in U(x, \varepsilon). \tag{71}$$

By (71), in the product structure $p^{-1}(U(x, \varepsilon))=S^{n-1}\times U(x, \varepsilon)$, we may regard each y_i is a constant cross-section and therefore the integral curve of the generalized vector field $<\delta_{y_j}, d_\rho f>$ starts from x is given by $\varphi_{y_j}(t)$ defined in the proof of lemma 13. Then, since $(\varphi_{y_1}(t_1), \cdots, \varphi_{y_n}(t_n))$, $-\varepsilon_1<t_1<\varepsilon_1, \cdots, -\varepsilon_n<t_n<\varepsilon_n$, give a local coordinate of M at x, the local cordinate of M at x is also given by

$$z \longrightarrow (f_1(z), \cdots, f_n(z)), \quad z\in U(x, \varepsilon),$$

by (71). or, in other word, the manifold structure of $M_{s,\rho}$ is given by $\{(U(x, \varepsilon), h_U)\}$, where h_U is given by

$$h_U(z)=(f_1(z),\ \cdots,\ f_n(z)).$$

Then, since $h_U h_V^{-1}$ is a differentiable map for any (U, V) by (70), we obtain the theorem.

Corollary 1. *$T^*(M_{s,\rho})$, the dual bundle of $T(M_{s,\rho})$, is equivalent to $\tau(M_{s,\rho})$, the tangent microbundle of $M_{s,}$.*

Corollary 2. *If $M=M_{s,\rho}$, that is, $l_x=dim.M$ for all $x\in M$, then M allows a differentiable structure.*

We assume the manifold structure of M is given by $\{(U, h_U)\}$, $h_U\colon U\to \boldsymbol{R}^n$. We take a $C(S^{n-1})$-differentiable function f of $\boldsymbol{R}^n$ at $h_U(x)$. Then we have

$$\begin{aligned}&f(h_U(r_{x,y,t}))\\ &=f(h_U(x))+d_\rho f(h_U(x), \frac{h_U(r_{x,y,t})-h_U(x)}{||h_U(r_{x,y,t}-h_U(x)||})||h_U(r_{x,y,t})-h_U(x)||\\ &+o(||h_U(r_{x,y,t})-h_U(x)||).\end{aligned}$$

Hence $h_U{}^*f(x)=f(h_U(x))$ is $C(S^{n-1})$-differentiable at x if and only if $h_U(r_{x,y,t})$ is a smooth curve with respect to t at $t=0$. Moreover, if $h_U(r_{x,y,t})$ is smooth at $t=0$ for any y, $y\in S_x$, then $h_U{}^*f$ is $C(S^{n-1}$-smooth at x if f is $C(S^{n-1})$-smooth (i.e. differentiable) at $h_U(x)$.

Since we know $h_U(r_{x,y,t})$ is smooth at $t=0$ for any y if and only if we have

$$h_U(r_{x,y,t})=h_U(\mathrm{x})+d_\rho h_U(x,\ y)t+o(t),$$

that is, h_U is $C(S^{n-1})$-differentiable at x with respect to the metric ρ and $d_\rho(h_U{}^*f)$ is continuous in x for smooth f if and only if $d_\rho h_U$ is continuous in x, we have by the corollary 2 of theorem 7,

Theorem 8. *If the manifold structure of M is given by $\{(U, h_U)\}$ and M allows a metric ρ such that ρ satisfies* (i), (ii) *of* § 1 *and h_U is $C(S^{n-1})$-smooth with respect to ρ, then M is smooth.*

Note. If M is smooth, then taking ρ to be the geodesic distance of a Riemannian metric of M, we have $M_{s,\rho}=M$ and this ρ satisfies the assumptions of theorem 8.

19. We set $M_{s,\rho}=M_{s,\rho,}0$ and for $k\geqq 0$, to define $M_{s,\rho,k}$ and $M_{\rho,k}$ by

$$\{x|\ l_x=n-k\}=M_{s,\rho,k},$$

$$\bigcup_{n\geqq k} M_{s,\rho,m}=M_{\rho,k}.$$

By definition, we have

$$M=M_{\rho,0},\quad M_{\rho,n+1}=M_{s,\rho,n+1}=\phi,$$

and also

(72)i $$M_{\rho,0}\supset M_{\rho,1}\supset \cdots \supset M_{\rho,n},$$

(72)ii $$M=\bigcup_{k=0}^{n} M_{s,\rho,k},\quad M_{s,\rho,i}\cap M_{s,\rho,j}=\phi,\quad i\neq j.$$

By lemma 14, we have

Lemma 16. *$M_{s,\rho,k}$ is open in $M_{\rho,k}$.*

Corollary. *To set*

$$T(M_{s,\rho,k})=\bigcup_{x\in M_{s,k,\rho}} l(S_x),$$

$T(M_{s,\rho,k})$ is a (tatal space of) vector bundle over $M_{s,\rho,k}$.

If $x\in M_{s,\rho,k}$, then we can take $C(S^{n-1})$-smooth functions $f_{1,x}, \cdots, f_{n-k,x}$ on $U(x, \varepsilon)$ for some $\varepsilon>0$ such that $\{d_\rho f_{1,x}(x), \cdots, d_\rho f_{n-k,x}(x)\}$ forms a basis of $l(S_x)$. Then, if $x'\in M_{s,\rho,k}$ and $\rho(x, x')$ is sufficiently small, $\{d_\rho f_{1,x}(x'), \cdots, d_\rho f_{n-k,x}(x')\}$ is a basis of $l(S_{x'})$. Moreover, we can choose $y_1(x')\in S_{x'}, \cdots, y_{n-k}(x')\in S_{x'}$ such that in the product structure of $\pi^{-1}(U(x, \varepsilon'))=U(x, \varepsilon')\times S^{n-1}$, $\varepsilon'<\varepsilon$, each $y_i(x')$ is mapped (a fixed) $y_i\in S^{n-1}$ and to set

(71)′ $$d_\rho f_{i,x}(x', y_j(x'))=a_{ij}(x'),$$

$(a_{ij}(x'))\in GL(n-k, \boldsymbol{R})$ and continuous in x′. We denote by y_i the cross-section of $s(M)$ defined on $U(x, \varepsilon')$ whose value at x' is $y_i(x')$.

In $U(x, \varepsilon')$, using the integral curves of the generalized vector fields $X_1, \cdots, X_{n-k}$ such that $rep. X_1=\delta_{y_1}, \cdots, rep. X_{n-k}=\delta_{y_{n-k}}$ starts from x, we can construct a closed subset $V(x)$ of $U(x, \varepsilon')$ such that $V(x)$ contains x and $V(x)$ is homeomorphic a neighborhood of the origin of $\boldsymbol{R}^{n-k}$. Moreover, to define a map $h_{x,k}$: $U(x, \varepsilon')\longrightarrow \boldsymbol{R}^{n-k}$ by

$$h_{x,k}(x')=(f_{1,x}(x'), \cdots, f_{n-k,x}(x')),\quad x'\in U(x, \varepsilon'),$$

we have a commutative diagram

$$\begin{tikzcd} & U(x,\varepsilon') \arrow[dr, "h_{x,k}"] & \\ V(x) \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3ex, "identity"] & & \boldsymbol{R}^{n-k}. \end{tikzcd}$$

But since to set

$$h_{x',k}=g_{x',x}h_{x,k},$$

$g_{x',x}$ is linear as a map of $\boldsymbol{R}^{n-k}$, $V(x)$ and $V(x')$ are changed by the linear map considering them to be subsets of $\boldsymbol{R}^{n-k}$ if x and x' are sufficiently near. Hence if $k\leqq n-1$, we have

