

Generalized Tangents of Curves and Generalized Vector Fields

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(Received October 30, 1971)

Introduction

The main purpose of this paper is to introduce the notion of generalized tangent of a curve γ given by $\varphi: \mathbf{I} \rightarrow M$, where M is an n -dimensional Paracompact topological manifold with a (fixed) metric ρ . Here ρ is assumed to satisfy (*). *If $\rho(x_1, x_2) \leq 1$, then there exists unique curve γ of M which joins x_1 and x_2 and*

$$\int_{\gamma} \rho = \rho(x_1, x_2).$$

(For the existence of such metric, see [4]). In the rest, we set

$$S_x = \{y \mid \rho(x, y) = 1\}.$$

By assumption, S_x is homeomorphic to S^{n-1} , the unit $(n-1)$ -sphere. Then the generalized tangent of γ at $a = \varphi(0)$ is defined to be a positive Radon measure on S_a and we show that for any positive Radon measure ξ on S_a , there exists a curve γ on M whose generalized tangent at a is ξ (§ 3, theorem 3).

More Precisely, to define the generalized tangent of γ , first we introduce the notion of $\mathcal{F}(S^{n-1})$ -smooth function at a , where $\mathcal{F}(S^{n-1})$ is a (fixed) function space on S^{n-1} such as $C(S^{n-1})$, $L^p(S^{n-1})$ (the measure on S^{n-1} is the standard volume element, that is given by $\sum_i x_i dx_i$ (cf. [5], [11])) or (if M is smooth or real analytic) $C^\infty(S^{n-1})$ or $C^0(S^{n-1})$, as follows: *A function f defined on some neighborhood of a is called to be $\mathcal{F}(S^{n-1})$ -smooth if f is written as*

$$f(x) = f(a) + g(\varepsilon_{a,x})\rho(a, x) + o(\rho(a, x)), \quad \rho(a, x) < 1,$$

and $g(y)$ belongs in $\mathcal{F}(S_a)$. Here $\varepsilon_{a,x}$ means the point y on S_a such that

$$x \in r_{a,y},$$

where $r_{a,y}$ is the curve of M which joins a and y and $\int_{r_{a,y}} \rho = 1$, and $\mathcal{F}(S_a)$ means the function space on S_a defined similarly as (using the measure induced from ρ (cf.

[3], [4])) $\mathcal{F}(S^{n-1})$.

Then the generalized tangent of γ at a is defined to be the element ξ of $\mathcal{F}^*(S_a)$, the dual space of $\mathcal{F}(S_a)$ which is determined by

$$\langle \xi, g \rangle = \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{1}{t} \{f(\rho(t)) - f(a)\} dt \right],$$

where γ is given by $\rho: I \rightarrow M$ and ρ is assumed to satisfy

- (i) $\rho(0) = a, \rho(t) \neq a, \text{ if } t \neq 0,$
- (ii) $\rho(a, \rho(t)) = \rho(t),$

f is an $\mathcal{F}(S^{n-1})$ -smooth function at a and $g \in \mathcal{F}(S_a)$ is given by

$$g(y) = \lim_{t \rightarrow 0} \frac{1}{t} (f(r_{a,y,t}) - f(a)),$$

where $r_{a,y,t}$ is given by

$$r_{a,y,t} \in r_{a,y}, \rho(a, r_{a,y,t}) = t.$$

We denote g by $d_\rho f$ or, $d_\rho f(a)$ or d_ρ, af .

We note that this definition of the generalized tangent depends on the choice of parameter t of γ (cf. n°11 of §3).

If $\mathcal{F}(S^{n-1})$ is taken to be $C(S^{n-1})$, the Banach space consisted by the continuous functions on S^{n-1} with the uniform convergence topology, then $C^*(S^{n-1})$ is the space of Radon measures on S^{n-1} (cf. [18]), and we can show that an element of $C^*(S_a)$ is expressed as a generalized tangent at a of a curve if and only if it is positive, that is $\langle \xi, g \rangle \geq 0$ if $g(y) \geq 0$ on S_a . For example, the Dirac measure on S_a is expressed as the generalized tangent at a of a curve γ which is smooth at a . Here a curve γ given by $\rho: I \rightarrow M, \rho(0) = a$, is called smooth at a if $\lim_{t \rightarrow 0} \varepsilon_{a,\varphi t}$ exists. The problem to characterize the element of $C^{\infty*}(S^{n-1})$, the space of distributions on S^{n-1} or $C^{\omega*}(S^{n-1})$, the space of analytic functionals on S^{n-1} , which is expressed as the generalized tangent of some curve, remains open.

We note that although the O.N. -basis of $L^2(S^{n-1})$ is given by spherical harmonics (cf. [5], [11]), a smooth function at a only represents a spherical function of degree 1. Hence, since the usual tangent of a smooth curve is defined only by using smooth functions, the usual tangents of smooth curves corresponds only this part of $L^2(S^{n-1})$. But the above result shows, if we use the $L^2(S^{n-1})$ -smooth functions, the generalized tangents covers the positive part of $L^2(S^{n-1})$.

As in the case of usual tangent vectors (cf. [6], [13]), to set

$$\mathfrak{X}_\varphi(f) = \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{1}{t} \{f(\varphi(t)) - f(a)\} dt \right],$$

where f is $\mathcal{F}(S^{n-1})$ -smooth at a , we have

$$(i) \quad \mathfrak{X}_\varphi(\alpha f_1 + \beta f_2) = \alpha \mathfrak{X}_\varphi(f_1) + \beta \mathfrak{X}_\varphi(f_2),$$

$$(ii) \quad \mathfrak{X}_\varphi(f_1 f_2) = f_1(a) \mathfrak{X}_\varphi(f_2) + f_2(a) \mathfrak{X}_\varphi(f_1),$$

and we also have

$$(iii) \quad \mathfrak{X}_\varphi(f) = 0 \text{ if } |f(x) - f(a)| = o(\rho(a, x)),$$

$$(iv) \quad \mathfrak{X}_\varphi(f) \geq 0 \text{ if } d_\rho f \geq 0.$$

On the other hand, if the map \mathfrak{X} from the space of $\mathcal{F}(S^{n-1})$ -smooth functions at a to \mathbf{R} , the real number field, satisfies (i), (ii) and (iii), then \mathfrak{X} is written as

$$f = \langle \xi, d_\rho f(a) \rangle,$$

by some $\xi \in \mathcal{F}^*(S_a)$. Hence if \mathfrak{X} also satisfies (iv), \mathfrak{X} is written as \mathfrak{X}_φ by some $\varphi : I \rightarrow M$. In some part, the globalization of these discussions are possible. To do this, first we construct the associate $\mathcal{F}(S^{n-1})$ -bundle of the tangent micro-bundle of M , which is denoted by $\mathcal{F}(s(M))$ and its dual bundle, which is denoted by $\mathcal{F}^*(s(M))$ (§ 1). (cf. [1], [9], [12]).

Next, we set

$$d_\rho f(x, y) = \lim_{t \rightarrow 0} \frac{1}{t} (f(r_{x,y,t}) - f(x)), \quad y \in S_x.$$

If $d_\rho f(x)$ is a continuous cross-section of $\mathcal{F}(s(M))$, then we call f is $\mathcal{F}(S^{n-1})$ -smooth on M (for $n = 1$, cf. [7], [8], [10]). We can show that the space of $\mathcal{F}(S^{n-1})$ -smooth functions on M (denoted by $C_{\mathcal{F}(S^{n-1})}(M)$) is dense in $C(M)$ or in $L^p_{loc}(M)$ (§ 2, theorem 1). (The measure on M by which $L^p(M)$ or $L^p_{loc}(M)$ is defined, is that of induced from ρ (cf. [3], [4])). Then a linear operator X of $C(M)$ which satisfies the following (i), (ii), (iii) is called an $\mathcal{F}(S^{n-1})$ -vector field on M .

$$(i). \quad X \text{ is a closed operator from } C_{\mathcal{F}(S^{n-1})}(M) \text{ into } C(M).$$

$$(ii). \quad (Xf)(a) = 0, \text{ if } |f(x) - f(a)| = o(\rho(a, x)) \text{ at } a.$$

$$(iii). \quad X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1).$$

We show that if X is an $\mathcal{F}(S^{n-1})$ -vector field on M , then X is written as

$$Xf(x) = \langle \xi(x), d_\rho f(x) \rangle, \quad x \in M,$$

where ξ is a continuous cross-section of $\mathcal{F}^*(s(M))$ (§ 2, theorem 2). Therefore, as usual vector field, we may identify X and a continuous cross-section of $\mathcal{F}^*(s(M))$. But an $\mathcal{F}(S^{n-1})$ -vector field X does not generate a 1-parameter group germ of M in general. For example, the theorem of Hille-Yosida shows that if M is compact and simply connected, the $C(S^{n-1})$ -vector field X corresponds to the cross-section m of $C^*(s(M))$ given by $m = m(x)$, $m(x)$ is the canonical measure on

S_x defined from the metric, does not generate any (equi-continuous) 1-parameter semi group of $C(M)$ or $L^p(M)$ (§ 2, exemple). (cf. [17], [18]). We note since $m(x)$ is positive, there exists a curve $\gamma = \gamma_x$ for any x , such that γ_x starts from x and whose generalized tangent at x is $m(x)$ (cf. § 3, exemple 2), if $n = 2$, γ_x is given by $r\theta = 1$.

As usual vector field, if X, Y are $\mathcal{F}(S^{n-1})$ -vector fields such that their compositions XY and YX are both possible, then

$$[X, Y] = XY - YX$$

is also an $\mathcal{F}(S^{n-1})$ -vector field of M . But the composition of $\mathcal{F}(S^{n-1})$ -vector fields may not be possible in general.

In § 1, we also construct associate $\mathcal{F}(\overbrace{S^{n-1} \times \dots \times S^{n-1}}^p)$ -bundle of the tangent microbundle of M . It is denoted by $\mathcal{F}(s^p(M))$. We denote by $A\mathcal{F}(s^p(M))$ the subbundle of $\mathcal{F}(s^p(M))$ whose fibre is consisted by those functions $f(y_1, \dots, y_p)$, $y_i \in S^{n-1}$, of $\mathcal{F}(S^{n-1} \times \dots \times S^{n-1})$ such that

$$f(y_{\sigma(1)}, \dots, y_{\sigma(p)}) = \text{sgn}(\sigma) f(y_1, \dots, y_p), \quad \sigma \in \gamma^p.$$

The cross-sections of these bundles are considered to be reductions of Alexander-Spanier cochains (cf. [1], [3], [14], [15]).

