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Generalized Tangents of Curves and Generalized Vector Fields

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Introduction

The main purpose of this paper is to introduce the notion of generalized tangent of a curve γ given by $\varphi: \mathrm{I} \rightarrow M$, where M is an n-dimensional Paracompact topological manifold with a (fixed) metric ρ. Here ρ is assumed to satisfy

(*). *If $\rho(x_1, x_2) \leqq 1$, then there exists unique curve γ of M which joins x_1 and x_2 and*

$$\int_\gamma \rho = \rho(x_1, x_2).$$

(For the existence of such metric, see [4]). In the rest, we set

$$S_x = \{y \mid \rho(x, y) = 1\}.$$

By assumption, S_x is homeomorphic to S^{n-1}, the unit $(n-1)$-sphere. Then the generalized tangent of γ at $a = \varphi(0)$ is defined to be a positive Radon measure on S_a and we show that for any positive Radon measure ξ on S_a, there exists a curve γ on M whose generalized tangent at a is ξ (§ 3, theorem 3).

More Precisely, to define the generalized tangent of γ, first we introduce the notion of $\mathscr{F}(S^{n-1})$-smooth function at a, where $\mathscr{F}(S^{n-1})$ is a (fixed) function space on S^{n-1} such as $C(S^{n-1})$, $L^p(S^{n-1})$ (the measure on S^{n-1} is the standard volume element, that is given by $\sum_i x_i \check{d} x_i$ (cf. [5], [11])) or (if M is smooth or real analytic) $C^\infty(S^{n-1})$ or $C^\omega(S^{n-1})$, as follows: *A function f defined on some neighborhood of a is called to be $\mathscr{F}(S^{n-1})$-smooth if f is written as*

$$f(x) = f(a) + g(\varepsilon_{a,x})\rho(a, x) + o(\rho(a, x)), \quad \rho(a, x) < 1,$$

and $g(y)$ belongs in $\mathscr{F}(S_a)$. Here $\varepsilon_{a,x}$ means the point y on S_a such that

$$x \in r_{a,y},$$

where $r_{a,y}$ is the curve of M which joins a and y and $\int_{r_{a,y}} \rho = 1$, and $\mathscr{F}(S_a)$ means the function space on S_a defined similarly as (using the measure induced from ρ (cf.

[3], [4]))$\mathscr{F}(S^{n-1})$.

Then the generalized tangent of γ at a is defined to be the element ξ of $\mathscr{F}^*(S_a)$, the dual space of $\mathscr{F}(S_a)$ which is determined by

$$<\xi,\ g>=\lim_{s\to 0}\frac{1}{s}[\lim_{h\to 0}\int_h^s\frac{1}{t}\{f(\rho(t))-f(a)\}dt],$$

where γ is given by $\rho: I\to M$ and ρ is assumed to satisfy

(i) $\rho(0)=a,\quad \rho(t)\neq a,\quad if\ t\neq 0,$

(ii) $\rho(a,\quad \rho(t))=o(t),$

f is an $\mathscr{F}(S^{n-1})$-smooth function at a and $g\in\mathscr{F}(S_a)$ is given by

$$g(y)=\lim_{t\to 0}\frac{1}{t}(f(r_{a,y,t})-f(a)),$$

where $r_{a,y,t}$ is given by

$$r_{a,y,t}\in r_{a,y},\quad \rho(a,\quad r_{a,y,t})=t.$$

We denote g by $d_\rho f$ *or*, $d_\rho f(a)$*or* $d_\rho,\ af$.

We note that this definition of the generalized tangent depends on the choice of parameter t of γ(cf. n°11 of § 3).

If $\mathscr{F}(S^{n-1})$ is taken to be $C(S^{n-1})$, the Banach space consisted by the continuous functions on S^{n-1} with the uniform convergence topology, then $C^*(S^{n-1})$ is the space of Radon measures on S^{n-1}(cf. [18]), and we can show that an element of $C^*(S_a)$ is expressed as a generalized tangent at a of a curve if and only if it is positive, that is $<\xi,\ g>\geqq 0$ if $g(y)\geqq 0$ on S_a. For example, the Dirac measure on S_a is expressed as the generalized tangent at a of a curve γ which is smooth at a. Here a curve γ given by $\rho: I\to M$, $\rho(0)=a$, is called smooth at a if $\lim_{t\to 0}\varepsilon_{a,\varphi t}$ exists. The problem to characterize the element of $C^{\infty *}(S^{n-1})$, the space of distributions on S^{n-1} or $C^{\omega *}(S^{n-1})$, the space of analytic functionals on S^{n-1}, which is expressed as the generalized tangent of some curve, remains open.

We note that although the O. N. -basis of $L^2(S^{n-1})$ is given by spherical harmonics (cf. [5], [11]), a smooth function at a only represents a spherical function of degree 1. Hence, since the usual tangent of a smooth curve is defined only by using smooth functions, the usual tangents of smooth curves corresponds only this part of $L^2(S^{n-1})$. But the above result shows, if we use the $L^2(S^{n-1})$-smooth functions, the generalized tangents covers the positive part of $L^2(S^{n-1})$.

As in the case of usual tangent vectors (cf. [6], [13]), to set

$$\mathfrak{X}_\varphi(f)=\lim_{s\to 0}\frac{1}{s}[\lim_{h\to 0}\int_h^s\frac{1}{t}\{f(\varphi(t))-f(a)\}dt],$$

where f is $\mathscr{F}(S^{n-1})$-smooth at a, we have

(i) $\mathfrak{X}_\varphi(\alpha f_1 + \beta f_2) = \alpha\mathfrak{X}_\varphi(f_1) + \beta\mathfrak{X}_\varphi(f_2)$,

(ii) $\mathfrak{X}_\varphi(f_1 f_2) = f_1(a)\mathfrak{X}_\varphi(f_2) + f_2(a)\mathfrak{X}_\varphi(f_1)$,

and we also have

(iii) $\mathfrak{X}_\varphi(f) = 0$ if $|f(x) - f(a)| = o(\rho(a, x))$,

(iv) $\mathfrak{X}_\varphi(f) \geqq 0$ *if* $d_\rho f \geqq 0$.

On the other hand, if the map $\mathfrak{X}$ from the space of $\mathscr{F}(S^{n-1})$-smooth functions at a to $\boldsymbol{R}$, the real number field, satisfies (i), (ii) and (iii), then $\mathfrak{X}$ is written as

$$f = <\xi,\ d_\rho f(a)>,$$

by some $\xi \in \mathscr{F}^*(S_a)$. Hence if $\mathfrak{X}$ also satisfies (iv), $\mathfrak{X}$ is written as $\mathfrak{X}_\varphi$ by some $\varphi : \boldsymbol{I} \to M$. In some part, the globalization of these discussions are possible. To do this, first we constract the associate $\mathscr{F}(S^{n-1})$-bundle of the tangent micro-bundle of M, which is denoted by $\mathscr{F}(s(M))$ and its dual bundle, which is denoted by $\mathscr{F}^*(s(M))$ (§ 1). (cf. [1], [9], [12]).

Next, we set

$$d_\rho f(x,\ y) = \lim_{t \to 0} \frac{1}{t}(f(r_{x,y,t}) - f(x)),\quad y \in S_x.$$

If $d_\rho f(x)$ is a continuous cross-section of $\mathscr{F}(s(M))$, then we call f is $\mathscr{F}(S^{n-1})$-smooth on M (for $n = 1$, cf. [7], [8], [10]). We can show that the space of $\mathscr{F}(S^{n-1})$-smooth functions on M (denoted by $C_{\mathscr{F}(S^{n-1})}(M)$ is dense in $C(M)$ or in $L^p_{loc}(M)$ (§ 2, theorem 1). (The measure on M by which $L^p(M)$ or $L^p_{loc}(M)$ is defined, is that of induced from ρ (cf. [3], [4])). Then a linear operator X of C(M) which satisfies the following (i), (ii), (iii) is called an $\mathscr{F}(S^{n-1})$-vector field on M.