(73)′ $$dim.\Big(\bigcup_{x\in M_{s,\rho,k}} V(x)\Big)=n-k.$$

Then, since $M_{s,\rho,k} \subset \bigcup_{x \in M_{s,\rho,k}} V(x)$, we obtain by (73)$'$

$$dim. M_{s,\rho,k} \leqq n-k, \quad k \leqq n-1. \tag{73}$$

On the other hand, since $h_U(r_{x,y,t})$ has $C(S^{n-1})$-tangent for all t, $0 \leqq t < 1$, by the method of the construction of ρ (cf. [3]), the $C(S^{n-1})$-tangent of $h_U(r_{x,y,t})$ takes the form $\delta_{y'}$ almost everywhere on, (0, 1) by theorem 5. Hence $M - M_{s,\rho,n}$ is dense in M by the proof of theorem 8. But since $M_{s,\rho,n}$ is a closed set of M, we have

$$dim. M_{s,\rho,n} \leqq 0. \tag{73$_0$}$$

Summalising these, we have

Lemma 18. *$dim. M_{s,\rho,k}$ is at most equal to $n-k$. Especially, $dim. M_{s,\rho}$ is equal to n and $dim. M_{s,\rho,n}$ is equal to 0 if $M_{s,\rho,n} \neq \phi$.*

Corollary. *$M_{s,\rho}$ is open dense in M.*

20. We set

$$M_{s,\rho,k}{}^\natural = \bigcup_{x \in M_{s,\rho,k}} V(x).$$

Then similarly as theorem 7, we have

Lemma 19. *If $M_{s,\rho,k} \neq \phi$, then $M_{s,\rho,k}{}^\natural$ allows the structure of an $(n-k)$-dimensional smooth manifold and it is a closed submanifold of $M - M_{\rho,k+1}$.*

Corollary. *To set*

$$M_{s,\rho}{}^\flat = M_{s,\rho} - M_{s,\rho} \cap (\bigcup_{m \geqq 1} M_{s,\rho,m}{}^\natural),$$
$$M_{s,\rho,k}{}^\flat = M_{s,\rho,k}{}^\natural - M_{s,\rho,k}{}^\natural \cap (\bigcup_{m \geqq k+1} M_{s,\rho,m}{}^\natural), \quad k \geqq 1,$$

we have

$$\begin{aligned} &M = M_{s,\rho}{}^\flat \cup M_{s,\rho,1}{}^\flat \cup \cdots \cup M_{s,\rho,n}{}^\flat, \\ &M_{s,\rho}{}^\flat \cap M_{s,\rho,k}{}^\flat = \phi, \quad k \geqq 1, \\ &M_{s,\rho,i} \cap {}^\flat M_{s,\rho,j}{}^\flat = \phi, \quad i \neq j. \end{aligned} \tag{74}$$

Here $dim. M_{s,\rho}{}^\flat = n$, $dim. M_{s,\rho,k}{}^\flat = n-k$ if $M_{s,\rho,k}{}^\flat \neq \phi$ and they are all smooth and $M_{s,\rho,k}{}^\flat$ is closed in $M - \cup_{m \geqq k+1} M_{s,\rho,m}{}^\flat$ (cf. [17]).

We note that by the definition of $M_{s,\rho,k}{}^\natural$, $M_{s,\rho,k}$ is dense in $M_{s,\rho,k}{}^\natural$ and therefore $M_{s,\rho,k} \cap M_{s,\rho,k}{}^\flat$ is dense in $M_{s,\rho,k}{}^\flat$.

Theorem 9. *To set $C_{C(S^{n-1})}{}^\infty(M)$ the space of $C(S^{n-1})$-smooth functions on M, $C_{C(S^{n-1})}{}^\infty(M)$ is dense in $C(M)$ by the compact open topology.*

Proof. First we note that by lemma 18 and the definitions of $M_{s,\rho,k}$ and $M_{s,\rho,k}{}^\flat$, for any continuous function f of $M_{s,\rho,k}{}^\flat$ and compact set K of $M_{s,\rho,k}{}^\flat$, there exists a $C(S^{n-1})$-smooth function g of M such that

(75) $$|f(x)-g(x)|<\varepsilon, \quad x\in K,$$

is hold for given $\varepsilon>0$.

We assume that in (74), we get

(74)′ $$M=M_{s,\rho}{}^{\flat}\cup M_{s,\rho,k_1}{}^{\flat}\cup\cdots\cup M_{s,\rho,k_m}{}^{\flat},$$
$$M_{s,\rho,k_i}{}^{\flat}=0,\quad 1\leqq k_1<k_2<\cdots<k_m\leqq n.$$

Let f be a continuous function of M and K a compact set of M they are both arbitary but fixed. Then by (75), there exists a $C(S^{n-1})$-smooth function g_m of M such that

$(76)_m$ $$|f(x)-g_m(x)|<\frac{\varepsilon}{2^m}, \quad x\in K\cap M_{s,\rho,k_m}{}^{\flat}$$

for given $\varepsilon>0$. Then, to set $f_m(x)=f(x)-g_m(x)$, there exists a $C(S^{n-1})$-smooth function $g_{m-1}(x)$ of M such that

$(76)_{m-1}$ $$|f_m(x)-g_{m-1}(x)|<\frac{\varepsilon}{2^{m-1}}, \quad x\in K\cap(M_{s,\rho,k_{m-}}{}^{\flat}\cap M_{s,\rho,k_m}{}^{\flat}).$$

In fact, there exists compact carrier $C(S^{n-1})$-smooth function $g_{m-1,0}(x)$ of M such that

$(76)_0$ $$|f_m(x)-g_{m-1,0}(x)|<\frac{\varepsilon}{2^m},$$
$$x\in K\cap(M_{s,\rho,k_{m-1}}{}^{\flat}-U(M_{s,\rho,k_m}{}^{\flat})),$$

for any neighborhood $U(M_{s,\rho,k_m}{}^{\flat})$ of $M_{s,\rho,k_m}{}^{\flat}$ in M. Hence, if $k_m=1$, we have $(76)_{m-1}$ by $(76)_0$ by virtue of theorem 9. On the oher hand, if $k_m\geqq 2$, then $M-M_{s,\rho,k_m}{}^{\flat}$ is connected. Therefore in $(76)_0$, we may assume

$$(M_{s,\rho,k_m}{}^{\flat}-V(\overline{M_{s,\rho,k_{m-1}}{}^{\flat}}\cap M_{s,\rho,k_m}{}^{\flat}))\cap car.(g_{m-1,0})=0,$$

for any neighborhood $V(\overline{M_{s,\rho,k_{m-1}}{}^{\flat}}\cap M_{s,\rho,k_m}{}^{\flat})$ of $\overline{M_{s,\rho,k_{m-1}}{}^{\flat}}\cap M_{s,\rho,k_m}{}^{\flat}$ in M. Hence by the continuity of $f_m(x)$, we have $(76)_{m-1}$. Then, repeating this, we have

$$|f_1(x)-g_0(x)|<\varepsilon, \quad x\in K, \quad f_1(x)=f(x)-\sum_{k=0}^{m-1}g_{m-k}(x).$$

Hence to set $g(x)=\sum_{k=0}^{m}g_k(x)$, we have the theorem.

We note that since the space of compact carrier smooth functions is dense in the space of compact carrier continous functions by the compact open topology and $\bigcup_{j=p}^{m}M_{s,\rho,k_j}{}^{\flat}$ is closed in M for all p, $1\leqq p\leqq m$, we also obtain

Theorem 9′. *Denoting the space of compact carrier $C(S^{n-1})$-smooth functions on*

M by $C_{C(S^{n-1}),\,0}{}^{\infty}(M)$, $C_{C(S^{n-1}),\,0}{}^{\infty}(M)$ is dense in $C_0(M)$, the space of compact carrier continuous functions on M, by the compact open topology.

Corollary. *For any locally finite open covering $\{U\}$ of M, there exists a partition of unity $\{e_U(x)\}$ of $C(S^{n-1})$-smooth functions on M subordinated to $\{U\}$.*

Proof. We take a partition of unity $\{f_U(x)\}$ of continuous functions subordinated to $\{U\}$. Then by theorem 9′, there exists a $C(S^{n-1})$-smooth function $e_U{}'(x)$ such that

$$car.\, e_U{}' \subset U,\quad e_U{}'(x) \geqq 0,\quad |f_U(x) - e_U{}'(x)| < \varepsilon_U,$$

where $\varepsilon_U > 0$ is arbitrary. Then, since $\{U\}$ is locally finite, taking ε_U sufficiently small, $e(x) = \Sigma_U e_U{}'(x)$ does not vanish at any point of M. Then to set $e_U(x) = e_U{}'(x)/e(x)$, we have the corollary.

