JOUR. FAC. SCI., SHINSHU UNIV., Vol. 7, pp. 119—122, December, 1972

Note on Results of D. A. R. Wallace

By KAORU MOTOSE

Department of Mathematics, Faculty of Science,
Shinshu University

(Recived October 31, 1972)

The purpose of this paper is to give a theorem relating to [6, Theorem 2] and slightly generalize several theorems of D. A. R. Wallace [7, Theorem], [8, Theorem], [9, Theorem 1]. We shall use the following conventions: Let G be a finite group, G' the commutator subgroup of G and P a p-Sylow subgroup of G. Moreover, R will represent a semi-primary ring with 1 such that the center of $\bar{R}=R/J(R)$ ($J(R)$ the Jacobson radical of R) contains the prime field of characteristic p, RG the group ring of G over R, and $(D)_n$ an arbitary simple component of $\bar{R}$, where D is a division ring with the center C.

The following Lemma is trivial by [3, Lemma 1], [2, Theorem 5.6.1] and [1, Corollary 69.10].

Lemma 1. (1) *$J(\bar{R}G)=\nu(J(RG))$, where ν is a ring homomorphism of RG onto $\bar{R}G$ defined by $\sum_{x\in G} a_x x \longrightarrow \sum_{x\in G}(a_x+J(R))x$ $(a_x\in R)$.*

(2) *there exists a ring isomorphism φ of $(D)_nG$ onto $(DG)_n$ defining by $\sum_{x\in G}(a_{kl}^{(x)})x \longrightarrow (\sum_{x\in G} a_{kl}^{(x)}x)$ $(a_{kl}^{(x)}\in D)$.*

(3) *$J(DG)=D\cdot J(CG)$.*

(4) *there exists a splitting field F for G such that F is finite dimensional separable over C, and hence $J(FG)=F\cdot J(CG)$.*

In the subsequent argument, we shall use notations which is used in Lemma 1. Concerning [6, Theorem 2], we obtain the following:

Theorem 2. *Let p be a divisor of $|G|$. Then, $J(RG)=J(R)G+J(RP)e$ with a central idempotent e of RG if and only if G is a Frobenius group with complement P and kernel N and $e=|N|^{-1}\sum_{x\in N} x$.*

Proof. The "if" part is evident by [4, Theorem]. We shall prove the "only if" part. If $J(RG)=J(R)G+J(RP)e$, then $(J(DG))_n=(J(DP))_n e^*$, where $e^*=\varphi\psi\nu(e)$ and ψ is a projection from $\bar{R}G$ to $(D)_nG$. Since e^* is a central in $(DG)_n$, e^* is a central idempotent of DG and hence $(J(DP))_n e^*=(J(DP)e^*)_n$. Thus, $J(DG)=J(DP)e^*$ and e^* is an element of CG. By Lemma 1, $J(FG)=J(FP)e^*$ and hence $[J(FG):F]\leq|P|-1$. On the orther hand, $[J(FG):F]\geq|P|-1$ by [6, Theorem 1] and so $[J(FG):F]=|P|-1$. Therefore, by [6, Theorem 2], G is a Frobenius group with complement

P and kernel N. Thus, by [4, Theorem], $J(\bar{R}P)\nu(e)=J(\bar{R}P)\nu(f)$, where $f=|N|^{-1}\sum_{x\in N}x$. Since $\bar{e}=\nu(e)$, $\bar{f}=\nu(f)$ are central idempotents of $\bar{R}G$, $(1-x)\bar{e}=(1-x)\bar{e}\bar{f}=(1-x)\bar{f}\bar{e}=(1-x)\bar{f}$ for every $x\in P$, and so $x(\bar{e}-\bar{f})=\bar{e}-\bar{f}$ for every $x\in P$. Thus, $\bar{e}-\bar{f}=\sum_{n\in N}\alpha_n\sigma n$, where $\sigma=\sum_{y\in P}y$ and $\alpha_n\in\bar{R}$. Noting that $P\cap zPz^{-1}=1$ for $z\in N-1$, we can see $\bar{e}=\bar{f}$. Let s be an element of RP such that $ef=sf$. Then, $(\nu(s)-1)\bar{f}=0$ by $\bar{e}=\bar{f}$, and so $\nu(s)=1$. Since sf *is* an idempotent of RG and f is a central idempotent of RG, $s^2=s$ by $(s^2-s)f=0$. Thus, $s-1$ is an idempotent of $J(R)G$ ($\subseteq J(RG)$) and hence $s=1$, Therefore, $e-f$ is an idempotent of $J(R)G$, which means $e=f$.