(i). *X is a closed operator from $C_{\mathscr{F}(S^{n-1})}(M)$ into $C(M)$.*

(ii). *$(Xf)(a) = 0$, if $|f(x) - f(a)| = o(\rho(a, x))$ at a.*

(iii). *$X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$.*

We show that if X is an $\mathscr{F}(S^{n-1})$-vector field on M, then X is written as

$$Xf(x) = <\xi(x),\ d_\rho f(x)>,\quad x \in M,$$

where ξ is a continuous cross-section of $\mathscr{F}^*(s(M))$ (§ 2, theorem 2). Therefore, as usual vector field, we may identify X and a continuous cross-section of $\mathscr{F}^*(s(M))$. But an $\mathscr{F}(S^{n-1})$-vector field X does not generate a 1-parameter group germ of M in general. For example, the theorem of Hille-Yosida shows that if M is compact and simply connected, the $C(S^{n-1})$-vector field X corresponds to the cross-section m of $C^*(s(M))$ given by $m = m(x)$, $m(x)$ is the cannonical measure on

S_x defined from the metric, does not generate any (equi-continuous) 1-parameter semi group of $C(M)$ or $L^p(M)$ (§2, exemple). (cf. [17], [18]). We note since $m(x)$ is positive, there exists a curve $\gamma=\gamma_x$ for any x, such that γ_x starts from x and whose generalized tangent at x is $m(x)$ (cf. §3, exemple 2), if $n=2$, γ_x is given by $r\theta=1$.

As usual vector field, if X, Y are $\mathscr{F}(S^{n-1})$-vector fields such that theire compositions XY and YX are both possible, then

$$[X, Y]=XY-YX$$

is also an $\mathscr{F}(S^{n-1})$-vector field of M. But the composition of $\mathscr{F}(S^{n-1})$-vector fields may not be possible in general.

In §1, we also construct associate $\mathscr{F}(\overbrace{S^{n-1}\times\cdots\cdots\times S^{n-1}}^{p})$-bundle of the tangent microbundle of M. It is denoted by $\mathscr{F}(s^p(M))$. We denote by $A\mathscr{F}(s^p(M))$ the subbundle of $\mathscr{F}(s^p(M))$ whose fibre is consisted by those functions $f(y_1, \cdots\cdots, y_p)$, $y_i\in S^{n-1}$, of $\mathscr{F}(S^{n-1}\times\cdots\cdots\times S^{n-1})$ such that

$$f(y_{\sigma(1)}, \cdots\cdots, y_{\sigma(p)})=\operatorname{sgn}(\sigma)f(y_1, \cdots\cdots, y_p),\quad \sigma\in\gamma^p.$$

The cross-sections of these bundles are considered to be reductions of Alexander-Spanier cochains (cf. [1], [3], [14], [15]).

For the cross-sections of $\mathscr{F}(s^p(M))$ and $A\mathscr{F}(s^p(M))$, we define the maps d_p and Ad_p by

$$\begin{aligned}
&d_pf(x, y_1, \cdots\cdots, y_{p+1})\\
&=\lim_{t\to 0}\frac{1}{t}[f(r_{x,y_1,t}, y_2, \cdots\cdots, y_{p+1})-f(x, y_2, \cdots\cdots, y_{p+1}),\\
&Ad_pf(x, y_1, \cdots\cdots, y_{p+1})\\
&=\frac{1}{p+1}\sum_{i=1}^{p+1}(-1)^{i+1}[\lim_{t\to 0}\frac{1}{t}(f(r_{x,y,it}, y_1, \cdots\cdots, y_{i+1}, \cdots\cdots, y_{p+1})-\\
&\qquad\qquad f(x, y_1, \cdots\cdots, y_{i-1}, y_{i+1}, \cdots\cdots, y_{p+1}))].
\end{aligned}$$

We call f is $\mathscr{F}(S^{n-1})$-smooth if d_pf (or Ad_pf) defines a continuous cross-section of $\mathscr{F}(s^{p+1}(M))$ (or $A\mathscr{F}(s^{p+1}(M))$). We note that to define $\int_\gamma f(x, y_1, \cdots\cdots, y_p)$ by

$$\begin{aligned}
&\int_\gamma f(x, y_1, \cdots\cdots, y_p)\\
&=\int_\gamma f(x, \varepsilon_{x,x_1}, \cdots\cdots, \varepsilon_{x,x_p})\,\rho(x, x_1)\cdots\cdots\rho(x, x_p),
\end{aligned}$$

where γ is a singular p-chain of M and the right hand side is the integration of

Alexander-Spanier cochain defined in [3], $\int_\gamma f$ is exists if f is $\mathscr{F}(S^{n-1})$-smooth and γ is given by $\varphi : I^p \to M$ where φ satisfies

$$\rho(\varphi(a_{J+1_i}),\ \varphi(a_J)) \leqq N|a_{j_i+1} - a_{j_i}|,$$

$$a_J = (a_{j_1},\ \cdots\cdots,\ a_{j_p}),\ \ a_{J+1_i} = (a_{j_1},\ \cdots\cdots,\ a_{j_{i-1}},\ a_{j_{i+1}},\ a_{j_{i+1}},\ \cdots\cdots,\ a_{j_p}).$$

for some $N > 0$. Since $Ad_\rho(Ad_\rho f) = 0$ if $Ad_\rho f$ is $\mathscr{F}(S^{n-1})$-smooth, we can obtain the analogy of de Rham's theorem by using the cross-sections of $A\mathscr{F}(s^p(M))$ and the Cech cohomology group of M. But the above shows that the analogy of de Rham's theorem is also obtained by using the singular homology group of M (cf. [15], [16]).

We note that if $M = \boldsymbol{R}^1$, the 1-dimensional euclidean space with the euclidean metric, then

$$d_\rho f(x) = (D_+ f(x),\ \ D_- f(x)),$$

where D_+ and D_- mean the right hand side and the left hand side derivations of f and the (fibre of $\mathscr{F}(s(\boldsymbol{R}^1))$ is $\boldsymbol{R} \oplus \boldsymbol{R}$. We know that f is smooth if and only if $D_+ f = D^- f$ at any point of $\boldsymbol{R}^1$, that is $d_\rho f$ defines a cross-section of the subbundle of $\mathscr{F}(s(\boldsymbol{R}^1))$ whose fibre is the diagonal of $\boldsymbol{R} \oplus \boldsymbol{R}$.

To generalize this, first we assume the metric ρ of M satisfies (*). *If* $\rho(x_1, x_2) \leqq 2$, *then there is unique path* γ *which joins* x_1 *and* x_2 *and*

$$\int_\gamma \rho = \rho(x_1,\ \ x_2).$$

Under this assumption, for any $y \in S_x$, there exists unique point $\hat{y}$ of S_x such that

$$\rho(y,\ \ \hat{y}) = 2.$$

We denote the quotient space of S_x obtained by identifying $\hat{y}$ and y by P_x. By definition, P_x is homeomorphic to $\boldsymbol{R}P^{n-1}$, the $(n-1)$-dimensional real projective space.