§7. $C(S^{n-1})$-smooth forms and de Rham's theorem.

21. Since the cotangent bundle $T(M_{s,\rho,k}{}^{\flat})$ of $M_{s,\rho,k}{}^{\flat}$ is given by

$$T(M_{s,\rho,k}{}^{\flat}) = \underset{x\in M_{s,\rho,k}{}^{\flat}}{} l(S_{x'}),$$

where x' is an element of $M_{s,\rho,k}{}^{\flat}$ such that $x \in V(x')$, if $e_1, \cdots, e_{n-k}$ are the d_ρ-smooth cross-sections of $T(M_{s,\rho,k}{}^{\flat})$ such that $\{e_1(x'), \cdots, e_{n-k}(x')\}$ form the basis of $l(S_{x'})$ if $x' \in U_k(x)$, a neighborhood of $x \in M_{s,\rho,k}{}^{\flat}$ in $M_{s,\rho,k}{}^{\flat}$, then for any m, $m<k$, there exists a neighborhood $U_m(x)$ of x in $\cup_{j\leqq m} M_{s,\rho,j}{}^{\flat}$ such that if $x'' \in U_m(x) \cap M_{s,\rho,m}{}^{\flat}$, then there exists a neighborhood $V_m(x'')$ of x'' in $U_m(x) \cap M_{s,\rho,m}{}^{\flat}$ and d_ρ-smooth cross-sections $f_1, \cdots, f_{k-m}$ of $T(M_{s,\rho,m}{}^{\flat})$ in $V_m(x'')$ such that $\{e_1(x''', \cdots, e_{n-k}(x'''), f_1(x'''), \cdots, f_{k-m}(x''')\}$ form the basis of $l(S_{x'})$ if $x''' \in V_m(x'')$. In the rest, as the element of $C(S_x)$, etc., we assume

$$\begin{aligned} &||e_i(x'||=1,\quad x' \in U_k(x),\quad i=1, \cdots, \mathrm{n}-k, \\ &||f_j(x''')||=1,\quad x''' \in V_m(x''),\quad j=1, \cdots, k-m. \end{aligned} \tag{77}$$

Definition. *A map φ from U, an open set of M, to $\cup_{k=0}^{m} \Lambda^p \mathrm{T}(M_{s,\rho,k}{}^{\flat})$ is called a d_ρ-smooth p-form (or a $C(S^{n-1})$-smooth p-form) on U if*

(i). *$\varphi | U \cap M_{s,\rho,k}{}^{\flat}$ is a d_ρ-smooth cross-section of $\Lambda^p T(M_{s,\rho,k}{}^{\flat})$ for each k.*

(ii). *Using the above notations, if we have*

$$\varphi | U \cap M_{s,\rho,k}{}^{\flat} = \sum_{i_1, \cdots, i_p} \varphi_{i_1, \cdots, i_p} e_{i_1} \wedge \cdots \wedge e_{i_p},$$

On $U_k(x)$, then to set

$$\varphi | V_m = \sum_{i_1, \cdots, i_p} \varphi_{i_1, \cdots, i_p,\, V_m} e_{i_1} \wedge \cdots \wedge e_{i_p} \tag{78}$$

$$+\sum_{i_1',\cdots,i_q' j_1,\cdots,j_{q-p}} \varphi_{i_1',\cdots,i_q' j_1,\cdots,j_{p-q}} \cdot e_{i_1'} \wedge \cdots \wedge e_{i_q'} \wedge f_{j_1} \wedge \cdots \wedge f_{j_{q-q}}, \quad q>0,$$

we have

(*a*). $\varphi_{i_1,\cdots,i_p,V_m}$ *is defined and* d_ρ-*smooth on* $U_m(x)$ *and we have*

$$\varphi_{i_1,\cdots,i_p,V_m} | U_k = \varphi_{i_1,\cdots,i_p},$$

for each $i_1, \cdots, i_p$ *and* V_m.

(b). *If a series* $\{x_j | x_j \in M_{s,\rho,m}{}^\flat\}$ *converges to some element of* $M_{s,\rho,k}{}^\flat$, *then to set*

$$\varphi' | V_m = \varphi | V_m - \sum_{i_1,\cdots i_p} (\varphi_{i_1,\cdots,i_p,V_m} | V_m) e_{i_1} \wedge \cdots \wedge e_{i_p},$$

we have

$$\lim_{j\to\infty}. \varphi'(x_j) = 0,$$

$$\lim_{j\to\infty}. d_\rho{}^k(\varphi')(x_j) = 0, \quad k \geqq 1.$$

Here $d_\rho{}^k(\Psi)$ *means* $\sum_{i_1,\cdots,i_p} d_\rho{}^k(\Psi_{i_1,\cdots,i_p})\, g_{i_1} \wedge \cdots \wedge g_{i_p}$, where $\Psi = \sum_{i_1,\cdots,i_p} \Psi_{i_1,\cdots,i_p} g_{i_1} \wedge \cdots \wedge g_{i_p}$ *and* $\lim._{j\to\infty} \varphi(x_j) = 0$, *etc. are defined by* $\lim._{j\to\infty} ||\varphi(x_j)|| = 0$, *etc., where* $||\varphi(x_j)||$ *and* $||d_\rho{}^k \varphi(x_j)||$ *are given by*

$$||\varphi(x_j)|| = \sum_{i_1,\cdots,i_p} |\varphi_{i_1,\cdots,i_p}(x_j)|,$$

$$||d_\rho{}^k \varphi(x_j)|| = \sum_{i_1,\cdots,i_p} \max_{y_1,\cdots,y_k}. |d_\rho{}^k \varphi(x_j, y_1, \cdots, y_k)|.$$

Here g_i *means either* $e_{i'}$ *or* $f_{i''}$.

We note that by (77), this definition does not depend on the choice of the basis of $l(S_x)$.

Note. Similarly, we can define d_ρ-k-smooth p-form on U. In this case, the condition (b) of (ii) is changed to

(b'). $\lim._{j\to\infty} \varphi'(x_j) = 0$ *and* $\lim._{j\to\infty} d_\rho{}^i \varphi'(x_j) = 0$ *if* $i \leqq k$.

Moreover, to set

$$M^k{}_{s,\rho,j} = \{x | \ \dim. l_k(S_x) = n - j\},$$

we can construct the $(n$-$k)$-dimensional smooth manifold $M^k{}_{s,\rho,j}{}^\flat$ similarly as $M_{s,\rho,j}{}^\flat$. Then using $M^k{}_{s,\rho,j}{}^\flat$, we can deine d_ρ-i-smooth p-form on U if $i \leqq k$.

By the definition of d_ρ-smooth forms, we have

Lemma 20. *If* φ *is a* d_ρ-smooth p-*form on* $M_{s,\rho,k}{}^\flat$, *then for any* $x \in M_{s,\rho,k}{}^\flat$, *there exists a neighborhood* $U(x)$ *of* x *in* M *such that there exists a* d_ρ-*smooth* p-*form*

$\tilde{\varphi}$ *on* $U(x)$ *such that*

(79) $$\tilde{\varphi}|M_{s,\rho,k}{}^{\flat}\cap U(x)=\varphi|M_{s,\rho,k}{}^{\flat}\cap U(x).$$

Note. We define a subspace $AC(\overbrace{S^{n-1}\times\cdots\times S^{n-1}}^{p})$ of $C(\overbrace{S^{n-1}\times\cdots\times S^{n-1}}^{p})$ by $\{f|\ f(y_{\sigma(1)},\cdots,y_{\sigma(p)})=\mathrm{sgn}(\sigma)f(y_1,\cdots,y_p),\ \sigma\in\mathfrak{S}^p\}$. Then using $AC(S^{n-1}\times\cdots\times S^{n-1})$ to be the fibre, we can construct a subbundle $AC^p(s(M))$ of $C^p(s(M))$ (cf. [4]). On the other hand, by the definition of $T(M_{s,\rho,k}{}^{\flat})$, we can consider $T(M_{s,\rho,k}{}^{\flat})$ to be a subbundle of $C(s(M_{s,\rho,k}{}^{\flat}))$. Therefore, $\cup_{k=0}^{n}\Lambda^pT(M_{s,\rho,k}{}^{\flat})$ is contained in $AC^p(s(M))$. Hence we can define a d_ρ-smooth p-form on U to be a d_ρ-smooth cross-section of $AC^p(s(M))$ such that whose value at x is contained in $\Lambda^pT(M_{s,\rho,k}{}^{\flat})$ if $x\in M_{s,\rho,k}{}^{\flat}$.