For the cross-sections of $\mathcal{F}(s^p(M))$ and $A\mathcal{F}(s^p(M))$, we define the maps d_ρ and Ad_ρ by

$$\begin{aligned} & d_\rho f(x, y_1, \dots, y_{p+1}) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} [f(r_{x, y_1, t}, y_2, \dots, y_{p+1}) - f(x, y_2, \dots, y_{p+1})], \\ & Ad_\rho f(x, y_1, \dots, y_{p+1}) \\ &= \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{i+1} \left[\lim_{t \rightarrow 0} \frac{1}{t} (f(r_{x, y, it}, y_1, \dots, y_{i+1}, \dots, y_{p+1}) - \right. \\ & \quad \left. f(x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{p+1})) \right]. \end{aligned}$$

We call f is $\mathcal{F}(S^{n-1})$ -smooth if $d_\rho f$ (or $Ad_\rho f$) defines a continuous cross-section of $\mathcal{F}(s^{p+1}(M))$ (or $A\mathcal{F}(s^{p+1}(M))$). We note that to define

$\int_\gamma f(x, y_1, \dots, y_p)$ by

$$\begin{aligned} & \int_\gamma f(x, y_1, \dots, y_p) \\ &= \int_\gamma f(x, \varepsilon_{x, x_1}, \dots, \varepsilon_{x, x_p}) \rho(x, x_1) \cdots \rho(x, x_p), \end{aligned}$$

where γ is a singular p -chain of M and the right hand side is the integration of

Alexander-Spanier cochain defined in [3], $\int_{\gamma} f$ is exists if f is $\mathcal{F}(S^{n-1})$ -smooth and γ is given by $\varphi : I^p \rightarrow M$ where φ satisfies

$$\begin{aligned} \rho(\varphi(a_{j+1i}), \varphi(a_j)) &\leq N |a_{j+1i} - a_{ji}|, \\ a_j &= (a_{j1}, \dots, a_{jp}), \quad a_{j+1i} = (a_{j1}, \dots, a_{ji-1}, a_{ji+1}, a_{j+1i}, \dots, a_{jp}). \end{aligned}$$

for some $N > 0$. Since $Ad_{\rho}(Ad_{\rho}f) = 0$ if $Ad_{\rho}f$ is $\mathcal{F}(S^{n-1})$ -smooth, we can obtain the analogy of de Rham's theorem by using the cross-sections of $A\mathcal{F}(s^p(M))$ and the Cech cohomology group of M . But the above shows that the analogy of de Rham's theorem is also obtained by using the singular homology group of M (cf. [15], [16]).

We note that if $M = \mathbf{R}^1$, the 1-dimensional euclidean space with the euclidean metric, then

$$d_{\rho}f(x) = (D_+f(x), D_-f(x)),$$

where D_+ and D_- mean the right hand side and the left hand side derivations of f and the (fibre of $\mathcal{F}(s(\mathbf{R}^1))$ is $\mathbf{R} \oplus \mathbf{R}$. We know that f is smooth if and only if $D_+f = D_-f$ at any point of \mathbf{R}^1 , that is $d_{\rho}f$ defines a cross-section of the subbundle of $\mathcal{F}(s(\mathbf{R}^1))$ whose fibre is the diagonal of $\mathbf{R} \oplus \mathbf{R}$.

To generalize this, first we assume the metric ρ of M satisfies (*). If $\rho(x_1, x_2) \leq 2$, then there is unique path γ which joins x_1 and x_2 and

$$\int_{\gamma} \rho = \rho(x_1, x_2).$$

Under this assumption, for any $y \in S_x$, there exists unique point \hat{y} of S_x such that

$$\rho(y, \hat{y}) = 2.$$

We denote the quotient space of S_x obtained by identifying \hat{y} and y by P_x . By definition, P_x is homeomorphic to $\mathbf{R}P^{n-1}$, the $(n-1)$ -dimensional real projective space.

For this P_x , if f is $\mathcal{F}(S^{n-1})$ -smooth at x and

$$d_{\rho}f(x, \hat{y}) = d_{\rho}f(x, y),$$

for any $y \in S_x$, then $d_{\rho}f$ may be considered to be an element of $\mathcal{F}(P_x)$. Here $\mathcal{F}(P_x)$ is defined similarly as $\mathcal{F}(S_x)$ and it is also considered to be a subspace of $\mathcal{F}(S_x)$ given by

$$\mathcal{F}(P_x) = \{g | g \in \mathcal{F}(S_x), g(y) = g(\hat{y})\}.$$

Since $\mathcal{F}(P_x)$ is isomorphic to $\mathcal{F}(RP^{n-1})$, we call f to be $\mathcal{F}(RP^{n-1})$ -smooth in this case. If M is \mathbf{R}^n , the n -dimensional euclidean space with the euclidean

metric, then f is $M(S^{n-1})$ -smooth at x if and only if f is one-sided differentiable at x along any line which ends at x and f is $M(\mathbf{R}P^{n-1})$ -smooth at x if and only if f is differentiable at x along any line which through x .

Since the total spaces of $s(M)$ and $s^p(M)$, the associate S^{n-1} and $\overbrace{S^{n-1} \times \cdots \times S^{n-1}}^p$ bundles of the tangent microbundle of M are given by

$$\begin{aligned} s(M) &= \{(x, y) \mid \rho(x, y) = 1, x \in M, (x, y) \in M \times M, \\ s^p(M) &= \{(x, y_1, \dots, y_p) \mid \rho(x, y_i) = 1, i = 1, \dots, p, x \in M, \\ &\quad (x, y_1, \dots, y_p) \in M \times \overbrace{M \times \cdots \times M}^p\}, \end{aligned}$$

we can construct the associate $\mathbf{R}P^{n-1}$ -bundle and $\overbrace{\mathbf{R}P^{n-1} \times \cdots \times \mathbf{R}P^{n-1}}^p$ -bundle of $\tau(M)$, the tangent microbundle of M , by taking $s(M)/\sim$ and $s^p(M)/\underset{p}{\sim}$ to be the total spaces. Here the equivalence relations \sim or $\underset{p}{\sim}$ are given by

$$\begin{aligned} (x, y) \underset{p}{\sim} (x', y') &\text{ if and only if } x = x' \text{ and } \rho(y, y') = 2, \\ (x, y_1, \dots, y_p) \underset{p}{\sim} &(x', y_1', \dots, y_p'), \\ &\text{ if and only if } x = x' \text{ and } \rho(y_i, y_i') = 2, i = 1, \dots, p, \end{aligned}$$

where ρ is assumed to satisfy (*). Then using $s(M)/\sim$ and $s^p(M)/\underset{p}{\sim}$, we can construct associate $\mathcal{F}(\mathbf{R}P^{n-1})$ -bundle and $\overbrace{\mathcal{F}(\mathbf{R}P^{n-1} \times \cdots \times \mathbf{R}P^{n-1})}^p$ -bundle of $\tau(M)$. They are denoted by $\mathcal{F}(s(M)/\sim)$ and $\mathcal{F}(s^p(M)/\underset{p}{\sim})$. We note that since we have

$$\begin{aligned} \mathcal{F}(S^{n-1}) &= \mathcal{F}(\mathbf{R}P^{n-1}) \oplus \check{\mathcal{F}}(\mathbf{R}P^{n-1}), \\ \mathcal{F}(\mathbf{R}P^{n-1}) &= \{g \mid g(y) = g(\hat{y})\}, \quad \check{\mathcal{F}}(\mathbf{R}P^{n-1}) = \{g \mid g(y) = -g(\hat{y})\}, \end{aligned}$$

we may consider $\mathcal{F}(s(M)/\sim)$ and $\mathcal{F}(s^p(M)/\underset{p}{\sim})$ are the subbundles of $\mathcal{F}(s(M))$ and $\mathcal{F}(s^p(M))$ and can be considered to be direct summands of them.

We note that using $\mathcal{F}(\mathbf{R}P^{n-1})$ -smooth functions and the bundles $\mathcal{F}(s(M)/\sim)$, $\mathcal{F}(s^p(M)/\underset{p}{\sim})$ and $A\mathcal{F}(s^p(M)/\underset{p}{\sim})$ ($A\mathcal{F}(s^p(M)/\underset{p}{\sim})$ is defined similarly as others), we can construct same theories as above.

Similarly, if $M = \mathbf{C}$, the complex number plane with the euclidean metric, f a holomorphic function, then

$$d_\rho f(a, y) = \frac{df}{dz}(a).$$

This suggests that if $\dim M = 2m$, then the condition (**), there exists associate $\mathbf{C}P^{m-1}$ -bundle of $\tau(M)$, may have some meaning for M .

The outline of this paper is as follows: In §1, we define the bundles

$\mathcal{F}(s)M$), $\mathcal{F}(s^p(M))$ and $A\mathcal{F}(s^p(M))$ and treat their properties. In § 2, we define $\mathcal{F}(S^{n-1})$ -smooth functions and $\mathcal{F}(S^{n-1})$ -vector fields. The generalized tangents of curves and their properties are stated in § 3.

Added in proof. In \mathbf{R}^n with the euclidean metric, $d_\rho f$ may be considered (one-sided) Gâteaux's differential Vf . Here Gâteaux's differential $Vf(x_0, h)$ is defined by

$$Vf(x_0, h) = \lim_{t \rightarrow 0} \frac{f(x_0 + th) - f(x_0)}{t}, \quad h \in M_2,$$

where f is a map from a Banach space M_1 to a Banach space M_2 . For the details and related notions with their applications, see Burysek, S. : *On symmetric G-differential and convex functionals in Banach spaces*, *Publ. Math., (Debrecen)*, 17, 1970, 145-161).

§ 1. Bundles $\mathcal{F}(s(M))$ and $\mathcal{F}(s^p(M))$.

1. We denote by M an n -dimensional connected paracompact topological manifold. On M , we fix a metric ρ by which the topology of M is given, and assume ρ satisfies the following (i), (ii), (iii) (For the existence of such metric, see [4]).

(i). *If $\rho(x_1, x_2) \leq 1$, then there exists unique path γ which joins x_1 and x_2 and*

$$\int_\gamma \rho = \rho(x_1, x_2).$$

(ii). *M is complete with respect to ρ .*

(iii). *The measure $m(\rho)$ induced from ρ on M is a positive Radon measure and satisfies*

$m(\rho)(E) \neq 0$, if E is measurable and contains some non empty open set.

For $x \in M$, we set

$$S_x = \{y \in M, \rho(x, y) = 1\}.$$

Since $\dim M = n$, S_x is homeomorphic to S^{n-1} , the unit $(n-1)$ -sphere (cf. [4]). We assume that for any x , ρ induces a metric ρ_x on S_x which is given by

$$\rho_x(y_1, y_2) = \inf_{\gamma, \gamma \text{ joins } y_1 \text{ and } y_2 \text{ in } S_x} \int_\gamma \rho.$$

The measure on S_x induced from ρ_x (cf. [3]) is denoted by $m = m(x)$. For this $m(x)$, we assume (cf. [4])

(i). *A Borel set of S_x is $m(x)$ -measurable and if E is $m(x)$ measurable and contains some non-empty open set of S_x , then*

$$m(x)(E) \neq 0.$$

(ii). $m(x)$ depends continuously on x .

Since S_x is compact, $m(x)(S_x)$ is finite. Hence, for the simplicity, we normalize $m(x)$ to satisfy $m(x)(S_x) = 1$.

Note. If M is a smooth manifold, ρ is the geodesic distance defined by a (complete) Riemannian metric on M , then $m(x)$ depends differentiably on x .

In $M \times M$, we set

$$(1) \quad s(M) = \{(x, y) | x \in M, \rho(x, y) = 1\}.$$

We define $\pi: s(M) \rightarrow M$ by $\pi(x, y) = x$. Then $\{s(M), \pi, M\}$ is the associate unit sphere bundle of the tangent microbundle of M (cf. [1], [9], [12]). We denote the transition function of $s(M)$ by $\{g_{UV}(x)\}$ if we consider the fibre of $s(M)$ at x to be S_x . We note that if we consider the fibre of $s(M)$ at x to be the measure space (S_x, m_x) , then the transition function of $s(M)$ should be replaced by $\{(g_{UV}(x), m_U(x)(g_{UV}(x)^*m_V(x))^{-1})\}$, where $m_U(x)$ is given by

$$m(x)(E) = \int_{h_{U,x}(E)} m_U(x) d\Omega.$$

Here $h_{U,x}$ is the local homeomorphism from $\pi^{-1}(U)$ to $U \times S^{n-1}$ and $d\Omega$ is the standard measure on S^{n-1} .