For this P_x, if f is $\mathscr{F}(S^{n-1})$-smooth at x and

$$d_\rho f(x,\ \ \hat{y}) = d_\rho f(x,\ \ y),$$

for any $y \in S_x$, then $d_\rho f$ may be considered to be an element of $\mathscr{F}(P_x)$. Here $\mathscr{F}(P_x)$ is defined similarly as $\mathscr{F}(S_x)$ and it is also considered to be a subspace of $\mathscr{F}(S_x)$ given by

$$\mathscr{F}(P_x) = \{g \mid g \in \mathscr{F}(S_x),\ \ g(y) = g(\hat{y})\}.$$

Since $\mathscr{F}(P_x)$ is isomorphic to $\mathscr{F}(RP^{n-1})$, we call f to be $\mathscr{F}(RP^{n-1})$-smooth in this case. If M is $\boldsymbol{R}^n$, the n-dimensional euclidean space with the euclidean metric, then f is $M(S^{n-1})$-smooth at x if and only if f is one-sided differentiable at x along any line which ends at x and f is $M(\boldsymbol{R}P^{n-1})$-smooth at x if and only if f is differentiable at x along any line which through x.

Since the tatal spaces of $s(M)$ and $s^p(M)$, the associate S^{n-1} and $\overbrace{S^{n-1}\times\cdots\cdots\times S^{n-1}}^{p}$ bundles of the tangent microbundle of M are given by

$$s(M)=\{(x,\ y)\,|\,\rho(x,\ y)=1,\ x\in M,\ (x,\ y)\in M\times M,$$
$$s^p(M)=\{(x,\ y_1,\ \cdots\cdots,\ y_p)\,|\,\rho(x,\ y_i)=1,\ i=1,\ \cdots\cdots,\ p,\ x\in M,$$
$$(x,\ y_1,\ \cdots\cdots,\ y_p)\in M\times\overbrace{M\times\cdots\cdots\times M}^{p}\},$$

we can construct the associate $\boldsymbol{R}P^{n-1}$-bundle and $\overbrace{\boldsymbol{R}P^{n-1}\times\cdots\cdots\times\boldsymbol{R}P^{n-1}}^{p}$-bundle of $\tau(M)$, the tangent microbundle of M, by taking $s(M)/\sim$ and $s^p(M)/\underset{p}{\sim}$ to be the tatal spaces. Here the equivalence relations $\sim$ or $\underset{p}{\sim}$ are given by

$$(x,\ y)\sim(x',\ y')\ \textit{if and only if}\ x=x'\ \textit{and}\ \rho(y,\ y')=2.$$

$$(x,\ y_1,\ \cdots\cdots,\ y_p)\underset{p}{\sim}(x',\ y_1',\ \cdots\cdots,\ y_p'),$$
$$\textit{if and only if}\ x=x,\ \textit{and}\ \rho(y_i,\ y_i')=2,\ i=1,\ \cdots\cdots,\ p,$$

where ρ is assumed to satisfy $(^*)'$. Then using $s(M)/\sim$ and $s^p(M)/\underset{p}{\sim}$, we can construct associate $\mathscr{F}(\boldsymbol{R}P^{n-1})$-bundle and $\mathscr{F}(\overbrace{\boldsymbol{R}P^{n-1}\times\cdots\cdots\times\boldsymbol{R}P^{n-1}}^{p})$-bundle of $\tau(M)$. They are denoted by $\mathscr{F}(s(M)/\sim)$ and $\mathscr{F}(s^p(M)/\underset{p}{\sim})$. We note that since we have

$$\mathscr{F}(S^{n-1})=\mathscr{F}(\boldsymbol{R}P^{n-1})\oplus\check{\mathscr{F}}(\boldsymbol{R}P^{n-1}),$$
$$\mathscr{F}(\boldsymbol{R}P^{n-1})=\{g\,|\,g(y)=g(\hat{y})\},\quad \check{\mathscr{F}}(\boldsymbol{R}P^{n-1})=\{g\,|\,g(y)=-g(\hat{y})\},$$

we may consider $\mathscr{F}(s(M)/\sim)$ and $\mathscr{F}(s^p(M)/\underset{p}{\sim})$ are the subbundles of $\mathscr{F}(s(M))$ and $\mathscr{F}(s^p(M))$ and can be considered to be direct summands of them.

We note that using $\mathscr{F}(\boldsymbol{R}P^{n-1})$-smooth functions and the bundles $\mathscr{F}(s(M)/\sim)$, $\mathscr{F}(s^p(M)/\underset{p}{\sim})$ and $A\mathscr{F}(s^p(M)/\underset{p}{\sim})$ ($A\mathscr{F}(s^p(M)/\underset{p}{\sim})$ is defined similarly as others), we can construct same theories as above.

Similarly, if $M=\boldsymbol{C}$, the complex number plane with the euclidean metric, f a holomorphic function, then

$$d_\rho f(a,\ y)=\frac{df}{dz}(a).$$

This suggests that if $dim.\,M=2m$, then the condition
$(^{**})$, *there exists associate* $\boldsymbol{C}P^{m-1}$-*bundle of* $\tau(M)$,
may have some meaning for M.

The outline of this paper is as follows: In §1, we define the bundles

$\mathscr{F}(s)M))$, $\mathscr{F}(s^p(M))$ and $A\mathscr{F}(s^p(M))$ and treat their properties. In § 2, we define $\mathscr{F}(S^{n-1})$-smooth functions and $\mathscr{F}(S^{n-1})$-vector fields. The generalized tangents of curves ant their properties are stated in § 3.

Added in proof. In $\boldsymbol{R}^n$ with the euclidean metric, $d_\rho f$ may be considered (one-sided) Gâteaux's differential Vf. Here Gâteaux's differential $Vf(x_0, h)$ is defined by

$$Vf(x_0, h) = \lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t}, \quad h \in M_2,$$

where f is a map from a Banach space M_1 to a Banach space M_2. For the details and related notions with their applications, see Burysek, S. : *On symmetric G-differential and convex functionals in Banach spaces, Publ. Math., (Debrecen)*, 17, 1970, 145–161).

§ 1. Bundles $\mathscr{F}(s(M))$ and $\mathscr{F}(s^p(M))$.

1. We denote by M an n-dimensional connected paracompact topological manifold. On M, we fix a metric ρ by which the topology of M is given, and assume ρ satisfies the following (i), (ii), (iii) (For the existence of such metric, see [4]).

(i). *If $\rho(x_1, x_2) \leq 1$, then there exists unique path γ which joins x_1 and x_2 and*

$$\int_\gamma \rho = \rho(x_1, x_2).$$

(ii). *M is complete with respect to ρ.*

(iii). *The measure $m(\rho)$ induced from ρ on M is a positive Radon measure and satisfies*

$m(\rho)(E) \neq 0$, *if E is measurable and containes some non empty open set.*

For $x \in M$, we set

$$S_x = \{y \mid y \in M, \ \rho(x, y) = 1\}.$$

Since $dim. M = n$, S_x is homeomorphic to S^{n-1}, the unit $(n-1)$-sphere (cf. [4]). We assume that for any x, ρ induces a metric ρ_x on S_x which is given by

$$\rho_x(y_1, y_2) = \inf_{\gamma,\ \gamma \text{ joins } y_1 \text{ and } y_2 \text{ in } S_x} \int_\gamma \rho.$$

The measure on S_x induced from ρ_x (cf. [3]) is denoted by $m = m(x)$. For this $m(x)$, we assume (cf. [4])

(i). *A Borel set of S_x is $m(x)$-measurable and if E is $m(x)$ measurable and contains some non-empty open set of S_x, then*

$$m(x)(E) \neq 0.$$

(ii). *$m(x)$ depends continuously on x.*

Since S_x is compact, $m(x)(S_x)$ is finite. Hence, for the simplicity, we normalize $m(x)$ to satisfy $m(x)(S_x)=1$.