22. As usual, we can define the addition and the multiplication of d_ρ-smooth p-form φ and q-form Ψ. Moreover, we can define the exterior differential $d\varphi$ of φ by

(80) $$(d\varphi)|M_{s,\rho,k}{}^{\flat}=d(\varphi|M_{s,\rho,k}{}^{\flat}).$$

Here, in the right hand side, d is taken in the usual sense. Then, by (b) of (ii) of the definition of d_ρ-smooth forms, d is well defined.

We note that, in the coordinate free form, regarding φ to be a cross-section of $AC^p(s(M))$, we obtain

(81) $$d\varphi=Ad_\rho\varphi.$$

Here $Ad_\rho\varphi$ is given by

$$\begin{aligned}&Ad_\rho\varphi(x,y_1,\cdots,y_{p+1})\\ &=\frac{1}{p+1}\sum_{i=1}^{p-1}(-1)^i\Big[\lim_{t\to 0}\Big\{\frac{1}{t}\varphi(r_{x,y_i,t},\ y_1,\cdots,y_{i-1},\ y_{i+1},\cdots,y_{p+1})\\ &\qquad -\varphi(x,\ y_1,\cdots,y_{i-1},\ y_{i+1},\cdots,y_{p+1})\Big\}\Big].\end{aligned}$$

By definition, in general, if f is a cross-section of $AC^p(s(M))$, then $Ad_\rho f$ is a cross-section of $AC^{p+1}(s(M))$ if it is defined.

Lemma 21. *If a d_ρ-*smooth *p-form φ satisfies $d\varphi=0$ on some neighborhood of x (in M), then there exists a neighborhood $U(x)$ of x in M and a $(p-1)$-form Ψ on $U(x)$ such that*

(82) $$\varphi=d\Psi,\ \ on\ U(x).$$

Proof. We assume $x\in M_{s,\rho,k}{}^{\flat}$. Then there is a neighborhood $U_k(x)$ of x in $M_{s,\rho,k}{}^{\flat}$ and a $(p-1)$-*form* Ψ_k on $U_k(x)$ such that

$(82)_k$ $$\varphi|U_k(x)=d\Psi_k,$$

by (usual) Poincaré lemma. We take a contractible neighborhood $U_{k-1}(x)$ of x in

$M_{s,\rho,k-1}{}^{b} \cup M_{s,\rho,k}{}^{b}$ and assume

$$U_k(x)=U_{k-1}(x)\,|\,M_{s,\rho,k}{}^{b}.$$

Then by lemma 20, there exists a $(p-1)$-form $\widetilde{\Psi}_k$ on $U_{k-1}(x)$ such that

$$\widetilde{\Psi}_k\,|\,U_k(x)=\Psi_k.$$

We set

$$\varphi_1=\varphi-d\widetilde{\Psi}_k,$$

on $U_{k-1}(x)$. Then, by definition, we have $d\varphi_1=0$. On the other hand, since $U_{k-1}(x)$ is contractible, the homology basis γ of $U_{k-1}(x)\text{-}M_{s,\rho,k}{}^{b}$ is taken to satisfy

(83) $$\max_{x\in|\gamma|}\rho(x, M_{s,\rho,k}{}^{b})<\varepsilon,$$

for any $\varepsilon>0$. Here $|\gamma|$ means the carrier of γ. Hence by the definition of d_ρ-smooth p-forms, we obtain

$$\int_\gamma \varphi_1=0,\ \textit{for any homology basis of } U_{k-1}(x)-M_{s,\rho,k}{}^{b}.$$

Therefore, by de Rham's theorem, there exists a$(p-1)$-form Ψ_{k-1}' on $U_{k-1}(x)$ such that

$(82)_{k-1}'$ $$\varphi_1=d\Psi_{k-1}',\ \text{on } U_{k-1}(x).$$

Then, by (b) of (ii) of the definition of d_ρ-smooth forms, to set α_ε to be the C^∞-function on $U_{k-1}(x)-M_{s,\rho,k}{}^{b}$ such that

$$\alpha_\varepsilon(x)=1,\ \ \rho(x,\ M_{s,\rho,k}{}^{b})>2\varepsilon,$$
$$\alpha_\varepsilon(x)=0,\ \ \rho(x,\ M_{s,\rho,k}{}^{b})<\varepsilon,\ \ 1\geqq\alpha_\varepsilon(x)\geqq0,$$

we have

(84) $$\varphi_1=\lim_{\varepsilon\to0} d(\alpha_\varepsilon\Psi_{k-1}').$$

Hence we may assume Ψ_{k-1}' vanishes on $M_{s,\rho,k}{}^{b}$ in the sence of C^∞-topology. Therefore, to set $\Psi_{k-1}=\Psi_{k-1}'+\widetilde{\Psi}_k$, we obtain

$(82)_{k-1}$ $$\varphi=d\Psi_{k-1},\ \textit{on } U_{k-1}(\mathrm{x}).$$

To repeat this, we have the lemma.

Note. Since we know

$$(k_a\delta+\delta k_a)f=f,\ \textit{on } U(a),\ \textit{a neighborhood of } a,$$

for the Alexander-Spanier cochain f on M, where $k_a f$ is given by

$$(k_a f)(x_0,\ x_1,\ \cdots,\ x_{p-1})=f(x_0,\ x_1,\ \cdots,\ x_{p-1},\ a),$$

to set

$$(\kappa_a\varphi)(x,\ y_1,\ \cdots,\ y_{p-1})$$
$$=\varphi(x,\ y_1,\ \cdots,\ y_{p-1},\ \varepsilon_{x,a})\rho(x,\ a),$$

for a cross-section φ of $C^p(s(M))$, we obtain for a smooth p-form on M regarding it to be a cross-section of $AC^p(s(M))$,

(85) $$d_\rho\kappa_a\varphi=\varphi+o(1),$$

if $d_\rho\varphi=0$, by (14)$'$.

23. By definition a d_ρ-smooth o-form φ^0 is a function on M and it satisfies

(i). *$\varphi^0|M_{s,\rho,\kappa}{}^\flat$ is smooth for each k.*

(ii). *If $\{x_j\}$ is a series of $M_{s,\rho,m}{}^\flat$ such that lim.$_{j\to\infty}x_j=x$, $x\in M_{s,\rho,k}{}^\flat$ $(m<k)$, then*

$$\lim_{j\to\infty} f(x_j)=f(x),\ \lim_{j\to\infty} d_\rho{}^k f(x(_j)=d_\rho{}^k f(x),\ \ k\geqq 1.$$

Hence, if $d_\varphi{}^0=0$, then φ^0 is a constant function. Therefore, denoting $\mathscr{C}^p$ the sheaf of germs of d_ρ-smooth p-forms on M, we have the fine resolution of the constant sheaf of real numbers $\boldsymbol{R}$ on M as follows

$$0\longrightarrow \boldsymbol{R}\longrightarrow \mathscr{C}^0\xrightarrow{d}\mathscr{C}^1\xrightarrow{d}\cdots\xrightarrow{d}\mathscr{C}^n\longrightarrow 0,$$

by virtue of lemma 21 and the corollary of theorem 9$'$. Hence we have de Rham's theorem of M, a paracompact topological manifold in the following form (cf. [13], [15]).