We denote by $\mathcal{F}(S^{n-1})$ a function space over S^{n-1} . In the rest, (S^{n-1}) means either of $C(S^{n-1})$ or $L^p(S^{n-1})$, $1 \leq p \leq \infty$, regarding them to be Banach spaces. Here $L^p(S^{n-1})$ is defined by $d\Omega$. (If M is smooth, or real analytic, then $C^\infty(S^{n-1})$, or $C^\omega(S^{n-1})$, is also taken as (S^{n-1})). Then by identifying $U \times C(S^{n-1}) \ni (x, f(y))$ and $(x, f(g_{UV}(x)y)) \in V \times C(S^{n-1})$, $x \in U \cap V$, we obtain the associate $C(S^{n-1})$ -bundle of $s(M)$. It is denoted by $C(s(M))$. Since $C(s(M))$ is a vector bundle over M with the fibre $C(S^{n-1})$, its dual bundle $C^*(s(M))$ is defined. $C^*(s(M))$ is a vector bundle over M with the fibre $C^*(S^{n-1})$, where $C^*(S^{n-1})$ is the space of Radon measures on S^{n-1} .

Lemma 1. *Regarding $m(x)$ to be a function on M , $m(x)$ is a cross-section of $C^*(s(M))$.*

Corollary. *We have*

$$(2) \quad m_U(x)(g_{UV}(x)^*m_V(x))^{-1} = 1.$$

By this corollary, although $\mathcal{F}(S^{n-1})$ is $L^p(S^{n-1})$, we can construct the associate $\mathcal{F}(S^{n-1})$ -bundle of $s(M)$ by identifying $U \times \mathcal{F}(S^{n-1}) \ni (x, f(y))$ and $(x, g_{UV}(x)^*f(y)) \in V \times \mathcal{F}(S^{n-1})$, $x \in U \cap V$. This bundle is denoted by $\mathcal{F}(s(M))$. The dual bundle of $\mathcal{F}(s(M))$ is denoted by $\mathcal{F}^*(s(M))$. By definition, the fibre of $\mathcal{F}^*(s(M))$ is $\mathcal{F}^*(S^{n-1})$, the dual space of $\mathcal{F}(S^{n-1})$. We denote the fibre of $\mathcal{F}(s(M))$ (and $\mathcal{F}^*(s(M))$) at x by $\mathcal{F}(S_x)$ (and $\mathcal{F}^*(S_x)$).

Definition. *An element of $\mathcal{F}^*(S_x)$ is called an $\mathcal{F}(S^{n-1})$ -vector at x .*

Note. If we regard S_x to be a measure space $(S_x, k(x))$, and define $L^p(S_x)$ by $k(x)$, then to define $K_U(x)$ similarly as $m_U(x)$, we obtain the associate $L^p(S^{n-1})$ -bundle of $s(M)$ by identifying $U \times L^p(S^{n-1}) \ni (x, f(y))$ and $(x, [k_U(x) (g_{UV}(x))^* k_V(x)]^{-1/p} g_{UV}(x)^* f(y)) \in V \times L^p(S^{n-1})$, $x \in U \cap V$.

2. In $\overline{M \times M \times \cdots \times M}^p$, we set

$$(1)' \quad s^p(M) = \{(x, y_1, \dots, y_p) \mid x \in M, \rho(x, y_i) = 1, i = 1, \dots, p\}.$$

To define $\pi : s^p(M) \rightarrow M$ by $\pi(x, y_1, \dots, y_p) = x$, $\{s^p(M), \pi, M\}$ is associate $\overline{S^{n-1} \times \cdots \times S^{n-1}}^p$ -bundle over M . If the fibre of $s(M)$ at x is considered to be the measure space $(S_x, m(x))$, then we consider the fibre of $s^p(M)$ at x to be the measure space $(S_x \times \cdots \times S_x, m(x) \otimes \cdots \otimes m(x))$. The transition functions $\{g_{UV}(x)\} = \{g_{UV}^p(x)\}$ of $s^p(M)$ is given by

$$g_{UV}^p(x)(y_1, \dots, y_p) = (g_{UV}(x)y_1, \dots, g_{UV}(x)y_p),$$

where $g_{UV}(x)$ in the right hand side is the transition function of $s(M)$.

We denote by $\overline{\mathcal{F}(S^{n-1} \times \cdots \times S^{n-1})}^p$ or $\mathcal{F}^p(S^{n-1})$ the function space over $S^{n-1} \times \cdots \times S^{n-1}$ which is of the same type with $\mathcal{F}(S^{n-1})$. That is $\mathcal{F}^p(S^{n-1})$ means either of $C(S^{n-1} \times \cdots \times S^{n-1})$ or $L^p(S^{n-1} \times \cdots \times S^{n-1})$ with the measure $m(x) \otimes \cdots \otimes m(x)$ in general and $C^\infty(S^{n-1} \times \cdots \times S^{n-1})$ or $C^w(S^{n-1} \times \cdots \times S^{n-1})$ is also considered if M is smooth or real analytic. By assumption, $\overline{\mathcal{F}(S^{n-1}) \otimes \cdots \otimes \mathcal{F}(S^{n-1})}^p$ is dense in $\mathcal{F}(S^{n-1} \times \cdots \times S^{n-1})$.

As $\mathcal{F}(s(M))$, we construct the associate $\mathcal{F}^p(S^{n-1})$ -bundle of $s^p(M)$. It is denoted by $\mathcal{F}(s^p(M))$. The dual bundle of $\mathcal{F}(s^p(M))$ is denoted by $\mathcal{F}^*(s^p(M))$. The fibres of $\mathcal{F}(s^p(M))$ and $\mathcal{F}^*(s^p(M))$ at x are denoted by $\mathcal{F}(S_x \times \cdots \times S_x)$ or $\mathcal{F}^p(S_x)$ and $\mathcal{F}^*(S_x \times \cdots \times S_x)$ or $\mathcal{F}^{p*}(S_x)$.

Definition. An element of $\mathcal{F}^{p*}(S_x)$ is called an $\mathcal{F}(S^{n-1})$ - p -vector at x .

For any $f \in \mathcal{F}^p(S^{n-1})$ and $\sigma \in \mathcal{F}^p$, we set

$$(3) \quad \sigma(f)(y_1, \dots, y_p) = f(y_{\sigma(1)}, \dots, y_{\sigma(p)}), \quad y_i \in S^{n-1}.$$

Then, since $\overline{\mathcal{F}(S^{n-1}) \otimes \cdots \otimes \mathcal{F}(S^{n-1})}$ is dense in $\mathcal{F}^p(S^{n-1})$, σ is continuous. Therefore, setting

$$A\mathcal{F}^p(S^{n-1}) = \{f \mid f \in \mathcal{F}^p(S^{n-1}), \sigma(f) = \text{sgn}(\sigma)f\},$$

$A\mathcal{F}^p(S^{n-1})$ is a closed subspace of $\mathcal{F}^p(S^{n-1})$. Since σ^* , the adjoint operator of σ , is σ^{-1} , we have

$$(A\mathcal{F}^p(S^{n-1}))^* = A\mathcal{F}^{p*}(S^{n-1}).$$

As we know

$$\begin{aligned} & \sigma(f(g_{UV}(x)y_1, \dots, g_{UV}(x)y_p)) \\ & = f(g_{UV}(x)y_{\sigma 1}, \dots, g_{UV}(x)y_{\sigma p}), \quad f \in \mathcal{F}^p(S^{n-1}), \end{aligned}$$

we obtain an $A\mathcal{F}^p(S^{n-1})$ -bundle over M to be a subbundle of $\mathcal{F}^p(s(M))$. This bundle is denoted by $A\mathcal{F}(s^p(M))$. Its dual bundle is denoted by $A\mathcal{F}^*(s^p(M))$. The fibres of $A\mathcal{F}(s^p(M))$ and $A\mathcal{F}^*(s^p(M))$ are denoted by $A\mathcal{F}^p(S_x)$ and $A\mathcal{F}^{p*}(S_x)$.

Note. Similarly, to set

$$S\mathcal{F}^p(S^{n-1}) = \{f \mid f \in \mathcal{F}^p(S^{n-1}), \sigma(f) = f\},$$

we can define an $S\mathcal{F}^p(S^{n-1})$ -bundle $S\mathcal{F}(s^p(M))$ to be a subbundle of $\mathcal{F}(s^p(M))$. Its dual bundle is denoted by $S\mathcal{F}^*(s^p(M))$. The fibres of $S\mathcal{F}(s^p(M))$ and $S\mathcal{F}^*(s^p(M))$ at x are denoted by $S\mathcal{F}^p(S_x)$ and $S\mathcal{F}^{p*}(S_x)$.

Definition. A (continuous) cross-section φ of $\mathcal{F}(s^p(M))$ is called a (continuous) $\mathcal{F}(S^{n-1})$ - p -cochain of M . If φ is a cross-section of $A\mathcal{F}(s^p(M))$, then φ is called an $\mathcal{F}(S^{n-1})$ - p -form of M .

Definition. A (continuous) cross-section of $\mathcal{F}^*(s^p(M))$ is called an $\mathcal{F}(S^{n-1})$ - p -vectorfield of M .

In general, we call an element of $\mathcal{F}^p(S_x) \otimes \mathcal{F}^{q*}(S_x)$ to be an $\mathcal{F}(S^{n-1})$ - (p, q) -tensor at x and a continuous cross-section of $\mathcal{F}(s^p(M)) \otimes \mathcal{F}^*(s^q(M))$ to be an $\mathcal{F}(S^{n-1})$ - (p, q) -tensorfield of M .

If M is smooth (or real analytic), then $\mathcal{F}(s^p(M))$ and $\mathcal{F}^*(s^q(M))$ allow the structure of smooth (or real analytic) vector bundles. Hence we can define smooth (or real analytic) $\mathcal{F}(S^{n-1})$ - p -cochain, etc..

3. We denote by $r_{x,y}$ the unique curve which joins x and y , $y \in S_x$ and satisfies

$$\int_{r_{x,y}} \rho = 1.$$

Then for any a , $0 \leq a \leq 1$, there exists unique point z in $r_{x,y}$ such that $\rho(x, z) = a$. We denote this z by $r_{x,y,a}$.

On the other hand, if $\rho(x, z) < 1$, then there exists unique point y of S_x such that $z \in r_{x,y}$. Or, in other word, x, z determines a point y of S_x . We denote this y by $\varepsilon_{x,z}$.

By definition, if $\rho(x, z) < 1$, then

$$(4) \quad r_{x, \varepsilon_{x,z}, \rho(x,z)} = z.$$

For an $\mathcal{F}(S^{n-1})$ - p -cochain $\varphi = \varphi(x, y_1, \dots, y_p)$ of M , we set

$$(5) \quad \begin{aligned} & \tilde{\varphi}(x, x_1, \dots, x_p) \\ & = \varphi(x, \varepsilon_{x,x_1}, \dots, \varepsilon_{x,x_p}) \rho(x, x_1) \cdots \rho(x, x_p), \end{aligned}$$

$$x_i \in M, \rho(x, x_i) < 1, i = 1, \dots, p.$$

Then $\tilde{\varphi}$ defines an Alexander-Spanier p -cochain of M . By definition, if φ is an $\mathcal{F}(S^{n-1})$ - p -form, then $\tilde{\varphi}$ is alternative in x_1, \dots, x_p .