The following contains [7, Theorem].

Theorem 3. *$J(RG)^2=0$ if and only if one of the following conditions is satisfied* :

(1) *$J(R)^2=0$ and G is a p'-group.*

(2) *$J(R)=0$, $p=2$ and $|G|=2m$, where m is odd.*

Proof. Let us assume that $J(RG)^2=0$, and distinguish between two cases.

Case 1. G is a p$'$-*group* : Then $J(RG)=J(R)G$ (cf. [3, Theorem 1]) and hence $J(R)^2=0$.

Case 2. p is a divisor of $|G|$: At first, we shall prove that $J(R)=0$. Notice that $\nu(R\sigma)$ is an ideal of square zero, where $\sigma=\sum_{x\in G}x$. Then, $R\sigma+J(R)G\subseteq J(RG)$ and so σ is an element of $J(RG)$. Thus, $J(R)\sigma\subseteq J(RG)^2=0$ and $J(R)=0$. By Lemma 1, $J(FG)^2=0$ and hence by [7, Theorem], $p=2$ and $|G|=2m$, where m is odd.

Next, we shall prove the converse. (1) implies that $J(RG)^2=(J(R)G)^2=0$. (2) implies that $J(RG)^2=0$ by [5, Theorem 16. 3].

The following is an extension of [8, Theorem].

Theorem 4. *$J(RG)$ is central in RG if and only if one of the following conditions is satisfied* :

(1) *RG is semi-simple.*

(2) *RG is commutative.*

(3) *$J(R)$ is central in R and G is an abelian p'-group.*

(4) *R is a direct sum of fields and $G'P$ is a Frobenius group with kernel G' and complement P.*

Proof. Let us assume that $J(RG)$ is central in RG, and distinguish between three cases.

Case 1. $J(R)=0$ and $(p, |G|)=1$: Then, RG is semi-simple (cf. [3, Theorem 1]).

Case 2. $J(R)\neq 0$: Since $J(R)G$ is contained in $J(RG)$ (cf. [3, Lemma 1]), $J(R)$ is central in R. For $0\neq j\in J(R)$ and $x, y\in G$, $j(xyx^{-1}y^{-1})-j=y(jx)x^{-1}y^{-1}-j=0$ and hence G is abelian. If p is a divisor of $|G|$, then for $1\neq x\in P$, $1-x$ is contained in $J(RG)$ (cf. [3, Theorem 2]). Hence, for $r, s\in R$, $r(s(1-x))=(s(1-x))r$ and R is commutative.

Case 3. $J(R)=0$ and $p \mid |G|$: If G is abelian, then, by making use of the same method as in Case 2, RG is commutative. Hence, we shall assume that G is not abelian. By Lemma 1, $(J(DG))_n$ is central in $(DG)_n$. Since DG is not semi-simple, $n=1$ and $J(DG)$ is central in DG. Thus, by Lemma 1, $J(FG)$ is central in FG and so, by [8, Theorem], $G'P$ is a Frobenius group with kernel G' and complement P. By [4, Theorem], $J(DG'P)=J(DP)e$, where $e=|G'|^{-1}\sum_{x' \in G} x'$. Thus, for $1 \neq x \in P$, $r, s \in D$, $(r(1-x)e)s=s(r(1-x)e)$ and D is a field. Hence, R is a direct sum of fields.

Next, we shall prove the converse. It is trivial that if one of the conditions (1), (2) is satisfied, then $J(RG)$ is central. (3) implies that $J(RG)=J(R)G$ (cf. [3, Theorem 1]) and hence $J(RG)$ is central in RG. (4) implies that $J(RG)=J(RG'P)G$ $=(J(RP)e)G=(J(RP)G)e \subseteq RGe$ (cf. [3, Theorem 1] and [4, Theorem]), where $e=|G'|^{-1} \sum_{x' \in G'} x'$. By [8, Lemma 5], RGe is central in RG. Hence, $J(RG)$ is central in RG.

The following is a generalization of [9, Theorem 1].

Theorem 5. *Let p be an odd prime. Then, $J(RG)$ is commutative if and only if one of the following conditions is satisfied :*

(1) *$J(RG)$ is central in RG.*

(2) *$J(R)$ is commutative and G is an abelian p'-group.*

(3) *$J(R)^2=0$ and G is a p'-group.*

(4) *R is a commutative ring with $J(R)^2=0$ and $G'P$ is a Frobenius group with kernel G' and complement P.*

Proof. Let us assume that $J(RG)$ is commutative, and distinguish between four cases.