Theorem 10. *To set $C^p(M)$ the group of d_ρ-*smooth *p-forms on M, then we have*

(86) $$H^p(M,\ \boldsymbol{R})=\mathfrak{Z}^p(M)/dC^{p-1}(M),$$

where $\mathfrak{Z}^p(M)$ is the kernel of d in $C^p(M)$.

On the other hand, since in a smooth manifold, a singular chain always homologous to a differentiable chain, and if $f:\sigma\longrightarrow M'$, M' is smooth, is a differentiable, then f satisfies (27), any (singular) cycle γ' of M is homologous to a cycle γ of M such that

(87) $$\gamma=\sum_i c_i f_i(\sigma),\ \textit{each } f_i \textit{ satisfies (27)},$$

by the decomposition (74). Hence we may consider a d_ρ-smooth form always integrable on the homology basis of M. Moreover, for a d_ρ-smooth form φ and a chain γ which is written in the form of (87), we obtain the Stokes' theorem

$$\int_{\partial\gamma}\varphi=\int_\gamma d_\varphi.$$

Hence in theorem 11, the pairing of $H_p(M, \boldsymbol{R})$ and $\mathfrak{Z}^p(M)/dC^{p-1}(M)$ is given by

$$<\bar{\gamma}, \bar{\varphi}>=\int_\gamma \varphi. \tag{88}$$

Here $\bar{\gamma}$ and $\bar{\varphi}$ mean the classes of γ and φ.

Note. We denote the groups of d_ρ-smooth cross-sections of $C^p(s(M))$ and alternative Alexander-Spanier p-cochains of M by $C^p(M)$ and $\mathfrak{C}^p(M)$. The subgroups of $\hat{C}^p(M)$ and $\mathfrak{C}^p(M)$ consisted by those chains φ that

$$\int_\gamma |\varphi|=0, \textit{ for any } \gamma \textit{ which is written as } (87),$$

by $\hat{C}^p{}_N(M)$ and $\mathfrak{C}^p{}_N(M)$. Here $|\varphi|$ is given by

$$|\varphi|(x, y_1, \cdots, y_p)=|\varphi(x, y_1, \cdots, y_p)|, \quad \varphi\in\hat{C}^p(M),$$
$$|\varphi|(x_0, x_1, \cdots, x_p)=|\varphi(x_0, x_1, \cdots, x_p)|, \quad \varphi\in\mathfrak{C}^p(M).$$

Then by (14)′ and (85), we have the commutative diagram

$$\begin{array}{ccccc}
C^{p+1}(M) & \xrightarrow{i} & C^{p+1}(M)/C^{p+1}{}_N(M) & \xrightarrow{\bar{k}} & \mathfrak{C}^{p+1}(M)/\mathfrak{C}^{p+1}{}_N(M) \\
{\scriptstyle d}\uparrow & & {\scriptstyle \bar{d}_\rho}\uparrow & & {\scriptstyle \bar{\delta}}\uparrow \\
\hat{C}^p(M) & \xrightarrow{i} & \hat{C}^p(M)/\hat{C}^p{}_N(M) & \xrightarrow{\bar{k}} & \mathfrak{C}^p(M)/\mathfrak{C}^p{}_N(M).
\end{array}$$

Here, i is the map induced from the inclusion, $\bar{d}_\rho$ and $\bar{\delta}$ are the maps induced from d_ρ and δ and $\bar{k}$ is the map induced from k. Here, k is given by

$$(k_\varphi)(x_0, x_1, \cdots, x_p)$$
$$=\varphi(x_0, \varepsilon_{x_0, x_1}, \cdots, \varepsilon_{x_0, x_p})\rho(x_0, x_1)\cdots\rho(x_0, x_p).$$

We also note that to define a subgroup $\mathfrak{C}^p{}_b(M)$ of $\mathfrak{C}^p(M)$ by

$$\mathfrak{C}^p{}_b(M)=\{\varphi|\int_\gamma|\varphi|<\infty \textit{ is } \gamma \textit{ is given by } (87)\}, \tag{89}$$

then we get

$$k(\hat{C}^p(M))\subset\mathfrak{C}^p{}_b(M).$$

Moreover, by Stokes' theorem (cf. [3]), if an element φ of $\mathfrak{C}^p{}_b(M)$ is written as δ, then we can take Ψ to be an element of $\mathfrak{C}^{p-1}{}_b(M)$. Hence denoting the sheaf of germes of the elements of $\mathfrak{C}^p{}_b(M)$ by $\mathfrak{C}^p{}_b$, we have a fine resolution

$$0\longrightarrow\boldsymbol{R}\longrightarrow\mathfrak{C}^0{}_b\xrightarrow{\delta}\mathfrak{C}^1{}_b\xrightarrow{\delta}\cdots\xrightarrow{\delta}\mathfrak{C}^p{}_b\longrightarrow\cdots,$$

because $\mathscr{C}^0$ is a subsheaf of $\mathfrak{C}^0{}_b$ and therefore the partition of unity subordinated to any locally finite open cevering of M by the functions of $\mathfrak{C}^0{}_b(M)$ is always possible. Hence we get

(86)′ $$H^p(M, \boldsymbol{R}) = \mathfrak{Z}^p{}_b(M)/\delta\mathfrak{C}^{p-1}{}_b(M),$$

where $\mathfrak{Z}^p{}_b(M)$ is the δ-kernel in $\mathfrak{C}^p{}_b(\mathrm{M})$ and we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}^p{}_b(M)/\delta\mathfrak{C}^{p-1}{}_b(M) & \xrightarrow{\text{the isomorphism of (86)}'} & \\ \uparrow \bar{k} & & H^p(M, \boldsymbol{R}) \\ B^p(M)/dC^{p-1}(M) & \xrightarrow{\text{the isomorphism of (86)}} & \end{array}$$

where k is the induced map from k.

As in (86), in (86)′, by the definition of $\mathfrak{C}^p(M)$ and the Stokes' theorem, we also have the pairing of $H_p(M, \boldsymbol{R})$ and $\mathfrak{Z}^p{}_p(M)/\delta\mathfrak{C}^{p-1}{}_b(M)$ by

(88)′ $$<\bar{\gamma}, \bar{\varphi}> = \int_\gamma \varphi,$$

where $\bar{\gamma}$ is the class of a chain γ of M and $\bar{\varphi}$ is the class of φ, an element of $\mathfrak{Z}^p{}_b(M)$.

§ 8. Generalized integral curves of generalized vector fields on manifolds.

24. We assume $x \in M_{s,p,k}$, $k \geqq 0$ and $k \neq n$, and take the $C(S^{n-1})$-smooth functions near x, $f_1, \cdots, f_{n-k}$ such that $d_\rho f_1(x'), \cdots, d_\rho f_{n-k}(x')$ form the basis of $l(S_{x'})$ if $x' \in M_{s,\rho,k}{}^\flat$ and sufficiently near to x. Then we can choose $y_1(x') \in S_{x'}, \cdots, y_{n-k}(x') \in S_{x'}$ such that each $y_i(x')$ depends continuously on x' and they satisfy

(71)″ $$d_\rho f_i(x', y_j(x')) = \delta_{ij}, \quad i, j = 1, \cdots, n-k.$$

Then to set

$$l^{*\prime}(S_{x'}) = \left\{ \sum_{i=1}^{n-k} c_i \delta_{y_i(x')} \,|\, c_i \in \boldsymbol{R}^n, \ i = 1, \cdots, n-k \right\},$$

we may consider $l^{*\prime}(S_{x'})$ to be the dual space of $l(S_{x'})$. Hence at x', we have

(42)′ $$C^*(S_{x'}) = l^{*\prime}(S_{x'}) \oplus l(S_{x'})^\perp.$$

Moreover, although $x' \notin M_{s,\rho,k}$, if $x' \in M_{s,\rho,k}{}^\flat$, then since we may consider $d_\rho f_1(x'), \cdots, d_\rho f_{n-k}(x')$ spann $T_{x'}$, the fibre of $T(M_{s,\rho,k}{}^\flat)$ at x', if x' is sufficiently near to x, we also have