Note. If M is a smooth manifold, ρ is the geodesic distance defined by a (complete) Riemannian metric on M, then $m(x)$ depends differentiably on x.

In $M\times M$, we set

$$s(M)=\{(x,\ y)\,|\,x\in M,\ \rho(x,\ y)=1\}. \tag{1}$$

We define π: $s(M)\to M$ by $\pi(x,\ y)=x$. Then $\{s(M),\ \pi,\ M\}$ is the associate unit sphere bundle of the tangent microbundle of M(cf. [1], [9], [12]). We denote the transition function of $s(M)$ by $\{g_{UV}(x)\}$ if we consider the fidre of $s(M)$ at x to be S_x. We note that if we consider the fibre of $s(M)$ at x to be the measure space $(S_x,\ m_x)$, then the transition function of $s(M)$ should be replaced by $\{(g_{UV}(x),$ $m_U(x)\,(g_{UV}(x)^*m_V(x))^{-1})\}$, where $m_U(x)$ is given by

$$m(x)(E)=\int_{h_{U,\ x(E)}} m_U(x)d\Omega.$$

Here $h_{U,x}$ is the local homeomorphism from $\pi^{-1}(U)$ to $U\times S^{n-1}$ and $d\Omega$ is the standard measure on S^{n-1}.

We denote by $\mathscr{F}(S^{n-1})$ a function space over S^{n-1}. In the rest, (S^{n-1}) means either of $C(S^{n-1})$ or $L^p(S^{n-1})$, $1\leqq p\leqq\infty$, regarding them to be Banach spaces. Here $L^p(S^{n-1})$ is defined by dΩ. (If M is smooth, or real analytic, then $C^\infty(S^{n-1})$, or $C^\omega(S^{n-1})$, is also taken as (S^{n-1})). Then by identifying $U\times C(S^{n-1})\in(x,\ f(y))$ and $(x,\ f(g_{UV}(x)y))\in V\times C(S^{n-1})$, $x\in U\cap V$, we obtain the associate $C(S^{n-1})$-bundle of $s(M)$. It is denoted by $C(s(M))$. Since $C(s(M))$ is a vector bundle over M with the fibre $C(S^{n-1})$, its dual bundle $C^*(s(M))$ is defined. $C^*(s(M))$ is a vector bundle over M with the fibre $C^*(S^{n-1})$, where $C^*(S^{n-1})$ is the space of Radon measures on S^{n-1}.

***Lemma* 1.** *Regarding $m(x)$ to be a function on M, $m(x)$ is a cross-section of $C^*(s(M))$.*

Corollary. *We have*

$$m_U(x)(g_{UV}(x)^*m_V(x))^{-1}=1. \tag{2}$$

By this corollary, although $\mathscr{F}(S^{n-1})$ is $L^p(S^{n-1})$, we can construct the associate $\mathscr{F}(S^{n-1})$-bundle of $s(M)$ by identifying $U\times\mathscr{F}(S^{n-1})\ni(x,\ f(y))$ and $(x,\ g_{UV}(x)^*$ $f(y))\in V\times\mathscr{F}(S^{n-1})$, $x\in U\cap V$. This bundle is denoted by $\mathscr{F}(s(M))$. The dual bundle of $\mathscr{F}(s(M))$ is denoted by $\mathscr{F}^*(s(M))$. By definftion, the fibre of $\mathscr{F}^*(s(M))$ is $\mathscr{F}^*(S^{n-1})$, the dual space of $\mathscr{F}(S^{n-1})$. We denote the fibre of $\mathscr{F}(s(M))$ (and $\mathscr{F}^*(s(M))$) at x by $\mathscr{F}(S_x)$ (and $\mathscr{F}^*(S_x)$).

Definition. *An element of $\mathscr{F}^*(S_x)$ is called an $\mathscr{F}(S^{n-1})$-vector at x.*

Note. If we regard S_x to be a measure space $(S_x, k(x))$, and define $L^p(S_x)$ by $k(x)$, then to define $K_U(x)$ similarly as $m_U(x)$, we obtain the associate $L^p(S^{n-1})$-bundle of $s(M)$ by identifying $U \times L^p(S^{n-1}) \ni (x, f(y))$ and $(x, [k_U(x)(g_{UV}(x)^* k_V(x))^{-1/p} g_{UV}(x)^* f(y)) \in V \times L^p(S^{n-1}), x \in U \cap V$.

2. In $\overbrace{M \times M \times \cdots\cdots \times M}^{p}$, we set

$$(1)' \qquad s^p(M) = \{(x, y_1, \cdots\cdots, y_p) | x \in M, \rho(x, y_i) = 1, i = 1, \cdots\cdots, p\}.$$

To define $\pi: s^p(M) \to M$ by $\pi(x, y_1, \cdots\cdots, y_p) = x$, $\{s^p(M), \pi, M\}$ is associate $\overbrace{S^{n-1} \times \cdots\cdots \times S^{n-1}}^{p}$-bundle over M. If the fibre of $s(M)$at x is considered to be the measure space $(S_x, m(x))$, then we consider the fibre of $s^p(M)$ at x to be the measure space $(S_x \times \cdots\cdots \times S_x, m(x) \otimes \cdots\cdots \otimes m(x))$. The transition functions $\{g_{UV}(x)\} = \{g_{UV}{}^p(x)\}$ of $s^p(M)$ is given by

$$g_{UV}{}^p(x)(y_1, \cdots\cdots, y_p) = (g_{UV}(x) y_1, \cdots\cdots, g_{UV}(x) y_p),$$

where $g_{UV}(x)$ in the right hand side is the transition function of $s(M)$.

We denote by $\mathscr{F}(\overbrace{S^{n-1} \times \cdots\cdots \times S^{n-1}}^{p})$ or $\mathscr{F}^p(S^{n-1})$ the function space over $S^{n-1} \times \cdots\cdots \times S^{n-1}$ which is of the same type with $\mathscr{F}(S^{n-1})$. That is $\mathscr{F}^p(S^{n-1})$ means either of $C(S^{n-1} \times \cdots\cdots \times S^{n-1})$ or $L^p(S^{n-1} \times \cdots\cdots \times S^{n-1})$ with the measure $m(x) \otimes \cdots\cdots \otimes m(x)$ in general and $C^\infty(S^{n-1} \times \cdots\cdots \times S^{n-1})$ or $C^\omega(S^{n-1} \times \cdots\cdots \times S^{n-1})$ is also considered if M is smooth or real analytic. By assumption, $\overbrace{\mathscr{F}(S^{n-1}) \otimes \cdots\cdots \otimes \mathscr{F}(S^{n-1})}^{p}$ is dense in $\mathscr{F}(S^{n-1} \times \cdots\cdots \times S^{n-1})$.

As $\mathscr{F}(s(M))$, we construct the associate $\mathscr{F}^p(S^{n-1})$-bundle of $s^p(M)$. It is denoted by $\mathscr{F}(s^p(M))$. The dual bundle of $\mathscr{F}(s^p(M))$ is denoted by $\mathscr{F}^*(s^p(M))$. The fibres of $\mathscr{F}(s^p(M))$ and $\mathscr{F}^*(s^p(M))$ at x are denoted by $\mathscr{F}(S_x \times \cdots\cdots \times S_x)$ or $\mathscr{F}^p(S_x)$ and $\mathscr{F}^*(S_x \times \cdots\cdots \times S_x)$ or $\mathscr{F}^{p*}(S_x)$.