Case 1. $J(R)=0$ and $(p, |G|)=1$: Then RG is semi-simple.

Case 2. $J(R)^2 \neq 0$: Then there exist two elements j, j' of $J(R)$ such that $jj' \neq 0$. Since $J(R)G$ is contained in $J(RG)$, $J(R)$ is commutative and hence, for $x, y \in G$, $jj'(xyx^{-1}y^{-1})-jj'=((j'y)(jx))x^{-1}y^{-1}-jj'=0$. Thus, G is abelian. If p is a divisor of $|G|$, then, for $1 \neq x \in P$, $r, s \in R$, $r(1-x) \cdot s(1-x)=s(1-x) \cdot r(1-x)$ and hence $(rs-sr)$ $(1-2x+x^2)=0$. Since p is odd, $rs=sr$ and R is commutative.

Case 3. $J(R)=0$ and $p \mid |G|$: If G is abelian, then by making use of the same method as in Case 2, R is commutative. Hence, we may assume that G is not abelian. Then, as in the proof of Theorem 4, R is a direct sum of division rings and $J(FG)$ is commutative. By [9, Theorem 1], $G'P$ is a Frobenius group with kernel G' and complement P. Thus, by [4, Theorem], $J(DG'P)=J(DP)e$, where $e=|G'|^{-1} \sum_{x' \in G'} x'$. For $r, s \in D$, $1 \neq x \in P$, $r(1-x)e \cdot s(1-x)e=s(1-x)e \cdot r(1-x)e$ and $(rs-sr)(1-2x+x^2)e=0$. Thus, D is commutative.

Case 4. $J(R)^2=0$ and $J(R) \neq 0$: We may assume that p is a divisor of $|G|$ and G is not abelian. Since $J(\bar{R}G)$ is commutative, $G'P$ is a Frobenius group with kernel G' and complement P. Thus, $N_G(P)=C_G(P)$ by $G'\cap N_G(P)=1$, and $|G'|$ is the number of p-Sylow subgroups of G. Hence, G is a semi-direct product of G' and $C_G(P)$. By [3, Theorem 1] and [4, Theorem], $J(RG)=J(RG'P)G=(J(R)G'P+J(RP)e)G=J(R)G+J(RP)Ge$, where $e=|G'|^{-1}\sum_{x'\in G'}x'$. Let x be an arbitary element of P different from 1. Then, for every $r, s\in R$, $r(1-x)e\cdot s(1-x)e=s(1-x)e\cdot r(1-x)e$ implies $(rs-sr)(1-2x+x^2)e=0$, which means that R is commutative.

Next, we shall prove the converse. By [3, Theorem 1], it is trivial that one of the conditions (1), (2) and (3) implies the commutativity of $J(RG)$. If (4) is satisfied, then, as was noted above, G is a semi-direct product of G' and $C_G(P)$. Moreover, $J(RG)=J(R)G+J(RP)Ge$, where $e=|G'|^{-1}\sum_{x'\in G}x'$. Noting that $C_G(P)$ is abelian, we shall easily verify the commutativity of $J(RG)$.

References

[1] C. W. CURTIS and I. REINER : *Representation theory of finite groups and associative algebras*, Wily, New York, 1962.

[2] N. JACOBSON : *Structure of rings*, Providence, 1956.

[3] K. MOTOSE : On group rings over semi-primary rings, Math. J. Okayama Univ. 14 (1969), 23–26.

[4] K. MOTOSE : On the radical of a group ring, Math. J. Okayama Univ. 15 (1971), 35–36.

[5] D. S. PASSMAN : Infinite group rings, Dekker, 1971.

[6] D. A. R. WALLACE : Note on the radical of a group algebra, Proc. Cambridge Philos. Soc. 54 (1958), 128–130.

[7] D. A. R. WALLACE : Group algebras with radicals of square zero, Proc. Glasgow Math. Assoc. 5 (1962), 158–159.

[8] D. A. R. WALLACE : Group algebras with central radicals, Proc. Glasgow Math. Assoc. 5 (1962), 103–108.

[9] D. A. R. WALLACE : On the commutativity of the radical of a group algebra, Proc. Glasgow Math. Assoc. 7 (1965), 1–8.