(90) $$C^*(S_{x'}) = l^{*\prime}(S_{x'}) \oplus T_{x'}{}^\perp.$$

We choose a locally finite open covering $\{U\}$ of $M_{s,\rho,k}{}^\flat$ such that for each U, the basis of $l(S_x)$, $x \in U$ (or the basis of T_x, if $x \notin M_{s,\rho,k}$ but $x \in M_{s,\rho,k}{}^\flat$) and the cross-sections $y_1{}^U, \cdots, y_{n-k}{}^U$ of $s(M)$ on U are given to satisfy (71)″. We set

$$(\delta)_U=(\delta_{y_1 U}, \cdots, \delta_{y_{n-k} U}),$$
$$(\delta)_U(x)=(\delta_{y_1 U}(x), \cdots, \delta_{y_{n-k} U}(x)).$$

Then, to set the transition functions of the tangent bundle of $M_{s,\rho,k}{}^{\natural}$ by $\{g_{UV}\}$, we have

$$(\delta)_U(x)=g_{UV}(x)((\delta_V)(x))+\xi_{UV}(x)\xi_{UV}(x)\in l(S_x).$$

Then since we have

$$\xi_{UV}(x)+g_{UV}(x)\xi_{VW}(x)+g_{UW}(x)\xi_{WU}(x)=0,$$
$$g_{UV}(x)\xi_{VU}(x)=-\xi_{UV}(x),$$

by definition, to take the partition of unity $\{e_{UV}\}$ of $C(S^{n-1})$-smooth functions on M subordinated to $\{U\cap V\}$, we have

$$\xi_{UV}(x)=\eta_U(x)-g_{UV}(x)\eta_V(x),$$
$$\eta_U(x)=\sum_{w\cap u\neq \flat} e_{WU}(x)\xi_{WU}(x).$$

Hence to set

$$l^*(S_x)_U=\{\textit{the subspace of } C^*(S_x) \textit{ spanned by the components of } (\delta)_U(x)-\eta_U(x)\},$$

$l^*(S_x)_U$ does not depend on the choice of U and it can be regarded to be the dual space of $l(S_x)$ (if $x\in M_{s,\rho,k}$ and if $x\notin M_{s,\rho,k}$ but $x\in M_{s,\rho,k}{}^{\flat}$, then $l^*(S_x)_U$ is regarded to be the dual space of T_x). Moreover, by definition, to set (denoting $l^*(Sx)$ instead of $l^*(S_x)_U$),

$$T^*(M_{s,\rho,k}{}^{\flat})=\bigcup_{x\in M_{s,p,k}{}^{\flat}} l^*(S_x),\quad n-1\geqq k\geqq 0,$$

$T^*(M_{s,p,k}{}^{\flat})$ is a (tatal space of) vector bundle over $M_{s,\rho,k}{}^{\flat}$ and it is the dual bundle of $(TM_{s,\rho,k}{}^{\flat})$.

We note that similarly, we can define $T^*(M_{s,p,k})$ and it is a (tatal space of) vector bundle over $M_{s,\rho,k}$. This $T^*(M_{s,\rho,k})$ is the dual bundle of $T(M_{s,\rho,k}) = \bigcup_{x\in M_{s,\rho,k}} l(S_x)$.

On the other hand, since $T(M_{s,\rho,k})$ and $C^*(s(M))|M_{s,\rho,k}$ are both vector bundles over $M_{s,\rho,k}$, to set

$$\mathrm{T}(M_{s,\rho,k})^{\perp}=\bigcup_{x\in M_{s,\rho,k}} l(S_x)^{\perp},\quad n-1\geqq k\geqq 0.$$

$T(M_{s,\rho,k})$ is also an (infinite dimensional) vector bundle over $M_{s,\rho,k}$ and we have

(91)′ $$C^*(s(M))|M_{s,\rho,k}=T^*(M_{s,\rho,k})\oplus T(M_{s,\rho,k})^{\perp},$$

by (42)$'$.

Similarly, to set

$$T(M_{s,\rho,k}{}^{\flat})=\bigcup_{x\in M_{s,\rho,k}{}^{\flat}} T_x{}^{\perp},\quad n-1\geq k\geq 1,$$

we get by (90)

$$C^*(s(M))\,|\,M_{s,\rho,k})^{\flat}=T^*(M_{s,\rho,k}{}^{\flat})\oplus T(M_{s,\rho,k}{}^{\flat})^{\perp}, \tag{92$'$}$$

If $k=n$, we know that $M_{s,\rho,n}{}^{\natural}=M_{s,\rho,n}$ and therefore $M_{s,\rho,n}{}^{\flat}\subset M_{s,\rho,n}$. If $x\in M_{s,\rho,n}$, then we define

$$l^{*\prime}(S_x)=l^*(S_x)=\{0\},$$

Then we have $T^*(M_{s,\rho,n})=0$, the 0-bundle, and $T(M_{s,\rho,n})=C^*(s(M))\,|\,M_{s,\rho,n}$. Hence (91)$'$ and (92)$'$ are hold true although $k=n$.

By (91)$'$, to set

$$\tau^{\#}(M)=\bigcup_{k=0}^{n} T(M_{s,\rho,k}),$$

$$\tau^{\#*}(M)=\bigcup_{k=0}^{n} T^*(M_{s,\rho,k}),\quad \tau^{\#}(M)^{\perp}=\bigcup_{k=0}^{n} T(M_{s,\rho,k})^{\perp},$$

we have

$$C^*(t(M))=\tau^*(M)\oplus\tau^{\#}(M)^{\perp}, \tag{91}$$

by (91)$'$.

Similarly, by (92)$'$, to set

$$\tau^{\natural}(M)=\bigcup_{k=0}^{n} T(M_{s,\rho,k}{}^{\flat}),$$

$$\tau^{\natural *}(M)=\bigcup_{k=0}^{n} T^*(M_{s,\rho,k}{}^{\flat}),\quad \tau^{\natural}(M)^{\perp}=\bigcup_{k=0}^{n} T(M_{s,\rho,k}{}^{\flat})^{\perp},$$

we also have

$$C^*(s(M))=\tau^{\natural *}(M)\oplus\tau^{\natural}(M)^{\perp}. \tag{92}$$

Note. Since we may assume the metric ρ of M defines a measure $\omega=\omega(x)$ on S_x, we may take $l'(S_x)=\{g\omega\,|\,g\in l(S_x)\}$ to be the model of the dual space of $l(S_x)$. Here, the pairing $<h, g>$, $h\in l(S_x)$ is given by

$$<h, g\omega(x)>=\int_{S_x} h.\,g\omega(x).$$

Then $\cup_{x\in M_{s,\rho,k}} l'(S_x)$ allows the structure of the dual bundle of $T(M_{s,\rho,k})$. But since the generalized tangent of a smooth curve takes the form δ_y in one hand, and no continuous curve takes the element of $l'(S_x)$ to be its generalized tangent at x on the other hand, the above construction of $T^*(M_{s,\rho,k})$ seems more natural.

25. Definition. *In (91) and (92), we denote the projections from $C^*(s(M))$ to $\tau^{\natural *}(M)$ (or to $\tau^{\# *}(M)$) and to $\tau^{\natural}(M)^{\perp}$ (or to $\tau^{\#}(M)^{\perp}$) by p_1 and p_2.*

Definition. *Let X be a generalized vector field on M, then we define the generalized vector fields $D(X)$ and $S(X)$ on M by*

$$rep.\,D(X)=p_1(rep.\,X),\quad rep.\,S(X)=p_2(rep.\,X).$$

By definition, we have

$$X=D(X)+S(X). \tag{93}$$

On the other hand, considering X to be a cross-section of $C^*(s(M))$, we define (the cross-sections of $C^*(s(M))$ on $M_{s,\rho,k}{}^{\flat}$)

$$X_k=X\,|\,M_{s,\rho,k}{}^{\flat},\quad n\geqq k\geqq 0. \tag{94}$$

Then to define $D(X_k)$ and $S(X_k)$ similarly as $D(X)$ and $S(X)$, we get

$$X_k=D(X_k)+S(X_k),\quad n\geqq k\geqq 0. \tag{93$'$}$$

We note that by definitions, we have

$$D(X_k)=D(X)\,|\,M_{s,\rho,k}{}^{\flat},\quad S(X_k)=S(X)\,|\,M_{s,\rho,k}{}^{\flat},$$

$$D(X_n)=0,\ \textit{the } 0\textit{-section on } M_{s,\rho,n}{}^{\flat}.$$

Since $C^*(s(M))\,|\,M_{s,\rho,k}{}^{\flat}\neq C^*(s(M_{s,\rho,k}{}^{\flat}))$ if $k\geqq 1$, X_k is not a generalized vector field on $M_{s,\rho,k}{}^{\flat}$ if $k\geqq 1$, but $D(X_k)$ is a (generalized) vector field on $M_{s,\rho,k}{}^{\flat}$ for all k, because $D(X_k)$ is a cross-section of $T^*(M_{s,\rho,k}{}^{\flat})$ on $M_{s,\rho,k}{}^{\flat}$ for all k.