Definition. *An element of $\mathscr{F}^{p*}(S_x)$ is called an $\mathscr{F}(S^{n-1})$-p-vector at x.*

For any $f \in \mathscr{F}^p(S^{n-1})$ and $\sigma \in \gamma^p$, we set

$$(3) \qquad \sigma(f)(y_1, \cdots\cdots, y_p) = f(y_{\sigma(1)}, \cdots\cdots, y_{\sigma(p)}), \quad y_i \in S^{n-1}.$$

Then, since $\mathscr{F}(S^{n-1}) \otimes \cdots\cdots \otimes \mathscr{F}(S^{n-1})$ is dense in $\mathscr{F}^p(S^{n-1})$, σ is continuous. Therefore, setting

$$A\mathscr{F}^p(S^{n-1}) = \{f | f \in \mathscr{F}^p(S^{n-1}), \sigma(f) = \mathrm{sgn}(\sigma) f\},$$

$A\mathscr{F}^p(S^{n-1})$ is a closed subspace of $\mathscr{F}^p(S^{n-1})$. Since σ^*, the adjoint operator of σ, is σ^{-1}, we have

$$(A\mathscr{F}^p(S^{n-1}))^* = A\mathscr{F}^{p*}(S^{n-1}).$$

As we know

$$\sigma(f(g_{UV}(x)y_1, \cdots\cdots, g_{UV}(x)y_p))$$
$$= f(g_{UV}(x)y_{\sigma 1}, \cdots\cdots, g_{UV}(x)y_{\sigma p}), \quad f \in \mathscr{F}^p(S^{n-1}),$$

we obtain an $A\mathscr{F}^p(S^{n-1})$-bundle over M to be a subbundle of $\mathscr{F}^p(s(M))$. This bundle is denoted by $A\mathscr{F}(s^p(M))$. Its dual bundle is denoted by $A\mathscr{F}^*(s^p(M))$. The fibres of $A\mathscr{F}(s^p(M))$ and $A\mathscr{F}^*(s^p(M))$ are denoted by $A\mathscr{F}^p(S_x)$ and $A\mathscr{F}^{p*}(S_x)$.

Note. Similarly, to set

$$S\mathscr{F}^p(S^{n-1}) = \{f \mid f \in \mathscr{F}^p(S^{n-1}), \ \sigma(f) = f\},$$

we can define an $S\mathscr{F}^p(S^{n-1})$-bundle $S\mathscr{F}(s^p(M))$ to be a subbundle of $\mathscr{F}(s^p(M))$. Its dual bundle is denoted by $S\mathscr{F}^*(s^p(M))$. The fibres of $S\mathscr{F}(s^p(M))$ and $S\mathscr{F}^*(s^p(M))$ at x are denoted by $S\mathscr{F}^p(S_x)$ and $S\mathscr{F}^{p*}(S_x)$.

Definition. *A (continuous) cross-section φ of $\mathscr{F}(s^p(M))$ is called a (continuous) $\mathscr{F}(S^{n-1})$-p-cochain of M. If φ is a cross-section of $A\mathscr{F}(s^p(M))$, then φ is called an $\mathscr{F}(S^{n-1})$-p-form of M.*

Definition. *A (continuous) cross-section of $\mathscr{F}^*(s^p(M))$ is called an $\mathscr{F}(S^{n-1})$-p-vectorfield of M.*

In general, we call an element of $\mathscr{F}^p(S_x) \otimes \mathscr{F}^{q*}(S_x)$ to be an $\mathscr{F}(S^{n-1})$-(p, q)-tensor at x and a continuous cross-section of $\mathscr{F}(s^p(M)) \otimes \mathscr{F}^*(s^q(M))$ to be an $\mathscr{F}(S^{n-1})$-(p, q)-tensorfield of M.

If M is smooth (or real analytic), then $\mathscr{F}(s^p(M))$ and $\mathscr{F}^*(s^q(M))$ allow the structure of smooth (or real analytic) vector bundles. Hence we can define smooth (or real analytic) $\mathscr{F}(S^{n-1})$-p-cochain, etc..

3. We denote by $r_{x,y}$ the unique curve which joins x and y, $y \in S_x$ and satisfies

$$\int_{r_{x,y}} \rho = 1.$$

Then for any a, $0 \leqq a \leqq 1$, there exists unique point z in $r_{x,y}$ such that $\rho(x, z) = a$. We denote this z by $r_{x,y,a}$.

On the other hand, if $\rho(x, z) < 1$, then there exists unique point y of S_x such that $z \in r_{x,y}$. Or, in other word, x, z determines a point y of S_x. We denote this y by $\varepsilon_{x,z}$.

By definition, if $\rho(x, z) < 1$, then

$$r_{x,\varepsilon_{x,z},\rho(x,z)} = z. \tag{4}$$

For an $\mathscr{F}(S^{n-1})$-p-cochain $\varphi = \varphi(x, y_1, \cdots\cdots, y_p)$ of M, we set

$$\tilde{\varphi}(x, x_1, \cdots\cdots, x_p) \tag{5}$$
$$= \varphi(x, \varepsilon_{x,x_1}, \cdots\cdots, \varepsilon_{x,x_p})\rho(x, x_1) \cdots \rho(x, x_p),$$

$$x_i \in M,\ \rho(x,\ x_i) < 1,\ i = 1,\ \cdots\cdots,\ p.$$

Then $\tilde{\varphi}$ defines an Alexander-Spanier p-cochain of M. By definition, if φ is an $\mathscr{F}(S^{n-1})$-p-form, then $\tilde{\varphi}$ is alternative in $x_1,\ \cdots\cdots,\ x_p$.

Definition. *If γ is a singular p-chain of M, then we define the integration $\int_\gamma \varphi$ of φ, an $\mathscr{F}(S^{n-1})$-p-cochain of M on γ by*

$$\int_\gamma \varphi = \int_\gamma \tilde{\varphi}. \tag{6}$$

Here the right hand side means the integral of the Alexander-Spanier cochain $\tilde{\varphi}$ on γ ([3]).

By the definition of the integral (cf. [3])), if φ is a $C(S^{n-1})$-p-cochain and γ is given by f: $\boldsymbol{I}^p \to M$ where f satisfies

$$\begin{aligned} &\rho(f(a_{J+1_i}),\ f(a_J)) \leqq N|a_{j_i+1} - a_{j_i}|, \\ &a_{J+1_i} = (a_{j_1},\ \cdots\cdots,\ a_{j\text{-}i_1},\ a_{j_i+1},\ a_{j_{i+1}},\ \cdots\cdots,\ a_{j_n}),\ a_J = (a_{j_1},\ \cdots\cdots,\ a_{j_n}), \end{aligned} \tag{7}$$

for some $N < 0$, then φ is absolutely integrable on γ. In fact, since S^{n-1} and γ both compact, to set

$$K = \max_{x \in \gamma}.\ (\max_{y_i \in S \times \cdots\cdots \times S_x}.\ |\varphi(x,\ y_1,\ \cdots\cdots,\ y_p)|),$$

K is finite, and for any partition Δ of $\boldsymbol{I}$, we have

$$\begin{aligned} &|\sum_J |\tilde{\varphi}\varphi(f(a_J),\quad f(a_{J+1}),\ \cdots\cdots,\ f(a_{J+1_p}))| \\ &\leqq KN^p(\sum_J |a_{j_1+1} - a_{j_1}| \cdots\cdots |a_{j_p+1} - a_{j_p}|) \leqq KN^p, \\ &\Delta \textit{ is given by } 0 = a_o < a_1 < \cdots\cdots < a_m < 1, \end{aligned}$$

which shows the absolute integrability of $\tilde{\varphi}$ on γ.

