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Remark to a Tauberian Theorem for Borel Summability

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It is shown in Hardy's book [1] as (a Tauberian) Theorem 147 that if a series

$$\sum_{n=1}^{\infty} a_n, \tag{1}$$

of finite order is summable by Borel's method, then it is also summable by Cesàro's method. On the other hand Hardy shows by an example that the converse of the above theorem does not hold in general ([1] p. 213).

But we have a Tauberian Theorem to ensure the converse; i. e.

Theorem A ([1] Th. 149)

Let us denote by σ_n *the* $(C, 1)$ *means of* (1). *If*

$$\sigma_n = s + o\left(\frac{1}{\sqrt{n}}\right),$$

then (1) *is summable* (B) *to* s.

Hardy at a time shows by the same example quoted above that this is a best possible theorem in the sense that the o in the theorem cannot be replaced by O. The example of Hardy is artificial but its series is unbounded and its general term a_n does not tend to 0.

The object of this paper is to prove the theorem below by showing a stronger example than Hardy's on appealing to a deep Tauberian Theorem.

Theorem B *In Theorem A we cannot replace the* o *by* O *even if the partial sum of* (1) *is bounded and* $a_n \to 0$ *as* $n \to \infty$.

Proof. First we prove that there exist series (1) such that

$$\sigma_n = O\left(\frac{1}{\sqrt{n}}\right),$$

$$s_n = O(1),$$

and $$a_n \to 0,$$

but are not summable (B), where s_n denotes the partial sum of (1).

Our example is

$$a_n = \sin(\sqrt{n}\,\theta) - \sin(\sqrt{n-1}\,\theta), \tag{2}$$

where $\theta \neq 0$ is any real constant.

For this a_n we have plainly

$$s_n = \sin(\sqrt{n}\,\theta) = O(1), \tag{3}$$

$$a_n = O\left(\frac{1}{\sqrt{n}}\right), \tag{4}$$

and

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} \sin(\sqrt{k}\,\theta). \tag{5}$$

But for instance Euler's summation formula ([1] p. 318) shows

$$\sum_{k=1}^{n} \sin(\sqrt{k}\,\theta) = \int_1^n \sin(\sqrt{t}\,\theta)\,dt + O(1)$$
$$+ \int_1^n (t - [t])\frac{\theta}{2\sqrt{t}}\cos(\sqrt{t}\,\theta)\,dt$$
$$= O(\sqrt{n}) + O(1) + O(\sqrt{n}) = O(\sqrt{n}),$$

and hence

$$\sigma_n = O\left(\frac{1}{\sqrt{n}}\right).$$

On the other hand Theorem 156 of [1] shows that if our series is summable (B), then it must be convergent in virtue of (4). But this is clearly a contradiction. Thus our example (2) proves the truth of Theorem B.

Finally we remark that one of more 'natural' examples than (2) may be

$$a_n = \frac{\sin(\sqrt{n}\,\theta)}{\sqrt{n}} \quad \text{or} \quad \frac{\cos(\sqrt{n}\,\theta)}{\sqrt{n}},$$

but the proof is then more difficult [2], and that it is possible, though more troublesome, to prove the non Borel-summability of our series for (2) without depending upon Theorem 156 of [1].

References

[1] HARDY, G. H. : *Divergent Series*, (Oxford, 1949).
[2] KANO, T. : On the behaviour of some exponential series, (to appear).