We also get to define the (not continuous) generalized vector fields X_k on M by

$$rep.\,\hat{X}_k(x)=X_k(x),\quad x\in M_{s,\rho,k}{}^{\flat},$$

$$rep.\,\hat{X}_k(x)=0,\quad x\notin M_{s,\rho,k}{}^{\flat},$$

then

$$X=\sum_{k=0}^{n}\hat{X}_k. \tag{95}$$

Similarly, to define $D(\hat{X}_k)$ and $S(\hat{X}_k)$ same as $\hat{X}_k$, we get

$$D(X)=\sum_{k=0}^{n}D(\hat{X}_k),\quad S(X)=\sum_{k=0}^{n}S(\hat{X}_k). \tag{95$'$}$$

We assume the smooth structure of $M_{s,\rho,k}{}^{\flat}$ is given by $\{(U,\ h_{k,\ U})\}$. Then to set *rep.* $X=\xi(x)$, if $h_{k,U}{}^{-1*}(\xi)$ satisfy the Lipschitz condition

$$||h_{k,U}{}^{-1*}(\xi)(a_1)-h_{k,U}{}^{-1*}(\xi)(a_2)||\leqq L||a_1-a_2||,$$

then $h_{k,V}{}^{-1*}(\xi)$ also satisfies the Lipschitz condition if a_1, a_2 both belongs in $h_{k,U}(U)\cap h_{k,V}(V)$.

Definition. *We call X_k satisfies the local Lipschitz condition on $M_{s,\rho,k}{}^\flat$* if $h_{k,U}{}^{-1*}(\xi)$ *satisfies the Lipschitz condition for all U. Here $\xi(x)=rep.X_k$ and the smooth structure of $M_{s,\rho,k}{}^\flat$ is given by $\{(U,h_{k,U})\}$.*

Definition. *We call X satisfies the local Lipschitz condition on M if each X_k satisfies the local Lipschitz condition on $M_{s,\rho,k}{}^\flat$, $0\leqq k\leqq n$.*

Theorem 11. *If $X=D(X)$ and X satisfies the Lipschitz condition on M, then X has the (unique) integral curve starts from x if $x\notin M_{s,\rho,n}{}^\flat$. Moreover, it $x\in M_{s,\rho,k}{}^\flat$, then the integral curve of X starts from x is contained in $M_{s,\rho,k}{}^\flat$.*

Proof. Since $M_{s,\rho,k}{}^\flat$ is smooth and $T^*(M_{s,\rho,k}{}^\flat)$ can be regarded to be the tangent bundle of $M_{s,\rho,k}{}^\flat$, $D(X)|M_{s,\rho,k}{}^\flat=D(X_k)$ can be regarded to be the usual vector field of $M_{s,\rho,k}{}^\flat$. Then, since $D(X_k)$ satisfies the Lipschitz condition by assumption, if $x\in M_{s,\rho,k}{}^\flat$, then $D(X_k)$ has the (unique) integral curve starts from x in $M_{s,\rho,k}{}^\flat$. Hence we have the theorem.

By theorem 11, if $X=D(X)$ and $X(x)\neq 0$, then we can solve the equation

(41)′ $\quad Xu=f,$

locally for continuous f. On the oher hand, by theorem 4, we have

Theorem 4′. *If $X=S(X)$, then Xf is equal to 0 almost everywhere on M with respect to m, the measure on M in duced from the metric ρ.*

We assume that on M the metric function $\rho(a,x)=f_a(x)$ is $C(S^{n-1})$-differentiable for any a. Then to set

$$u(x)=g(\varepsilon_{a,x})f_a(x)e_a(x),\quad x\neq a,\quad u(a)=0,$$

where $g(y)$ is a function in $C(S_a)$ such that $\langle X(a),\ g\rangle=1$ and $e_a(x)$ is a $C(S^{n-1})$-smooth function on M such that

$$e_a(a)=1,\quad car.\,e_a\subset\bar{B}_a=\{x'\,|\,\rho(x,x')\leqq 1\},$$

we have

(44)′ $\quad Xu(a)=1.$

Moreover, if f_a is $C(S^{n-1})$-smooth in $\bar{B}_a$ except at a and g is $C(S^{n-2})$-smooth, then we get

(44)″ $\quad Xu(x)=0,\ x\neq a,$

if $X=S(X)$. In this case, to set

$$l^1_{loc.}(M)=\left\{f\,|\sum_{x\in K}|f(x)|<\infty \text{ for any compact set } K \text{ of } M\right\},$$

we have

$$X(C_{(C(S^{n-1})}(M))\supset l^1_{loc.}(M),$$

if $X=S(X)$ and there exists a cross-section $e(x)$ of $C(s(M))$ such that

$$(96)_i \qquad <\xi(x), e(x)>=1,\ \ \xi(x)=rep.X,$$

$$(96)_{ii} \qquad ||e(x)||\leqq A,\ \ ||d_{\rho,y}e(x)||\leqq B,\ \ x\in M,$$

by theorem 3. Therefore, if $f_a(x)$ is $C(S^{n-1})$-smooth on $\overline{B}_a-a$, for all a, then we can construct $C_{X,0}(M)$ similarly as $C_{X,0}(U)$ in n^0 13, if $X=S(X)$ and satisfies $(96)_i$ and $(96)_{ii}$. Hence we can solve the equation $(41)'$ locally as an element of $F(M, C_{X,0}(M)^*)$.

We note that if M is smooth and ρ is the geodesic distance of a Riemannian metric of M, then $f_a(x)$ is smooth on $M-\{a\}$.

26. Since $\tau^\natural(M)^\perp$ is a subset of $C^*(s(M))$, we can define the projection $\pi : \tau^\natural(M)^\perp \longrightarrow M$ by

$$\pi^\natural=\pi|\tau^\natural(M)^\perp.$$

We also denote by $\pi^{-1}(\tau^\natural(M)^\perp)$ the induced $C^*(S^{n-1})$-bundle over $\tau^\natural(M)^\perp$ from $C^*(s(M))$, Then, for a function f on M, or a cross-section ξ of $C^*(s(M))$ from M, we can define a function $\pi^{\natural *}(f)$ on $\tau^\natural(M)^\perp$ or a cross-section $\pi^{\natural *}(\xi)$ of $\pi^{-1}(\tau^\natural(M)^\perp)$ from $\tau^\natural(M)^\perp$.