Definition. If γ is a singular p -chain of M , then we define the integration $\int_{\gamma} \varphi$ of φ , an $\mathcal{F}(S^{n-1})$ - p -cochain of M on γ by

$$(6) \quad \int_{\gamma} \varphi = \int_{\gamma} \tilde{\varphi}.$$

Here the right hand side means the integral of the Alexander-Spanier cochain $\tilde{\varphi}$ on γ ([3]).

By the definition of the integral (cf. [3]), if φ is a $C(S^{n-1})$ - p -cochain and γ is given by $f: I^p \rightarrow M$ where f satisfies

$$(7) \quad \begin{aligned} \rho(f(a_{J+1_i}), f(a_J)) &\leq N |a_{j_i+1} - a_{j_i}|, \\ a_{J+1_i} &= (a_{j_1}, \dots, a_{j_{i-1}}, a_{j_i+1}, a_{j_{i+1}}, \dots, a_{j_n}), \quad a_J = (a_{j_1}, \dots, a_{j_n}), \end{aligned}$$

for some $N < \infty$, then φ is absolutely integrable on γ . In fact, since S^{n-1} and γ both compact, to set

$$K = \max_{x \in \gamma} (\max_{y_i \in S^1 \times \dots \times S^1} |\varphi(x, y_1, \dots, y_p)|),$$

K is finite, and for any partition Δ of I , we have

$$\begin{aligned} & \left| \sum_J |\tilde{\varphi}(f(a_J), f(a_{J+1}), \dots, f(a_{J+p}))| \right. \\ & \leq KN^p \left(\sum_J |a_{j_1+1} - a_{j_1}| \dots |a_{j_p+1} - a_{j_p}| \right) \leq KN^p, \\ & \Delta \text{ is given by } 0 = a_0 < a_1 < \dots < a_m < 1, \end{aligned}$$

which shows the absolute integrability of $\tilde{\varphi}$ on γ .

Note. This is also true if φ is an $M(S^{n-1})$ - p -cochain and it seems to be true for $L^q(S^{n-1})$ - p -cochains if we change the definition of the integral of Alexander-Spanier cochains to the Lebesgue type.

Definition. For an $\mathcal{F}(S^{n-1})$ - p -cochain $\varphi = \varphi(x, y_1, \dots, y_p)$ of M , we define

$$(8) \quad \begin{aligned} & d_{\rho} \varphi(x, y_1, \dots, y_{p+1}) \\ & = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} (\varphi(x, y_1, \alpha, y_2, \dots, y_{p+1}) - \varphi(x, y_2, \dots, y_{p+1})). \end{aligned}$$

By definition, if $d_{\rho} \varphi(x, y_1, \dots, y_{p+1})$ exists as an element of $\mathcal{F}^p(S_x)$ for any x and continuous in x , then $d_{\rho} \varphi$ is an $\mathcal{F}(S^{n-1})$ - $(p+1)$ -cochain of M .

Definition. An $\mathcal{F}(S^{n-1})$ - p -cochain φ is called $\mathcal{F}(S^{n-1})$ -smooth if $d_{\rho} \varphi$ is a (con-

tinuous) $\mathcal{F}(S^{n-1})$ - $(p+1)$ -cochain of M .

In general we define $d_\rho^m \varphi$ by

$$d_\rho^m \varphi = d_\rho(d_\rho^{m-1} \varphi),$$

and call φ to be $\mathcal{F}(S^{n-1})$ - m -smooth if $d_\rho^m \varphi$ is a (continuous) $\mathcal{F}(S^{n-1})$ - $(p+m)$ -cochain of M . If φ is $\mathcal{F}(S^{n-1})$ -smooth for all m , then we call φ to be $\mathcal{F}(S^{n-1})$ - ∞ -smooth.

Definition. For an $\mathcal{F}(S^{n-1})$ - p -form $\varphi = \varphi(x, y_1, \dots, y_p)$ of M , we define

$$(8)' \quad \begin{aligned} & Ad_\rho \varphi(x, y_1, \dots, y_{p+1}) \\ &= \frac{1}{p+1} \left[\sum_{i=1}^{p+1} (-1)^{i+1} \left(\lim_{a \rightarrow 0} \frac{1}{a} (\varphi(r_{x, y_i, a}, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{p+1}) - \right. \right. \\ & \quad \left. \left. - \varphi(x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{p+1})) \right) \right]. \end{aligned}$$

By definition, if φ is $\mathcal{F}(S^{n-1})$ -smooth, then $Ad_\rho \varphi$ is an $\mathcal{F}(S^{n-1})$ - $(p+1)$ -form and if φ is $\mathcal{F}(S^{n-1})$ -3-smooth, then

$$(9)' \quad Ad_\rho(Ad_\rho \varphi) = 0.$$

By (9)', denoting $C^p(M, \mathcal{F}(S^{n-1}))$ the space of $\mathcal{F}(S^{n-1})$ - ∞ -smooth $\mathcal{F}(S^{n-1})$ - p -forms on M , $\{\sum_{p \geq 0} C^p(M, \mathcal{F}(S^{n-1})), Ad_\rho \varphi\}$ is a differential complex and we can show the analogy of de Rham's theorem. Because we know

$$(9)' \quad Ad_\rho \tilde{\varphi} = \delta \tilde{\varphi},$$

where δ is the coboundary homomorphism in the Alexander-Spanier cochain. By (9)', we also have

$$(10) \quad \int_{\partial \gamma} \varphi = \int_\gamma Ad_\rho \varphi,$$

if φ is an $\mathcal{F}(S^{n-1})$ - p -form (cf. [3]).

Note. By (10), we have especially

$$\int_\gamma d_\rho f = \int_{\gamma'} d_\rho f, \text{ if } \gamma \text{ and } \gamma' \text{ start from same point and end at same point.}$$

Because for a function f , we have

$$Ad_\rho f = d_\rho f.$$

Therefore, we may write $\int_a^x d_\rho f$ if $\rho(a, x)$ is small and we obtain

$$(10)' \quad \int_a^x d_\rho f = f(x) - f(a).$$

§ 2. Generalized vector fields.

4. Definition. A function f on some neighborhood of x is called to be $\mathcal{F}(S^{n-1})$ -smooth at x if $(d_{\rho, x}f)(y) = d_{\rho}f(x, y)$, x is fixed, defines a function of $\mathcal{F}(S_x)$.

By definition, we have

Lemma 2. f is $\mathcal{F}(S^{n-1})$ -smooth at a if and only if f is written as

$$(11) \quad f(x) = f(a) + g(\varepsilon_{a, x})\rho(a, x) + o(\rho(a, x)),$$

where x belongs in $U(a)$, a neighborhood of a and $g(y)$ is an element of $\mathcal{F}(S_x)$.

For example, if $M = \mathbf{R}^n$, n -dimensional euclidean space, ρ is the euclidean metric of \mathbf{R}^n and f is smooth at a , then f is written as

$$f(x) = f(a) + \left(\sum_i \frac{\partial f(a)}{\partial x_i} (x_i - a_i) / \|x - a\| \right) \|x - a\| + o(\|x - a\|),$$

where $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$ and $\|x\| = \sqrt{\sum_i x_i^2}$. Then since $g(y) = \sum_i$

$\frac{\partial f(a)}{\partial x_i} y_i$, $y = (y_1, \dots, y_n)$, $\|y\| = 1$, belongs for any $\mathcal{F}(S^{n-1})$, f is $\mathcal{F}(S^{n-1})$ -smooth at a for any $\mathcal{F}(S^{n-1})$.

Definition. A function f on some neighborhood of x is called to be $\mathcal{F}(S^{n-1})$ - m -smooth at x if

$$(d_{\rho, x}^m f)(y_1, \dots, y_m) = d_{\rho}^m f(x, y_1, \dots, y_m), \quad x \text{ is fixed},$$

defines a function of ${}^m(S_x)$. If f is $\mathcal{F}(S^{n-1})$ - m -smooth at x for any m , then we call f is $\mathcal{F}(S^{n-1})$ - ∞ -smooth at x .

For example, if $M = \mathbf{R}^n$, ρ is the euclidean metric of \mathbf{R}^n and f is of class C^m at a , then f is $\mathcal{F}(S^{n-1})$ - m -smooth at a for any $\mathcal{F}(S^{n-1})$. In fact, in this case, we get

$$\begin{aligned} & (d_{\rho, x}^m f)(y_1, \dots, y_m) \\ &= \frac{1}{m!} \sum_{i_j \leq n} \frac{\partial^m f(a)}{\partial x_{i_1} \dots \partial x_{i_m}} y_{1, i_1} \dots y_{m, i_m}, \\ & y_i = (y_{i, 1}, \dots, y_{i, n}), \quad \|y_i\| = 1, \quad i = 1, \dots, m. \end{aligned}$$

We denote by $\mathcal{F}(M)$ the function space on M either of $C(M)$ or $L^p(M)$, $1 \leq p \leq \infty$, if M is compact and either of $C(M)$, $C_b(M)$, the space of bounded continuous functions on M , $L^p(M)$, $1 \leq p \leq \infty$ and $L^p_{loc}(M)$, $1 \leq p \leq \infty$ if M is not compact. Here, M is considered to be a measure space with the measure $m(\rho)$, the induced measure from the metric.

We assume the manifold structure of M is given by $\{(U, h_U) | h_U : U \rightarrow \mathbf{R}^n\}$, then we have

Lemma 3. *If we have*

$$(12) \quad ||h_U(a) - h_U(x)|| = O(\rho(a, x)),$$

for any $a, x \in M$ and $U \in \{U\}$, where a is regarded to be fixed and x to be a variable, then the space of $\mathcal{F}(S^{n-1})$ -smooth functions on M is dense in $\mathcal{F}(M)$.

Proof. If f is a smooth function on \mathbf{R}^n with compact carrier, then the function h_U^*f on M given by

$$\begin{aligned} h_U^*f(x) &= f(h_U(x)), & x \in U, \\ h_U^*f(x) &= 0, & x \notin U, \end{aligned}$$

is an $\mathcal{F}(S^{n-1})$ -smooth function on M by (12) and lemma 2. Hence we obtain the lemma since M is paracompact.

Corollary. *Under the same assumptions about M and ρ , for any locally finite open covering $\{V\}$ of M , there exists a partition of unity by $\mathcal{F}(S^{n-1})$ -smooth functions $\{e_V(x)\}$ subordinated to $\{V\}$ for any $\mathcal{F}(S^{n-1})$.*

Theorem 1. *A paracompact topological manifold M always has a metric ρ such that the space of $\mathcal{F}(S^{n-1})$ -smooth functions by ρ on M is dense in $\mathcal{F}(M)$ if $\mathcal{F}(M)$ is either of $C(M)$, $C_b(M)$ or $L^p_{loc}(M)$, $1 \leq p \leq \infty$.*

Proof. We take the metric ρ of M constructed in [4]. Then, since

$$0 < \int_{h_U(r)} ||\xi - \eta|| < \infty, \text{ if and only if } 0 < \int_r \rho < \infty,$$

to set

$$A = \{a | a \in M, a \text{ does not satisfy (12)}\},$$

A is a discrete set of M . Hence for any $a \in A$, there exists a neighborhood $U(a)$ of a such that $U(a) \cap A = \{a\}$. For this $U(a)$, we set

$$C_a(U(a)) = \{f | f \text{ is continuous on } U(a) \text{ and } f(a) = 0\}.$$

By definition, we have

$$(13) \quad C(U(a)) = \mathbf{R} \oplus C_a(U(a)),$$

where \mathbf{R} is the space of constant functions on $U(a)$.