Note. This is also true if φ is an $M(S^{n-1})$-p-cochain and it seems to be true for $L^q(S^{n-1})$-p-cochains if we change the definition of the integral of Alexander-Spanier cochains to the Lebesgue type.

Definition. *For an $\mathscr{F}(S^{n-1})$-p-cochain $\varphi = \varphi(x,\ y_1,\ \cdots\cdots,\ y_p)$ of M, we define*

$$\begin{aligned} &d_\rho\varphi(x,\ y_1,\ \cdots\cdots,\ y_{p+1}) \\ &= \lim_{a \to 0}.\ \frac{1}{a}(\varphi(r_{x, y_1, a},\ y_2,\ \cdots\cdots,\ y_{p+1}) - \varphi(x,\ y_2,\ \cdots\cdots,\ y_{p+1})). \end{aligned} \tag{8}$$

By definition, if $d_\rho\varphi(x,\ y_1,\ \cdots\cdots,\ y_{p+1})$ exists as an element of $\mathscr{F}^p(S_x)$ for any x and continuous in x, then $d_\rho\varphi$ is an $\mathscr{F}(S^{n-1})$ - $(p+1)$-cochain of M.

Definition. *An $\mathscr{F}(S^{n-1})$-p-cochain φ is called $\mathscr{F}(S^{n-1})$-smooth if $d_\rho\varphi$ is a (con-*

tinuous) $\mathscr{F}(S^{n-1})$ - $(p+1)$-*cochain of* M.

In general we befine $d_\rho{}^m\varphi$ dy

$$d_\rho{}^m\varphi = d_\rho(d_\rho{}^{m-1}\varphi),$$

and call φ to be $\mathscr{F}(S^{n-1})$-m-smooth if $\mathrm{d}_\rho{}^m\varphi$ is a (continuous) $\mathscr{F}(S^{n-1})$ - $(p+m)$-cochain of M. If φ is $\mathscr{F}(S^{n-1})$-smooth for all m, then we call φ to be $\mathscr{F}(S^{n-1})$ $-\infty$-smooth.

Definition. *For an* $\mathscr{F}(S^{n-1})$-p-*form*$\varphi = \varphi(x, y_1, \cdots\cdots, y_p)$ *of* M, *we define*

$$(8)' \qquad Ad_\rho\varphi(x, y_1, \cdots\cdots, y_{p+1})$$
$$=\frac{1}{p+1}\Big[\sum_{i=1}^{p+1}(-1)^{i+1}(\lim_{a\to 0}\frac{1}{a}(\varphi(r_{x,y_i,a}, y_1, \cdots\cdots, y_{i-1}, y_{i+1}, \cdots\cdots, y_{p+1})-$$
$$-\varphi(x, y_1, \cdots\cdots, y_{i-1}, y_{i+1}, \cdots\cdots, y_{p+1}))\Big].$$

By definition, if φ is $\mathscr{F}(S^{n-1})$-smooth, then $Ad_\rho\varphi$ is an $\mathscr{F}(S^{n-1})-(p+1)$-form and if φ is $\mathscr{F}(S^{n-1})$-3-smooth, then

$$(9)' \qquad Ad_\rho(Ad_\rho\varphi) = 0.$$

By (9)$'$, denoting $C^p(M, \mathscr{F}(S^{n-1}))$ the space of $\mathscr{F}(S^{n-1})$-∞-smooth $\mathscr{F}(S^{n-1})$-p-forms on M, $\{\sum_{p\geqq 0}C^p(M, \mathscr{F}(S^{n-1})), Ad_\rho\varphi\}$ is a differential complex and we can show the analogy of de Rham's theorem. Because we know

$$(9)' \qquad \widetilde{Ad_\rho\varphi} = \delta\tilde{\varphi},$$

where δ is the coboundary homomorphism in the Alexander-Spanier cochain. By (9), we also have

$$(10) \qquad \int_{\partial r}\varphi = \int_r Ad_\rho\varphi,$$

if φ is an $\mathscr{F}(S^{n-1})$-p-form (cf. [3]).

Note. By (10), we have especially

$$\int_r d_\rho f = \int_{r'} d_\rho f, \textit{ if } \gamma \textit{ and } \gamma' \textit{ start from same point and end at same point.}$$

Because for a function f, we have

$$Ad_\rho f = d_\rho f.$$

Therefore, we may write $\int_a^x d_\rho f$ if $\rho(a, x)$ is small and we obtain

$$(10)' \qquad \int_a^x d_\rho f = f(x) - f(a).$$

§ 2. Generalized vector fields.

4. ***Definition.*** *A function f on some neighborhood of x is called to be $\mathscr{F}(S^{n-1})$-smooth at x if $(d_{\rho,x}f)(y)=d_\rho f(x, y)$, x is fixed, defines a function of $\mathscr{F}(S_x)$.*

By definition, we have

Lemma 2. *f is $\mathscr{F}(S^{n-1})$-smooth at a if and only if f is written as*

$$f(x)=f(a)+g(\varepsilon_{a,x})\rho(a, x)+o(\rho(a, x)), \tag{11}$$

where x belongs in $U(a)$, a neighborhood of a and $g(y)$ is an element of $\mathscr{F}(S_x)$.

For example, if $M=\boldsymbol{R}^n$, n-dimensional euclidean space, ρ is the euclidean metric of $\boldsymbol{R}^n$ and f is smooth at a, then f is written as

$$f(x)=f(a)+(\sum_i \frac{\partial f(a)}{\partial x_i}(x_i-a_i)/||x-a||)||x-a||+o(||x-a||),$$

where $=(x_1, \cdots\cdots, x_n)$, $a=(a_1, \cdots\cdots, a_n)$ and $||x||=\sqrt{\sum_i x_i^2}$. Then since $g(y)=\sum_i \frac{\partial f(a)}{\partial x_i}y_i$, $y=(y_1, \cdots\cdots, y_n)$, $||y||=1$, belongs for any $\mathscr{F}(S^{n-1})$, f is $\mathscr{F}(S^{n-1})$-smooth at a for any $\mathscr{F}(S^{n-1})$.

Definition. *A function f on some neighborhood of x is called to be $\mathscr{F}(S^{n-1})$-m-smooth at x if*

$$(d_{\rho,x}{}^m f)(y_1, \cdots\cdots, y_m)=d_\rho{}^m f(x, y_1, \cdots\cdots, y_m), \ x \text{ is fixed},$$

defines a function of ${}^m(S_x)$. If f is $\mathscr{F}(S^{n-1})$-m-smooth at x for any m, then we call f is $\mathscr{F}(S^{n-1})$-∞-smooth at x.

For example, if $M=\boldsymbol{R}^n$, ρ is the euclidean metric of $\boldsymbol{R}^n$ and f is of class C^m at a, then f is $\mathscr{F}(S^{n-1})$-m-smooth at a for any $\mathscr{F}(S^{n-1})$. In fact, in this case, we get

$$\begin{aligned}&(d_{\rho,x}{}^m f)(y_1, \cdots\cdots, y_m)\\ &=\frac{1}{m!}\sum_{i\, j\leqq n}\frac{\partial^m f(a)}{\partial x_{i_1}\cdots\cdots\partial x_{i_m}}y_{1,i_1}\cdots\cdots y_{m,i_m},\\ &y_i=(y_{i,1}, \cdots\cdots, y_{i,n}),\ ||y_i||=1,\ i=1, \cdots\cdots, m.\end{aligned}$$

We denote by $\mathscr{F}(M)$ the function space on M either of $C(M)$ or $L^p(M)$, $1\leqq p\leqq\infty$, if M is compact and either of $C(M)$, $C_b(M)$, the space of bounded continuous functions on M, $L^p(M)$, $1\leqq p\leqq\infty$ and $L^p{}_{loc}(M)$, $1\leqq p\leqq\infty$ if M is not compact. Here, M is considered to be a measure space with the measure $m(\rho)$, the induced measure from the metric.