We assume $x\in M_{s,\rho,k}{}^\flat$ and take a neighborhood U of x in $M_{s,\rho,k}{}^\flat$ such that there exists a homeomorphism $\iota_U(x)$ from U onto a neighborhood of the origin of $l^*(S_x)$. Then, since $C^*(S_x)=l^*(S_x)\oplus T_x{}^\perp$, we can define a homeomorphism $\iota_U(x)^\#$ from $UxT_x{}^\perp$ onto a neighborhood of the origin of $C^*(S_x)$ by

$$\iota_U(x)^\#(\xi)=\iota_U(x)(p_1(\xi)+p_2(\xi).$$

Then, since $T_x{}^\perp$ is the fibre of $T(M_{s,\rho,k}{}^\flat)$ at x, there is a homeomorphism φ_U : $\pi_s{}^{\natural -1}(U)\longrightarrow UxT_x{}^\perp$ and we obtain the map

$$(97) \qquad \varphi_U\iota_U(x)^\# : \pi^{\natural -1}(U)\longrightarrow C^*(S_x).$$

Hence, if ξ is a generalized vector field, then to set

$$\xi_{U,x}=(\varphi_U\iota_U(x)^\#)^*(\pi^{\natural *}(\xi)|\pi^{\natural -1}(U)),$$

$\xi_{U,x}$ is a map from $C^*(S_x)$ to $C^*(S_x)$. Hence, if $||\xi_{U,x}||$ is continuous and saitsfies the Lipschitz condition

$$(98) \qquad ||\xi_{U,x}(\zeta_1)-\xi_{U,x}(\zeta_2)||\leqq L||\zeta_1-\zeta_2||,$$

where $||\xi||$ is the norm of ζ in $C^*(S_x)$, then the equation

$$(61)' \qquad \frac{d\Psi_{U,x}(t)}{dt}=\xi_{U,x}(\Psi_{U,x}(t)),$$

has the unique solution under the initial condition $\Psi_{U,x}(0)=\alpha$, locally. We denote the solution of (61)′ with the initial condition

$$\Psi_{U,x}(0)=\varphi_U\iota_U(x)^{\#}((x, 0))=\iota_U(x)(x),$$

by $\Psi_{U,x,x}$.

We note that although $\xi_{U,x}$ satisfy the Lipschitz condition (98), $\xi_{V,x}$ may not satisfy the Lipschitz condition in general. But, since $M_{s,\rho,k}{}^{\flat}$ allows the structure of a smooth manifold, and $l^*(S_x)$ is the fibre of the tangent bundle of $M_{s,\rho,k}{}^{\flat}$ at x and $T(M_{s,\rho,k}{}^{\flat})^{\perp}$ is the associate $l_*(S_x)^{\perp}$-bundle of the tangent bundle of $M_{s,\rho,k}{}^{\flat}$, we may consider $\varphi_U\iota_U(x)^{\#}$ to be a smooth map. Then, since

$$(99) \qquad \xi_{V,x}=(\varphi_V\iota_V(x)^{\#})^*(\varphi_U\iota_U(x)^{\#})^{*-1}\xi_{U,x},$$

$\xi_{V,x}$ also satisfies the Lipschitz condition.

On the other hand, since $(\varphi_V\iota_V(x)^{\#*}(\varphi_U\iota_U(x)^{\#})^{*-1}$ is a map from (an open set of) $C^*(S_x)$ to an (open set of) $C^*(S_x)$ and does not depend on t, we have

$$(100) \qquad \frac{d}{dt}(\varphi_V\iota_V(x)^{\#})^*(\varphi_U\iota_U(x)^{\#})^{*-1}\Psi(t))$$

$$=(\varphi_V\iota_V(x)^{\#})^*(\varphi_U\iota_U(x)^{\#})_{\#}{}^{-1}\Big(\frac{d}{dt}\Psi(t)\Big),$$

for all $C^*(S_x)$-valued C^1-class function $\Psi(t)$. Hence, if $\xi_{U,x}$ satisfies the Lipschitz condition (98), then by the uniqueness of the solution of (61)′, we have by (99) and (100),

$$(101) \qquad \Psi_{V,x,x}(t)=\varphi_V\iota_V(x)^{\#})^*\varphi_U\iota_U(x)^{\#})^{*-1}\Psi_{U,x,x}(t).$$

By (101) and the definitions of φ_U and $\iota_U(x)^{\#}$, we also have

$$(101)' \qquad \pi^{\natural}\varphi_U\Psi_{U,x,x}(t)=\pi^{\natural}\varphi_V\Psi_{V,x,x}(t).$$

Summarising these, we obtain

Theorem 12. *If X is a continuous generalized vector field on M such that X satisfies the (local) Lipschitz condition on M, then X has the integral curve $\Psi_x(t)$ starts from x in the space $\pi^{\natural}(M)^{\perp}$ uniquely. This integral curve satisfies*

$$(102) \qquad \pi^{\natural}\Psi_x(t)\in M_{s,\rho,k}{}^{\flat},$$

if $x\in M_{s,\rho,k}{}^{\flat}$.

Corollary. *If X is a continuous generalized vector field on M and satisfies the* *(local) Lipschitz condition on M, then X defines a local 1-parameter group of transformations $\{T_t\}$ of M such that $\{T_t\}$ is smooth in t and*

$(103)_i$ $\quad T_0$ *is the identity and* $T_tT_s=T_{t+s}$,

$(103)_{ii}$ $\quad T_t(M)\subset\tau^\natural(M)^\perp$,

$(103)_{iii}$ $\quad \pi^\natural(T_t(M_{s,\rho,k}{}^\flat))\subset M_{s,\rho,k}{}^\flat$,

$(103)_{iv}$ $\quad \dfrac{d}{dt}(T_t{}^*f)=\pi^{\natural *}(X)(T_t{}^*f).$

Note 1. If $X=S(X)$, then $\pi^\natural\Psi_x(t)=x$ for any t and x. On the other hand, if $X=D(X)$, then $\pi^\natural\Psi_x(t)$ is the usual integral curve of X starts from x, and, we may identify $\{T_t|M_{s,\rho,k}{}^\flat\}$, the restriction of the above $\{T_t\}$ on $M_{s,\rho,k}{}^\flat$, and the usual (local) 1-parameter group of transformations of $\mathrm{M}_{s,\rho,k}{}^\flat$ generated by X for each k. Therefore, if $X=D(X)$, then we may consider

$(103)_{ii}'$ $\quad T_t(M)\subset M.$

Note 2. If M is smooth, then we have $\mathrm{M}_{s,\rho}=M$ if we take ρ to be the geodesic distance of a Riemannian metric of M. Then we have

$$\tau^\natural(M)=T^*(M),\ \textit{the cotangent bundle of } M.$$

Hence $\tau^\natural(M)^\perp=T^*(M)^\perp$ is a fibre bundle over M with the typical fibre $l(S^{n-1})^\perp$. Therefore, $\tau^\natural(M)^\perp$ is a smooth Banach manifold ([6], [12]), but it is not C^1-smooth by the theorem of Restrepo ([6], [14]).

Note 3. By [3], we may consider the manifold structure $\{U,\ h_U\}$ of M is given to satisfy

(i). *Jf* $x,\ y\in U$, then

$$\rho(x, y)\leqq A||h_U(x)-h_U(y)||^\alpha,\quad \alpha\leqq\log.2/\log.(2n+2),$$

for some $A>0$.

(ii). *The components of* $h_Uh_V{}^{-1}$ *are the functions of bounded variations and* log. 2/log. (2n+2)-*Hölder continuous for each* $(U,\ V)$.

Hence, to set $rep.X=\xi(x)$, if $h_U{}^*(\xi)$ satisfies

$(98)'$ $\quad ||h_U{}^*(\xi)(a_1)-h_U{}^*(\xi)(a_2)||\leqq L||a_1-a_2||^\alpha,$

for $\alpha\leqq\log.2/\log.(2n+2)$ and for some $L>0$, then $h_V{}^*(\xi)$ also satisfies $(98)'$ for some $L'>0$. Therefore, we may define

Definition. *We call a generalized vector field X with $rep.X=\xi(x)$ to be (locally) α-Hölder continuous for $\alpha\leqq\log.2/\log.(2n+2)$, if $h_U{}^*(\xi)$ satisfies $(98)'$ for each U.*

As in n°16, we definec

Definition. *The integral curve of X in $\tau^{\natural}(M)^{\perp}$ starts from x is called the generalized integral curve of X starts from x.*

Then, as in n°16, we obtain

Theorem 12′. *If a generalized vector field X on M is continuous on M, then X has the generalized integral curve starts from x if and only if $D(X)$ has the (usual) integral curve starts from x.*

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