We take a neighborhood system $\{V_n(a)\}$ of a in $U(a)$ such that

$$V_n(a) \subset V_{n+1}(a), \quad \bigcap_n V_n(a) = \{a\},$$

and denote

$$C_n(U(a)) = \{f \mid f \text{ is continuous on } U(a) \text{ and } f|_{V_n(a)} = 0\}.$$

Then by lemma 3, $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $C_n(U(a))$ for any n . Hence $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $C_a(U(a))$ because $\cup_n C_n(U(a))$ is dense in $C_a(U(a))$. But, since a constant function is $\mathcal{F}(S^{n-1})$ -smooth for any $\mathcal{F}(S^{n-1})$, $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $C(U(a))$ by (13).

For each $a \in A$, we take a neighborhood $V(a)$ such that $V(a) \subseteq U(a)$ and set

$$V(A) = \cup_{a \in A} V(a), \quad U(A) = \cup_{a \in A} U(a).$$

Then we have

$$(14) \quad V(A) \subseteq U(A).$$

By lemma 3, we know that $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $CM - V(A)$, and by 14, we can set

$$f = f_1 + f_2, \quad \text{car. } f_1 \subset M - V(A), \\ f_2 = \sum_{a \in A} f_{2,a}, \quad \text{car. } f_{2,a} \subset U(a),$$

for any continuous function f of M . Hence $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $C_b(M)$. Since $C_b(M)$ is dense in $L^p_{loc}(M)$, $1 \leq p \leq \infty$, we have the theorem.

Note. If the total measure of M by $m(\rho)$, the induced measure of ρ , is finite, then $\mathcal{F}(S^{n-1})$ -smooth functions are dense in $L^p(M)$, $1 \leq p \leq \infty$, although M is not compact.

5. We denote the space of $\mathcal{F}(S^{n-1})$ -smooth functions on M by $C_{\mathcal{F}(S^{n-1})}(M)$. If M is not compact, then the subspace of $C_{\mathcal{F}(S^{n-1})}(M)$ consisted by bounded $\mathcal{F}(S^{n-1})$ -smooth functions on M is denoted by $C_{\mathcal{F}(S^{n-1}),b}(M)$. We assume that $C_{\mathcal{F}(S^{n-1})}(M)$ is dense in $C(M)$.

Lemma 4. $C_{\mathcal{F}(S^{n-1})}(M)$ and $C_{\mathcal{F}(S^{n-1}),b}(M)$ are both rings with the unit.

Proof. If f_1 and f_2 are $\mathcal{F}(S^{n-1})$ -smooth at a , then we may set

$$f_i(x) = f_i(a) + g_i(\varepsilon_{a,x})\rho(a,x) + o(\rho(a,x)), \quad i = 1, 2, \quad x \in U(a).$$

Hence we have

$$f_1(x)f_2(x) \\ = f_1(a)f_2(a) + \{f_1(a)g_2(\varepsilon_{a,x}) + f_2(a)g_1(\varepsilon_{a,x})\}\rho(a,x) + o(\rho(a,x)),$$

for $x \in U(a)$. Since $f_1(a)g_2(\varepsilon_{a,x}) + f_2(a)g_1(\varepsilon_{a,x})$ belongs in (S_a) , f_1f_2 is $\mathcal{F}(S^{n-1})$ -smooth

at a . On the other hand, since we know $d_\rho 1 = 0$, where 1 is the constant function with the value 1 , 1 is $\mathcal{F}(S^{n-1})$ -smooth for any $\mathcal{F}(S^{n-1})$. Therefore we obtain the lemma.

Definition. A closed operator X defined in $C(M)$ with the range in $C(M)$ is called an $\mathcal{F}(S^{n-1})$ -vector field of M if it satisfies the following (i), (ii) (iii).

- (i). X is defined on $C_{\mathcal{F}(S^{n-1})}(M)$.
- (ii). If $|f(x) - f(a)| = o(\rho(a, x))$ at a , then $(Xf)(a)$ is equal to 0 .
- (iii). $X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$.

Lemma 5. If $\xi = \xi(x)$ is an $\mathcal{F}(S^{n-1})$ -1-vector field of M , then to set

$$(Xf)(x) = \langle \xi(x), d_\rho f(x) \rangle, \quad x \in M,$$

X is an $\mathcal{F}(S^{n-1})$ -vector field of M . Here $\langle \xi, \varphi \rangle$, $\xi \in {}^* \mathcal{F}(S_x)$, $\varphi \in \mathcal{F}(S_x)$, means the value of ξ at φ .

Proof. By the definition of d_ρ , d_ρ has the following properties.

- (i). If $\{f_n\}$ converges to f in $C(M)$ and $\{d_\rho f_n\}$ converges normally to some $\mathcal{F}(S^{n-1})$ -1-cochain φ , then f is $\mathcal{F}(S^{n-1})$ -smooth and $d_\rho f = \varphi$.
- (ii). $(d_\rho f)(a) = 0$ if $|f(x) - f(a)| = o(\rho(a, x))$ at a .
- (iii). If f_1 and f_2 are both $\mathcal{F}(S^{n-1})$ -smooth, then

$$d_\rho(f_1 f_2) = f_1 d_\rho f_2 + f_2 d_\rho f_1.$$

Hence we have the theorem.

Note. A series of $\mathcal{F}(S^{n-1})$ -1-cochains $\varphi_m(x, y)$ is called converges normally to $\varphi(x, y)$ if the series of functions on M given by $\{|\varphi_m(x, y) - \varphi(x, y)|\}_x$ converges uniformly to 0 on any compact set of M . Here $|\varphi(x, y)|_x$ means the norm of $\varphi(x)$, $\varphi(x)(y) = \varphi(x, y)$, in $\mathcal{F}(S_x)$.

By the definition of $\mathcal{F}(S^{n-1})$ -vector fields, we have

Lemma 6. If X is an $\mathcal{F}(S^{n-1})$ -vector field of M , then X satisfies the following (14) and (15).

$$(14) \quad X_c = 0, \quad \text{where } c \text{ is a constant function of } M.$$

$$(15) \quad (Xf_1)(a) = (Xf_2)(a), \quad \text{if } |f_1(x) - f_2(x)| = o(\rho(a, x)).$$

Theorem 2. If X is an $\mathcal{F}(S^{n-1})$ -vector field of M , then there exists an $\mathcal{F}(S^{n-1})$ -1-vector field $\xi(x)$ of M such that

$$(16) \quad (Xf)(x) = \langle \xi(x), d_\rho f(x) \rangle, \quad x \in M.$$

Such $\xi(x)$ is determined uniquely from X if A , the set defined in the proof of theorem 1, is the empty set.

Proof. We use same notations as in the proof of theorem 1 and first assume $x \notin A$. Then the map

$$d_{\rho, x} : C_{\mathcal{F}(S^{n-1})}(M) \rightarrow \mathcal{F}(S_x),$$

given by $(d_{\rho, x}f)(y) = d_{\rho}f(x, y)$, is onto. Then we define

$$(17) \quad \langle \xi(x), g \rangle = (Xf)(x), \quad d_{\rho, x}f = g, \quad g \in \mathcal{F}(S^{n-1}).$$

By lemma 2 and (15), (17) is well defined and since $d_{\rho, x}$ is onto, $\xi(x)$ is an element of $\mathcal{F}^*(S_x)$ by closed graph theorem because X is a closed operator. Since Xf is continuous for any $f \in C_{\mathcal{F}(S^{n-1})}(M)$, $\xi(x)$ is continuous in x , $x \in M-A$. Moreover, since $M-A$ is dense in M and Xf is continuous on M , $\lim_{x_n \rightarrow a} \xi(x_n) = \xi(a)$ exists as an element of $d_{\rho, a}(C_{\mathcal{F}(S^{n-1})}(M))^*$ for any $a \in A$. Hence (by the theorem of Hahn-Banach), we may consider $\xi(a)$ to be an element of $\mathcal{F}^*(S_a)$ and ξ is continuous at a . Therefore we obtain the theorem.

By lemma 4 and theorem 2, there is a 1 to 1 correspondence between the set of $\mathcal{F}(S^{n-1})$ -vector fields of M and the set of $\mathcal{F}(S^{n-1})$ -1-vector fields of M . Hence we identify them.

Note 1. If X, Y are $\mathcal{F}(S^{n-1})$ -vector fields of M such that their compositions XY and YX are both defined, then $[X, Y] = XY - YX$ also satisfies the conditions (ii), (iii) of $\mathcal{F}(S^{n-1})$ -vector fields.

Note 2. Let X be a closed operator with the domain $\mathcal{D}(X) \subset C(M)$ and the range is in $C(M)$ such that

- (i). $\mathcal{D}(X)$ is a dense subring of $C(M)$ with the unit.
- (ii). If f_1, f_2 are in $\mathcal{D}(X)$, then $X(f_1f_2) = f_1X(f_2) + f_2X(f_1)$.

Then we call X is a generalized vector field of M . If X also satisfies

- (iii). $(Xf)(a) = 0$ if $|f(x) - f(a)| = o(\rho(x, a))$,

for a (fixed) metric ρ of M , then we call X is a generalized vector field of M with respect to ρ .

Since X is closed, to define the topology of $\mathcal{D}(X)$ by taking

$$U(f, V, W) = \{g \mid g \in \mathcal{D}(X), g \in V, Xg \in W\},$$

where V and W are the neighborhoods of f and Xf in $C(M)$, as the neighborhood basis of $f \in \mathcal{D}(X)$, $\mathcal{D}(X)$ is a complete space and to set

$$\mathfrak{I}_a(X) = \{f \mid f \in \mathcal{D}(X), f(a) = Xf(a) = 0\},$$

$\mathfrak{I}_a(X)$ is a closed ideal of $\mathcal{D}(X)$ by this topology. Hence setting

$$\mathcal{F}_a(X) = (\mathcal{D}(X) \cap I_a(M)) / \mathfrak{I}_a(X), \quad I_a(M) = \{f \mid f \in C(M), f(a) = 0\},$$

we can set

$$Xf(a) = \langle \xi(a), d_X f(a) \rangle, \quad \xi(a) \in \mathcal{F}_a(X)^*,$$

where $d_X f(a)$ is the class of $f \cdot f(a)$ in $\mathcal{F}_a(X)$.

If X is a generalized vector field of M with respect to ρ , then we have

$$\mathfrak{S}_a(X) \supset \{f \mid f \in \mathcal{F}(X), |f(x)| = o(\rho(a, x))\}.$$

6. For an $\mathcal{F}(S^{n-1})$ -vector field X given by $Xf = \langle \xi, d_\rho f \rangle$ and t , $0 \leq t \leq 1$, we set

$$(18) \quad U_{X,t}(f)(x) = \langle \xi(x), f(r_{x,y,t}) \rangle.$$

Here $f(r_{x,y,t})$ is regarded to be a function of y , $y \in S_x$. Since f is continuous, $f(r_{x,y,t})$ is continuous on S_x . Hence $U_{X,t}(f)$ is well defined for any X .