We assume the manifold structure of M is given by $\{(U, h_U) | h_U : U \to \boldsymbol{R}^n\}$, then we have

Lemma 3. *If we have*

$$\|h_U(a) - h_U(x)\| = O(\rho(a, x)), \tag{12}$$

for any a, $x \in M$ and $U \in \{U\}$, where a is regarded to be fixed and x to be a variable, then the space of $\mathscr{F}(S^{n-1})$-smooth functions on M is dense in $\mathscr{F}(M)$.

Proof. If f is a smooth function on $\boldsymbol{R}^n$ with compact carrier, then the function $h_U{}^*f$ on M given by

$$h_U{}^*f(x) = f(h_U(x)), \quad x \in U,$$
$$h_U{}^*f(x) = 0, \qquad x \notin U,$$

is an $\mathscr{F}(S^{n-1})$-smooth function on M by (12) and lemma 2. Hence we obtain the lemma since M is paracompact.

Corollary. *Under the same assumptions about M and ρ, for any locally finite open covering $\{V\}$ of M, there exists a partition of unity by $\mathscr{F}(S^{n-1})$-smooth functions $\{e_V(x)\}$ subordinated to $\{V\}$ for any $\mathscr{F}(S^{n-1})$.*

Theorem 1. *A paracompact topological manifold M always has a metric ρ such that the space of $\mathscr{F}(S^{n-1})$-smooth functions by ρ on M is dense in $\mathscr{F}(M)$ if $\mathscr{F}(M)$ is either of $C(M)$, $C_b(M)$ or $L^p{}_{loc}(M)$, $1 \leqq p \leqq \infty$.*

Proof. We take the metric ρ of M constructed in [4]. Then, since

$$0 < \int_{hU(r)} \|\xi - \eta\| < \infty, \text{ if and only if } 0 < \int_r \rho < \infty,$$

to set

$$A = \{a |\ a \in M,\ a \text{ does not satisfy } (12)\},$$

A is a discreet set of M. Hence for any $a \in A$, there exists a neighborhood $U(a)$ of a such that $U(a) \cap A = \{a\}$. For this $U(a)$, we set

$$C_a(U(a)) = \{f | f \text{ is continuous on } U(a) \text{ and } f(a) = 0\}.$$

By definnition, we have

$$C(U(a)) = \boldsymbol{R} \oplus C_a(U(a)), \tag{13}$$

where $\boldsymbol{R}$ is the space of constant functions on $U(a)$.

We take a neighborhood system $\{V_n(a)\}$ of a in $U(a)$ such that

$$V_n(a) \Subset V_{n+1}(a), \quad \bigcap_n V_n(a) = \{a\},$$

and denote

$$C_n(U(a)) = \{f \mid f \text{ is continuous on } U(a) \text{ and } f \mid V_n(a) = 0\}.$$

Then by lemma 3, $\mathscr{F}(S^{n-1})$-smooth functions are dense in $C_n(U(a))$ for any n. Hence $\mathscr{F}(S^{n-1})$-smooth functions are dense in $C_a(U(a))$ because $\cup_n C_n(U(a))$ is dense in $C_a(U(a))$. But, since a constant function is $\mathscr{F}(S^{n-1})$-smooth for any $\mathscr{F}(S^{n-1})$, $\mathscr{F}(S^{n-1})$-smooth functions are dense in $C(U(a))$ by (13).

Foe each $a \in A$, we take a neighborhood $V(a)$ such that $V(a) \Subset U(a)$ and set

$$V(A) = \bigcup_{a\in A} V(a), \quad U(A) = \bigcup_{a\in A} U(a).$$

Then we have

(14) $\qquad V(A) \Subset U(A).$

By lemma 3, we know that $\mathscr{F}(S^{n-1})$-smooth functions are dense in $CM - V(A))$, and by 14, we can set

$$f = f_1 + f_2, \ \ car. \ f_1 \subset M - V(A),$$
$$f_2 = \sum_{a\in A} f_{2,a}, \ \ car. \ f_{2,a} \subset U(a),$$

for any continuous function f of M. Hence $\mathscr{F}(S^{n-1})$-smooth functions are dense in $C_b(M)$. Since $C_b(M)$ is dense in $L^p{}_{loc}(M)$, $1 \leqq p \leqq \infty$, we have the theorem.

Note. If the tatal measure of M ay $m(\rho)$, the induced measure of ρ, is finite, then $\mathscr{F}(S^{n-1})$-smooth functions are dense in $L^p(M)$, $1 \leqq p \leqq \infty$, although M is not compact.

5. We denote the space of $\mathscr{F}(S^{n-1})$-smooth functions on M by $C_{\mathscr{F}(S^{n-1})}(M)$. If M is not compact, then the subspace of $C_{\mathscr{F}(S^{n-1})}(M)$ consisted by bounded $\mathscr{F}(S^{n-1})$-smooth functions on M is denoted by $C_{\mathscr{F}(S^{n-1}),b}(M)$. We assume that $C_{\mathscr{F}(S^{n-1})}(M)$ is dense in $C(M)$.

Lemma 4. *$C_{\mathscr{F}(S^{n-1})}(M)$ and $C_{\mathscr{F}(S^{n-1}),b}(M)$ are both rings with the unit.*

Rroof. If f_1 and f_2 are $\mathscr{F}(S^{n-1})$-smooth at a, then we may set

$$f_i(x) = f_i(a) + g_i(\varepsilon_{a,x})\rho(a,x) + o(\rho(a,x)), \ \ i = 1, \ 2, \ \ x \in U(a).$$

Hence we have

$$\begin{aligned} & f_1(x) f_2(x) \\ &= f_1(a)f_2(a) + \{f_1(a)g_2(\varepsilon_{a,x}) + f_2(a)g_1(\varepsilon_{a,x})\}\rho(a, \ x) + o(\rho(a, \ x)), \end{aligned}$$

for $x \in U(a)$. Since $f_1(a)g_2(\varepsilon_{a,x}) + f_2(a)g_1(\varepsilon_{a,x})$ belongs in (S_a), f_1f_2 is $\mathscr{F}(S^{n-1})$-smooth at a. On the other hand, since we know $d_\rho 1=0$, where 1 is the constant function with the value 1, 1 is $\mathscr{F}(S^{n-1})$-smooth for any $\mathscr{F}(S^{n-1})$. Therefore we obtain the lemma.

Definition. *A closed operator X defined in $C(M)$ with the range in $C(M)$ is called an $\mathscr{F}(S^{n-1})$-vector field of M if it satisfies the following (i), (ii) (iii).*

(i). *X is defined on $C_{\mathscr{F}(S^{n-1})}(M)$.*

(ii). *If $|f(x)-f(a)|=o(\rho(a, x))$ at a, then $(Xf)(a)$ is equal to 0.*

(iii). $X(f_1 f_2)=f_1 X(f_2)+f_2 X(f_1)$.

Lemma 5. *If $\xi=\xi(x)$ is an $\mathscr{F}(S^{n-1})$-1-vector field of M, then to set*

$$(Xf)(x)=<\xi(x),\ d_\rho f(x)>,\ x\in M,$$

X is an $\mathscr{F}(S^{n-1})$-vector field of M. Here $<\xi,\ \varphi>$, $\xi\in{}^*\mathscr{F}(S_x)$, $\varphi\in\mathscr{F}(S_x)$, means the value of ξ at φ.