By definition, $U_{X,t}$ is defined on $C(M)$ and a bounded linear operator of $C(M)$ if M is compact. We also know that $\lim_{t \rightarrow t_0} U_{X,t}(f)$ converges normally to $U_{X,t_0}(f)$. Therefore, if M is compact, then $U_{X,t}$ is strongly continuous in t . Moreover, we know

$$(19) \quad \lim_{t \rightarrow 0} \frac{1}{t} (U_{X,t} - U_{X,0})f = Xf, \quad \text{if } f \in C_{\mathcal{F}(S^{n-1})}(M).$$

We note that

$$U_{X,0}f(x) = \langle \xi(x), 1 \rangle f(x),$$

where 1 is the constant function with the value 1 on S_x .

(19) shows that there is a curve in $L(C(M), C(M))$, the space of (bounded) linear operators of $C(M)$ (with the strong topology), such that whose tangent at its starting point is X .

For $U_{X,t}$, we set

$$T_{X,a,t} = \exp\left(\frac{t}{a}(U_{X,a} - U_{X,0})\right), \quad t \geq 0.$$

Then $\{T_{X,a,t}\}$ is a 1-parameter semi-group of $C(M)$ with the generating operator $(1/a)(U_{X,a} - U_{X,0})$. Hence if $\lim_{a \rightarrow 0} T_{X,a,t}$ exists, then to set its limit by $T_{X,t}$, $T_{X,t}$ is a 1-parameter semi-group with the generating operator X . But this limit does not exist in general. In fact, there exists an $\mathcal{F}(S^{n-1})$ -vector field which does not generate any 1-parameter semi-group of $C(M)$ or $L^p(M)$, $1 \leq p \leq \infty$.

Example. We assume that M satisfies

(i). $H^1(M, \mathbf{R})$ vanishes.

(ii). M is compact.

To define a $C(S^{n-1})$ -1-form $\varphi(x, y)$ on M by $\varphi(x, y) = \lambda$, an (arbitrary) constant, we get

$$d_\rho \varphi = 0.$$

Hence by (i), there exists a $C(S^{n-1})$ -smooth function n on M such that

$$(d_\rho h)(x, y) = \varphi(x, y).$$

Let X be the $C(S^{n-1})$ -vector field on M given by

$$Xf(x) = \langle m(x), d_\rho f(x) \rangle, \quad m(x) \text{ is the canonical measure on } S_x.$$

Then we have for the above h ,

$$Xh = \lambda, \text{ the constant function with the value } \lambda \text{ on } M.$$

For this h , we set $k = \exp.(h) = \sum_m (h)^m / m!$. Then we get

$$Xk = \lambda k.$$

This shows λ is a proper value of X in $C(M)$ (or in $L^p(M)$, $1 \leq p \leq \infty$, because $C(M)$ is contained in $L^p(M)$ since M is compact), Since M is compact, $C(M)$ is a Banach space. Then by the theorem of Hille-Yosida ([17], [18]), X can not generate any (equi-continuous) 1-parameter semi-group of $C(M)$ (or $L^p(M)$), because λ is arbitrary.

In general, if an $L^2(S^{n-1})$ -vector field X is given by

$$Xf = \langle \xi(x), d_\rho f(x) \rangle, \quad \xi(x) \neq 0 \text{ for any } x \in M,$$

and M is compact, then X does not generate any 1-parameter semi-group of $C(M)$ (or $L^p(M)$, $1 \leq p \leq \infty$). In fact, in this case, we may set

$$L^2(S_x) = (\xi(x))^\perp \oplus \mathbf{R}\xi(x),$$

and denote the projection to $\mathbf{R}\xi(x)$ by $P_{\xi(x)}$. Then a cross-section f of the bundle $\cup_{x \in X} \mathbf{R}\xi(x)$ is considered to be a function f of M by setting

$$f \mathfrak{H}(x) = a, \text{ if } f(x) = a \frac{\xi(x)}{||\xi(x)||}.$$

(We note that this also shows that a fuction of M always defines a crosssection of $\cup_{x \in X} \mathbf{R}\xi(x)$). Then by the befinition of X , we have

$$Xf(x) = ||\xi(x)|| (P_{\xi(x)} d_\rho f) \mathfrak{H}(x).$$

We define $P_\varepsilon d_\rho f$ by $(P_\varepsilon d_\rho f)(x) = P_{\xi(x)} d_{\rho, x} f$. Then $P_\varepsilon d_\rho C L^2(S^{n-1})$ is dense in the space of the cross-sections of $\cup_{x \in X} \mathcal{R}\xi(x)$, for any constant function λ and $\varepsilon > 0$, there exists an $L^2(S^{n-1})$ -smooth function $f_{\lambda, \varepsilon}$ such that

$$||Xf_{\lambda, \varepsilon} - \lambda|| < \varepsilon.$$

This means λ is at least continuous spectre of X , because M is compact. Hence by the theorem of Hille-Yosida, we have the assertion.

Note. The generating operator of a 1-parameter semi-group $\{T_t\}$ is an $\mathcal{F}(S^{n-1})$ -vector field of M , if and only if $\{T_t\}$ satisfies

$$(20) \quad T_t(f_1 f_2) - (T_t f_1)(T_t f_2) = o(t), \text{ if } f_1, f_2 \in C \mathcal{F}(S^{n-1})(M).$$

7. In this n^o, we give some definitions about X , an $\mathcal{F}(S^{n-1})$ -vector field on M .

Definition. X is called to be 0 at a , $a \in M$, if $(Xf)(a) = 0$ for all $\mathcal{F}(S^{n-1})$ -smooth functions.

By definition, if X is given by $Xf = \langle \xi(x), d_\rho f(x) \rangle$, then X is 0 at a if and only if $\xi(a) = 0$ as an element of $\mathcal{F}^*(S_a)$. As usual, we set

$$\text{car. } X = \overline{\{x \mid X \text{ is not 0 at } x\}}.$$

Definition. For X , we set

$$(21) \quad \text{CAR.}(X) = \overline{\cup_{x \in M} \text{car. } \xi(x)}, \text{ if } (Xf)(x) = \langle \xi(x), d_\rho f(x) \rangle.$$

By definition, $\text{CAR. } X$ is a (closed) subset of $s(M)$ and we have

$$(22) \quad \pi(\text{CAR. } X) = \text{car. } X.$$

We note that if M is smooth and X is a usual vector field on M regarded to be a $C(S^{n-1})$ -vector field on M and does not vanish at any point of M , then $\text{CAR. } X$ is a cross-section of $s(M)$ (cf. n^o9).

Definition. X is called to be positive if X is given by $Xf = \langle \xi(x), d_\rho f(x) \rangle$ and

$$\xi(x) \geq 0 \text{ for any } x \in M.$$

As usual, we call $X \geq Y$ if $X - Y \geq 0$. Then since

$$(\text{sup. } \{X\})f = \langle \text{sup. } \{\xi_\alpha(x)\}, d_\rho f(x) \rangle,$$

if $\{X_\alpha\}$ is upper (or lower) bounded, then $\text{sup. } \{X_\alpha\}$ (or $\text{inf. } \{X_\alpha\}$) exists to be an

$\mathcal{F}(S^{n-1})$ -vector field. Especially, we may define $X^+ = \max. (X, 0)$ and $X^- = (-X)^+$ for any $\mathcal{F}(S^{n-1})$ -vector field X and we have

$$(23) \quad X = X^+ - X^-.$$

We note that if $Xf = \langle \xi(x), d_\rho f(x) \rangle$, then

$$(X^+f)(x) = \langle (\xi(x))^+, d_\rho f(x) \rangle, \quad (X^-f)(x) = \langle (\xi(x))^-, d_\rho f(x) \rangle,$$

where $(\xi(x))^+$ is $\max. (\xi(x), 0)$ and $(\xi(x))^-$ is $(\xi(x))^+$.

Note. Since the space of $\mathcal{F}(S^{n-1})$ -vector field of M is a vector space, these shows that this space has the structure of (complete) vector lattice. Hence to fix an $\mathcal{F}(S^{n-1})$ -vector field Y , $Yf = \langle \gamma(x), d_\rho f \rangle$, the Radon-Nykodim partition of any $\mathcal{F}(S^{n-1})$ -vector field X , $Xf = \langle \xi(x), d_\rho f \rangle$ with respect to Y is possible. It corresponds to the Radon-Nykodim partition of $\xi(x)$ with respect to $\gamma(x)$.

Definition. If $\mathcal{F}(S^{n-1})$ -vector fields X_1 and X_2 are given by $(X_i f)(x) = \langle \xi_i(x), d_\rho f(x) \rangle$, $i = 1, 2$, and $Y = [X_1, X_2]$ is defined to be an $\mathcal{F}(S^{n-1})$ -vector field of M , then we denote

$$(24) \quad \gamma(x) = [\xi_1(x), \xi_2(x)].$$

Here Y is given by $(Yf)(x) = \langle \gamma(x), d_\rho f(x) \rangle$.

We note that if x is fixed in (24), then (24) defines the bracket product for some elements of $\mathcal{F}^*(S_x)$. Or, in other word, $\mathcal{F}^*(S_x)$ contains (as a dense subset), a Lie pseudoalgebra.

§ 3. Generalized tangent of a curve.

8. We denote the set of germs of $\mathcal{F}(S^{n-1})$ -smooth functions of M at a , $a \in M$, by $C_{\mathcal{F}(S^{n-1}), *, a}(M)$.

Lemma 7. If $\mathcal{F}(S^{n-1})$ -smooth functions f_1 and f_2 defines same germ in $C_{\mathcal{F}(S^{n-1})}(M)$ and $|f_1(x) - f_1(a)| = o(\rho(x, a))$, then $|f_2(x) - f_2(a)|$ is also $o(\rho(x, a))$.

By this lemma, we can say $|f(x) - f(a)|$ is $o(\rho(x, a))$ although f is regarded to be an element of $C_{\mathcal{F}(S^{n-1}), *, a}(M)$.

Definition. A linear map \mathfrak{X} from $C_{\mathcal{F}(S^{n-1}), *, a}(M)$ to R is called an $\mathcal{F}(S^{n-1})$ -vector of M at a if it satisfies the following (i), (ii), (iii).

- (i). $\mathfrak{X}(f_1 f_2) = f_1(a)\mathfrak{X}(f_2) + f_2(a)\mathfrak{X}(f_1)$.
- (ii). $\mathfrak{X}(f) = 0$, if $|f(x) - f(a)| = o(\rho(a, x))$.
- (iii). $\mathfrak{X}(f) = (Xf)(a)$, where X is an $\mathcal{F}(S^{n-1})$ -vector field of $U(a)$, a neighborhood of a .

By (iii) and theorem 2, we have

Theorem 2'. For any $\mathcal{F}(S^{n-1})$ -vector \mathfrak{X} of M at a , there exists an element ξ of $\mathcal{F}^*(S_a)$ such that

$$\mathfrak{X}(f) = \langle \xi, d_{\rho, a} f \rangle,$$

and such ξ is determined uniquely by \mathfrak{X} . Conversely, if $\xi \in \mathcal{F}^*(S_a)$, then $\langle \xi, d_{\rho, a} f \rangle$ is an $\mathcal{F}(S^{n-1})$ -vector of M at a .

Let γ be a curve of M given by $\varphi : I \rightarrow M$ such that

$$(25) \quad \varphi(0) = a, \quad \varphi(t) \neq a \text{ if } t > 0.$$

$$(25)' \quad \rho(a, \varphi(t)) = 0(t).$$

Then we set

$$(26) \quad \mathfrak{X}_{\varphi}(f) = \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{1}{t} [f(\varphi(t)) - f(a)] dt \right],$$

where f is an $\mathcal{F}(S^{n-1})$ -smooth function at a .

By (25) and (25)', we have

$$(26)' \quad \mathfrak{X}_{\varphi}(f) = \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{\rho(a, \varphi(t))}{t} (d_{\rho, a} f)(\varepsilon_{a, \varphi(t)}) dt \right].$$

Lemma 8. If $\mathfrak{X}_{\varphi}(f)$ exists for all $\mathcal{F}(S^{n-1})$ -smooth functions at a , then \mathfrak{X}_{φ} is an $\mathcal{F}(S^{n-1})$ -vector of M at a .

Proof. By (26)', we only need to show (i). But, since we know

$$\begin{aligned} & (d_{\rho, a}(f_1 f_2))(\varepsilon_{a, \varphi(t)}) \\ &= f_1(a)(d_{\rho, a} f_2)(\varepsilon_{a, \varphi(t)}) + f_2(a)(d_{\rho, a} f_1)(\varepsilon_{a, \varphi(t)}), \end{aligned}$$

we have (i) by (26)'.

Definition. If \mathfrak{X}_{φ} is defined on $C_{\mathcal{F}(S^{n-1}), *, a}(M)$, then γ is called $\mathcal{F}(S^{n-1})$ -smooth at a .

By theorem 2' and lemma 8, If \mathfrak{X}_{φ} is defined on the space of $\mathcal{F}(S^{n-1})$ -smooth functions at a , then there exists an element $\xi = \xi(\varphi)$ of $\mathcal{F}^*(S_a)$ such that

$$\mathfrak{X}_{\varphi}(f) = \langle \xi(\varphi), d_{\rho, a} f \rangle.$$

We note that since $C^*(S_a)$ contains $L^p(S_a)$ for all p , we may consider ξ to be a Radon-measure on S_a .

Definition. $\xi(\varphi)$ is called the generalized tangent of γ at a .

Note. If M is smooth, real analytic or real algebraic, then to take $C^{\infty}(S_a)$, $C^{\omega}(S_a)$ or $C^{alg.}(S_a)$ as $\mathcal{F}(S_a)$, we may define the generalized tangent for wider class of curves. Here $C^{alg.}(S^{n-1})$, the model of $C^{alg.}(S_a)$, is given by

$$\text{Calg.}(S^{n-1}) = \mathbf{R}[x_1, \dots, x_n]/(x_1^2 + \dots + x_n^2 - 1),$$

which is dense in $C(S^{n-1})$ or in $L^p(S^{n-1})$ (cf. [5], [11]).

9. In this n^o, we give some examples of the generalized tangent.

Example 1. We assume γ is smooth at a , that is

$$\lim_{t \rightarrow 0} \varepsilon_{a, \varphi(t)} = y, \quad y \in S_a,$$

$$\lim_{t \rightarrow 0} \frac{\rho(a, \varphi(t))}{t} = c, \quad c \text{ is a (positive) real number,}$$

both exists and f is $C(S^{n-1})$ -smooth at a , then we have by the mean value theorem

$$\begin{aligned} & \int_h^s \frac{\rho(a, \varphi(t))}{t} (d_{\rho, a} f)(\varepsilon_{a, \varphi(t)}) dt \\ &= \frac{\rho(a, \varphi(s_0))}{s_0} (d_{\rho, a} f)(\varepsilon_{a, \varphi(s_0)})(s - h), \quad h < s_0 < s. \end{aligned}$$

Hence we have

$$\mathfrak{X}_{\varphi}(f) = c(d_{\rho, a} f)(y).$$

Therefore, denoting the Dirac measure of S_a concentrated at y by δ_y , we get

$$(27) \quad \mathfrak{X}_{\varphi}(f) = \langle c\delta_y, d_{\rho, a} f \rangle.$$

We note that if f is smooth at a , then $\mathfrak{X}_{\varphi}(f)$ coincide to the usual definition of the (one-sided) derivation of f along γ .

Note. If M is smooth and X is a usual vector field of M which does not vanish at any point of M , then at any point a of M , X has a smooth integral curve γ_a given by $\varphi_a: I \rightarrow M$, $\varphi_a(0) = a$, and

$$(Xf)(a) = \mathfrak{X}_{\varphi_a}(f).$$

Hence we have by (27)

$$(Xf)(a) = \langle c(a)\delta_{y(a)}, d_{\rho, a} f \rangle.$$

Hence we have

$$(28) \quad \text{CAR. } X = \bigcup_{a \in M} y(a).$$

Since $y(a)$ depends continuously on a , CAR. X is a (continuous) cross-section of $s(M)$.

In the following two examples, we need the following

Lemma 9. *If $g(t)$ is a continuous periodic function on \mathbf{R}^1 with the period T , then*

$$(29) \quad \lim_{s \rightarrow \infty} s \int_s^{\infty} \frac{g(t)}{t^2} dt = \frac{1}{T} \int_0^T g(t) dt.$$

Proof. We define a periodic function $e_{[a,b]}(t)$, $0 \leq a < b \leq T$, with the period T by

$$\begin{aligned} e_{[a,b]}(t) &= 1, \quad t \in [a + nT, b + nT], \text{ for some integer } n, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then for $0 \leq a' \leq a < b \leq b' \leq T$, to set

$$v_{m,a',b'}^{a,b} = \frac{b' - a'}{b - a} (t - (mT + a)) + mT + a', \quad mT \leq v_{m,a',b'}^{a,b} \leq (m+1)T,$$

we have

$$e_{[a,b]}(v_{m,a',b'}^{a,b}) = e_{[a',b']}(t), \quad mT \leq v_{m,a',b'}^{a,b} \leq (m+1)T.$$

Hence we get

$$\int_{mT}^{\infty} \frac{e_{[a,b]}(t)}{t^2} dt = \frac{b-a}{b'-a'} \int_{mT}^{\infty} \frac{e_{[a',b']}(t)}{t^2} dt.$$

Then, since we know

$$\lim_{\substack{a' \rightarrow 0 \\ b' \rightarrow T}} s \int_s^{\infty} \frac{e_{[a',b']}(t)}{t^2} dt = s \int_s^{\infty} \frac{dt}{t^2},$$

we obtain

$$\lim_{s \rightarrow \infty} s \int_s^{\infty} \frac{e_{[a,b]}(t)}{t^2} dt = \frac{|b-a|}{T}.$$

Then, since $g(t)$ is bounded and (uniformly) continuous, we have

$$\begin{aligned} & \lim_{s \rightarrow \infty} s \int_s^{\infty} \frac{g(t)}{t^2} dt \\ &= \lim_{s \rightarrow \infty} \left[\lim_{|a_{i+1} - a_i| \rightarrow 0} \sum_i s \int_s^{\infty} g(a_i) \frac{e_{[a_i, a_{i+1}]}(t)}{t^2} dt \right] \\ &= \lim_{|a_{i+1} - a_i| \rightarrow 0} \sum_i g(a_i) \left[\lim_{s \rightarrow \infty} s \int_s^{\infty} \frac{e_{[a_i, a_{i+1}]}(t)}{t^2} dt \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{|a_{i+1}-a_i| \rightarrow 0} \sum_i g(a_i) \frac{|a_{i+1}-a_i|}{T} \\
&= \frac{1}{T} \int_0^T g(t) dt.
\end{aligned}$$

Here, $0=a_0 < a_1 < \dots < a_m < a_{m+1} = T$ is a partition of $[0, T]$.

Example 2. Let M be \mathbf{R}^2 with the euclidean metric, a the origin $0 = (0, 0)$ of \mathbf{R}^2 and γ is given by $\varphi: I \rightarrow \mathbf{R}^2$, where φ is given by

$$\varphi(t) = (t \cos(\frac{1}{t}), t \sin(\frac{1}{t})), \quad t > 0,$$

$$\varphi(0) = 0.$$

Hence, if we use the polar coordinate (r, θ) of \mathbf{R}^2 , $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, then γ is given by

$$r\theta = 1, \quad r > 0.$$

Then, if $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ is parametrized by θ and g is continuous on S^1 , we get

$$\begin{aligned}
&\lim_{s \rightarrow 0} \frac{1}{s} [\lim_{h \rightarrow 0} \int_h^s \frac{\rho(0, \varphi(t))}{t} g(\varepsilon_{0, \varphi t}) dt] \\
&= \lim_{s \rightarrow 0} \frac{1}{s} [\lim_{h \rightarrow 0} \int_h^s g(\frac{1}{t}) dt] = \lim_{u \rightarrow \infty} u \int_u^\infty \frac{g(v)}{v^2} dv,
\end{aligned}$$

Hence by lemma 9, we have

$$(30) \quad \mathfrak{X}_\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} (d_{\rho, 0} f)(\theta) d\theta.$$

Or, in other word, the generalized tangent of the curve $r\theta = 1$ at 0 is the standard measure of S^1 .

Example 3. We take M and ρ same as above and take φ to be

$$\varphi(t) = (t, t \sin(\frac{1}{t})), \quad t > 0, \quad \varphi(0) = 0, \quad \text{the origin of } \mathbf{R}^2.$$

By bdefinition, we have

$$\frac{\rho(0, \varphi(t))}{t} = \sqrt{1 + \sin^2(\frac{1}{t})}, \quad \varepsilon_{0, \varphi(t)} = \tan^{-1}(\sin(\frac{1}{t})).$$

Hence we have by lemma 9,

$$\begin{aligned}
& \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{\rho(0, \varphi(t))}{t} g(\varepsilon_0, \varphi(t)) dt \right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + \sin^2 v} g(\tan^{-1}(\sin(v))) dv \\
&= \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} g(\theta) \frac{1}{\cos^2 \theta \sqrt{\cos(2\theta)}} d\theta.
\end{aligned}$$

Therefore, the generalized tangent of the curve $x \sin(1/x)$ at the origin is the measure on S^1 concentrated on $-(\pi/4) \leq \theta \leq \pi/4$ with the weight $(1/\pi)(1/\cos^2 \theta \sqrt{\cos(2\theta)})$.

Note. If γ is given by $(-t, t \sin(1/t))$, $t > 0$, then the generalized tangent of γ at the origin is similar as above but has carrier on $3\pi/4 \leq \theta \leq 5\pi/4$.

10. Lemma 10. *The generalized tangent of a curve at a is a positive measure on S_a .*

Proof. If ξ is the generalized tangent of $\varphi: I \rightarrow M$, then we have

$$\int_{S_a} g(y) d\xi = \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{\rho(a, \varphi(t))}{t} g(\varepsilon_a, \varphi(t)) dt \right].$$

Hence if $g \geq 0$ on S_a , then $\int_{S_a} g(y) d\xi \geq 0$. Therefore ξ is a positive measure.

Lemma 11. If the parameter of γ is changed to ct instead of t , c is a constant, then the generalized tangent ξ of γ at a is changed to $c\xi$. In general, if the parameter of γ is changed to $\alpha(t)$ and

$$\lim_{t \rightarrow 0} \frac{\alpha(t)}{t} = c,$$

then the generalized tangent ξ of γ at a changes to $c\xi$.

By this lemma, we may assume the generalized tangent ξ of γ at a satisfies

$$(31) \quad \xi(S_a) = 1.$$

Theorem 3. *If ξ is a positive measure on S_a , then there exists a curve of M starting from a such that whose generalized tangent at a is ξ .*

Proof. Since the proof for $n = 1$ is similar, we assume $n \geq 2$.

First we note that the problem is local, we may assume $M = \mathbf{R}^n$ with the euclidean metric and a is the origin $0 = (0, \dots, 0)$ of \mathbf{R}^n . Hence S_a is the unit $(n-1)$ -sphere S^{n-1} .

We take a positive measure ξ of S^{n-1} such that $\xi(S^{n-1}) = 1$. By lemma 11, this is not restrictive.

We choose a countable dense subset $\{y_p\}$ of S^{n-1} such that

$$(32) \quad y_p \neq \pm y_q, \text{ if } p \neq q.$$

For this $\{y_p\}$, we divide S^{n-1} by Borel sets $\{E_p^q\}$ as follows:

$$(33) \quad S^{n-1} = \bigcup_{p \leq q} E_p^q, \quad E_{p'}^q \cap E_{p''}^q = \emptyset, \text{ if } p' \neq p'', \quad y_p \in E_p^q.$$

$$(33)' \quad \lim_{q \rightarrow \infty} \text{dia. } (E_p^q) = 0.$$

Here *dia.* (E_p^q) means the diameter of E_p^q . Hence, if $g(y)$ is a continuous function of S^{n-1} , then

$$(34) \quad \int_{S^{n-1}} g(y) d\xi = \lim_{q \rightarrow \infty} \sum_{p \leq q} g(y_p) \xi E_p^q.$$

On the other hand, for the above $\{E_p^q\}$ and ξ , we take a series of (positive) real numbers $\{t_{q,p}\}$, $p \leq q$, as follows:

$$(35) \quad t_{q,p} > t_{q,p+1}, \text{ if } p+1 \leq q, \quad t_{q,q} > t_{q+1,1},$$

$$(35)' \quad \lim_{q \rightarrow \infty} t_{q,p} = 0,$$

$$(35)'' \quad \sum_{q: t_{q,p} \leq s} \frac{1}{s} |(t_{q,p} - t_{q,p+1}) - \xi(E_p^q)| \leq \frac{s}{2^p}, \quad s > 0.$$

This is possible because $\xi(S^{n-1}) = 1$ and $\sum_p \sum_{q: t_{q,p} \leq s} (1/s) |t_{q,p} - t_{q,p+1}| = 1 - (s - t_{q_0, p_0})/s$ is sufficiently near to 1. Here, t_{q_0, p_0} is the largest $t_{q,p}$ which is smaller than s .

Using this $\{t_{q,p}\}$, we set

$$\begin{aligned} \Psi(t_{q,p}) &= t_{q,p} y_p, \\ \Psi(t) &= \frac{t_{q,p} - t}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p+1}) + \frac{t - t_{q,p+1}}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p}), \\ &\text{if } t_{q,p} > t > t_{q,p+1}, \\ \Psi(t) &= \frac{t_{q,q} - t}{t_{q,q} - t_{q+1,1}} \Psi(t_{q+1,1}) + \frac{t - t_{q+1,1}}{t_{q,q} - t_{q+1,1}} \Psi(t_{q,q}), \\ &\text{if } t_{q,q} > t > t_{q+1,1}, \\ \Psi(0) &= 0. \end{aligned}$$

Then since $\|y_p\| = 1$, we have the definition of $\Psi(t)$ and (32),

$$(36)'' \quad \|\Psi(t)\| \leq |t|,$$

$$(36)' \quad \Psi(t) \neq 0, \text{ if } t \neq 0.$$

We also note that by the definition of $\Psi(t)$, $\Psi(t)$ is continuous for all t , $0 \leq t \leq 1$.

By (36)', to define $\varphi(t)$ by

$$(37) \quad \varphi(t) = \frac{\Psi(t)}{|\Psi(t)|} t, \quad t > 0, \quad \Psi(0) = 0,$$

$\varphi(t)$ is also continuous in t and satisfies similar conditions as (36)' and.

$$(36) \quad ||\varphi(t)|| = t.$$

By (36) and the mean value theorem, if $\{y_p\}$ satisfies

$$(32)' \quad \lim_{p \rightarrow \infty} ||y_{p+1} - y_p|| = 0,$$

then we have for this $\varphi(t)$,

$$\begin{aligned} & \int_{t_{q,p+1}}^{t_{q,p}} \frac{||\varphi(t)||}{t} g(\varepsilon_0, \varphi(t)) dt \\ &= g(y_p)(t_{q,p} - t_{q,p+1}) + o(|t_{q,p} - t_{q,p+1}|). \end{aligned}$$

Hence we have

$$(38) \quad \begin{aligned} & \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{||\varphi(t)||}{t} g(\varepsilon_0, \varphi(t)) dt \right] \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \sum_p g(y_p) \left(\sum_{q: t_{q,p} \leq s} (t_{q,p} - t_{q,p+1}) \right). \end{aligned}$$

On the other hand, by (35)'', we obtain

$$\begin{aligned} & \left| \sum_{p \leq a} \sum_{t_{q,p} \leq s} g(y_p) \xi(E_p^q) - \frac{1}{s} \sum_p g(y_p) \left(\sum_{q: t_{q,p} \leq s} (t_{q,p} - t_{q,p+1}) \right) \right| \\ & \leq \sum_p \frac{s}{2^p} = s. \end{aligned}$$

Then, by (34) and (38), we get

$$\begin{aligned} & \int_{S^{n-1}} g(y) d\xi \\ &= \lim_{s \rightarrow 0} \frac{1}{s} \left[\lim_{h \rightarrow 0} \int_h^s \frac{||\varphi(t)||}{t} g(\varepsilon_0, \varphi(t)) dt \right], \end{aligned}$$

for this $\varphi(t)$. Therefore the curve γ given by $\varphi: I \rightarrow M$, has the generalized tangent at the origin and it is equal to ξ . Hence we have the theorem.

Note. Since $C^*(S^{n-1})$ contains $L^p(S^{n-1})$, a positive linear functional of $L^p(S^{n-1})$ always expressed as the generalized tangent of some curve.

Example 1. If ξ is the Dirac measure of S^{n-1} concentrated at y_1 , $y_1 \in S^{n-1}$, then $\{t_{q,p}\}$ is given by

$$t_{q,1} = \frac{1}{2^q}, \quad t_{q,p} = \frac{1}{2^q} - \left(1 - \frac{1}{2^{p-1}}\right) \frac{1}{8^q}, \quad 2 \leq p \leq q.$$

Example 2. If ξ is the standard measure of S^{n-1} , then we take E_p^q to satisfy $\xi(E_p^q) = 1/q$, Then we can take $\{t_{q,p}\}$ to be

$$t_{q,p} = \frac{1}{q+1} + \frac{q+1-p}{p+1} \left(\frac{1}{q(q+1)}\right).$$

We note that although the curve $\varphi(t) = y_1 t$ has the generalized tangent δy_1 , it is not given by the above method.

11. We denote by $H^+(I)$ the group of orientation preserving homeomorphisms of $I = [0, 1]$. The subgroup of $H^+(I)$ consisted by those homeomorphisms that are the identity map on $[0, \varepsilon]$ for some $\varepsilon > 0$, is denoted by $H_\varepsilon(I)$. Then we set

$$H_*^+(I) = H^+(I)/H_\varepsilon(I).$$

$H_*^+(I)$ is the group of germs of the (orientation preserving) homeomorphisms of I (cf. [2]).

If $\alpha \in H^+(I)$, then by the theorem of Radon-Nykodim, there exists a (positive) measurable function m_α on I which does not vanish almost everywhere on I , such that

$$(39) \quad \int_a^b \mu(\alpha(t)) dt = \int_{\alpha(a)}^{\alpha(b)} \mu(u) m_\alpha(u) du,$$

where $\mu(t)$ is an (arbitrary) measurable function on I . We note that this $m_\alpha(t)$ also satisfies

$$(40) \quad \int_0^1 m_\alpha(t) dt = 1.$$

Conversely, if $m(t)$ is a positive measurable function on I such that to satisfy (40) and does not vanish almost everywhere on I , then $\int_0^t m(u) du$ is an element of $H^+(I)$. Moreover, we know that

(i). If $\alpha_1, \alpha_2 \in H^+(I)$ and $\alpha_1(\alpha_2)$ is the composition of α_1 and α_2 in $H^+(I)$, then

$$(41) \quad m_{\alpha_1(\alpha_2)} = \alpha_2^*(m_{\alpha_1})m_{\alpha_2}, \quad \alpha^*m(t) \text{ means } m(\alpha(t)).$$

(ii). α belongs in $H_\varepsilon(I)$ if and only if $m_\alpha(t) = 1$, $0 \leq t < \varepsilon$, for some $\varepsilon > 0$.

Hence to denote the set of all positive measurable functions on I which do

not vanish almost everywhere on I and satisfy (40) by $\mathcal{M}^+(I)$ and to define a multiplication $m_1 * m_2$ for $m_1, m_2 \in \mathcal{M}^+(I)$ by

$$(42) \quad m_1 * m_2 = \alpha_2^*(m_1)m_2, \quad \alpha_2(t) = \int_0^t m_2(u)du,$$

$\mathcal{M}^+(I)$ is isomorphic to $H^+(I)$ and to set

$$\mathcal{M}_e(I) = \{m \mid m \in \mathcal{M}^+(I), m(t) = 1, 0 \leq t < \varepsilon, \text{ for some } \varepsilon > 0\},$$

we have

$$(43) \quad \mathcal{M}_*(I) \cong H_*^+(I), \quad \mathcal{M}_*(I) = {}^+(I) / \mathcal{M}_e(I).$$

For $\varphi: I \rightarrow M$, and $\alpha \in H^+(I)$, we set

$$\alpha^*(\varphi)(t) = \varphi(\alpha(t)).$$

Then the image of φ and $\alpha^*(\varphi)$ is same. Moreover, we know if $\alpha \in H_e(I)$, then φ has the generalized tangent at its starting point if and only if $\alpha^*(\varphi)$ has the generalized tangent at its starting point and we have by lemma 10,

$$(44) \quad \mathfrak{X}_\varphi(f) = \mathfrak{X}_{\alpha^*(\varphi)}(f).$$

By (44), we have

$$(44)' \quad \mathfrak{X}_{\alpha^*(\varphi)} = \mathfrak{X}_{\beta^*(\varphi)}, \quad \text{if } \alpha \equiv \beta \text{ mod. } H_e(I).$$

By (43), (44)' and theorem 3, we can define an operation of the element m of $\mathcal{M}_*(I)$ to $\mathcal{D}_+^*(S^{n-1})$, the set of positive linear functionals of $\mathcal{D}(S^{n-1})$ by

$$(45) \quad \langle m(\xi), g \rangle = \mathfrak{X}_{\alpha^*(\varphi)}(f),$$

where, assuming the starting point of φ is a , $d_{\rho, a}f = g$, $\mathfrak{X}_\varphi(f) = \langle \xi, g \rangle$ and the class of m in $\mathcal{M}_*(I)$ is m . Then, since the change of parameter of γ corresponds to the operation of $\mathcal{M}_*(I)$, we may consider the generalized tangent of γ to be an element of $\mathcal{F}_+^*(S^{n-1}) / \mathcal{M}_*(I)$.

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