Proof. By the definition of d_ρ, d_ρ has the following properties.

(i). *If $\{f_n\}$ converges to f in $C(M)$ and $\{d_\rho f_n\}$ converges normally to some $\mathscr{F}(S^{n-1})$-1-cochain φ, then f is $\mathscr{F}(S^{n-1})$-smooth and $d_\rho f=\varphi$.*

(ii). $(d_\rho f)(a)=0$ *if* $|f(x)-f(a)|=o(\rho(a, x))$ *at* a.

(iii). *If f_1 and f_2 are both $\mathscr{F}(S^{n-1})$-smooth, then*

$$d_\rho(f_1 f_2)=f_1 d_\rho f_2+f_2 d_\rho f_1.$$

Hence we have the theorem.

Note. A series of $\mathscr{F}(S^{n-1})$-1-cochains $\varphi_m(x, y)$ is called convergses normally to $\varphi(x, y)$ if the series of functions on M given by $\{||\varphi_m(x, y)-\varphi(x, y)||_x\}$ converges uniformlly to 0 on any compact set of M. Here $||\varphi(x, y)||_x$ means the norm of $\varphi(x)$, $\varphi(x)(y)=\varphi(x, y)$, in $\mathscr{F}(S_x)$.

By the definition of $\mathscr{F}(S^{n-1})$-vector fields, we have

Lemma 6. *If X is an $\mathscr{F}(S^{n-1})$-vector field of M, then X satisfies the following (14) and (15).*

(14) $\qquad X_c=0,$ *where c is a constant function of M.*

(15) $\qquad (Xf_1)(a)=(Xf_2)(a),$ *if* $|f_1(x)-f_2(x)|=o(\rho(a, x))$.

Theorem 2. *If X is an $\mathscr{F}(S^{n-1})$-vector field of M, then there exists an $\mathscr{F}(S^{n-1})$-1-vector field $\xi(x)$ of M such that*

$$(Xf)(x)=<\xi(x),\ d_\rho f(x)>,\ x\in M. \tag{16}$$

Such $\xi(x)$ is determined uniquely from X if A, the set defined in the proof of theorem 1, is the empty set.

Proof. We use same notations as in the proof of theorem 1 and first assume $x \notin A$. Then the map

$$d_{\rho,x} : C_{\mathscr{F}(S^{n-1})}(M) \to \mathscr{F}(S_x),$$

given by $(d_{\rho,x}f)(y) = d_\rho f(x, y)$, is onto. Then we define

(17) $$<\xi(x), g> = (Xf)(x), \ d_{\rho,x}f = g, \ g \in \mathscr{F}(S^{n-1}).$$

By lemma 2 and (15), (17) is well defined and since $d_{\rho,x}$ is onto, $\xi(x)$ is an element of $\mathscr{F}^*(S_x)$ by closed graph theorem because X is a closed operator. Since Xf is continuous for any $f \in C_{\mathscr{F}(S^{n-1})}(M)$, $\xi(x)$ is continuous in x, $x \in M\text{-}A$. Moreover, since $M\text{-}A$ is dense in M and Xf is continuous on M, $lim._{x_n \to a}$ $\xi(x_n) = \xi(a)$ exists as an element of $d_{\rho,a}(C_{\mathscr{F}(S^{n-1})}(M))^*$ for any $a \in A$. Hence (by the theorem of Hahn-Banach), we may consider $\xi(a)$ to be an element of $\mathscr{F}^*(S_a)$ and ξ is continuous at a. Therefore we obtain the theorem.

By lemma 4 and theorem 2, there is a *1* to *1* correspondence between the set of $\mathscr{F}(S^{n-1})$-vector fields of M and the set of $\mathscr{F}(S^{n-1})$-1-vector fields of M. Hence we identify them.

Note 1. If X, Y are $\mathscr{F}(S^{n-1})$-vector fields of M such that their compositions XY and YX are both deined, then $[X, Y] = XY - YX$ also satisfies the conditions (ii), (iii) of $\mathscr{F}(S^{n-1})$-vector fields.

Note. 2. Let X be a closed operator with the domain $\mathscr{D}(X) \subset C(M)$ and the range is in $C(M)$ such that

(i). *$\mathscr{D}(X)$ is a dense subring of $C(M)$ with the unit.*

(ii). *If f_1, f_2 are in $\mathscr{D}(X)$, then $X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$.*

Then we call X is a generalized vector field of M. If X also satisfies

(iii). $(Xf)(a) = 0$ *if* $|f(x) - f(a)| = o(\rho(x, a))$,

for a (fixed) metric ρ of M, then we call X is a generalized vector field of M with respect to ρ.

Since X is closed, to define the topology of $\mathscr{D}(X)$ by taking

$$U(f, V, W) = \{g \mid g \in \mathscr{D}(X), \ g \in V, \ Xg \in W\},$$

where V and W are the neighborhoods of f and Xf in $C(M)$, as the neighborhood basis of $f \in (X)$, (X) is a complete space and to set

$$\mathfrak{J}_a(X) = \{f \mid f \in \mathscr{D}(X), \ f(a) = Xf(a) = 0\},$$

$\mathfrak{J}_a(X)$ is a colsed ideal of $\mathscr{D}(X)$ by this topology. Hence setting

$$\mathscr{F}_a(X) = ((X) \cap I_a(M))/\mathfrak{J}_a(X), \ I_a(M) = \{f \mid f \in C(M), \ f(a) = 0\},$$

we can set

$$Xf(a)=<\xi(a),\ d_Xf(a)>,\ \ \xi(a)\in\mathscr{F}_a(X)^*,$$

where $d_Xf(a)$ is the class of $f\text{-}f(a)$ in $\mathscr{F}_a(X)$.

If X is a generalized vector field of M with respect to ρ, then we have

$$\mathfrak{J}_a(X)\supset\{f|\ f\in\mathscr{F}(X),\ \ |f(x)|=o(\rho(a,x))\}.$$

6. For an $\mathscr{F}(S^{n-1})$-vector field X given by $Xf=<\xi,\ d_\rho f>$ and t, $0\leqq t\leqq 1$, we set

$$U_{X,t}(f)(x)=<\xi(x),\ f(r_{x,y,t})>. \tag{18}$$

Here $f(r_{x,y,t})$ is regarded to be a function of y, $y\in S_x$. Since f is continuous, $f(r_{x,y,t})$ is continuous on S_x. Hence $U_{X,t}(f)$ is well defined for any X.

By definition, $U_{X,t}$ is defined on $C(M)$ and a bounded linear operator of $C(M)$ if M is compact. We also know that $lim._{t\to t_0}U_{X,t}(f)$ converges normally to $U_{X,t_0}(f)$. Therefore, if M is compact, then $U_{X,t}$ is strongly continuous in t. Moreover, we know

$$\lim_{t\to 0}\frac{1}{t}(U_{X,t}-U_{X,0})f=Xf,\ \ if\ f\in C_{\mathscr{F}(S^{n-1})}(M). \tag{19}$$

We note that

$$U_{X,0}f(x)=<\xi(x),\ 1>f(x),$$

where 1 is the constant function with the value 1 on S_x.

(19) shows that there is a curve in $L(C(M),\ C(M))$, the spce of (bounded) linear operators of $C(M)$ (with the strong topology), such that whose tangent at its starting point is X.

For $U_{X,t}$, we set

$$T_{X,a,t}=\exp.(\frac{t}{a}(U_{X,a}-U_{X,0})),\ \ t\geqq 0.$$

Then $\{T_{X,a,t}\}$is a 1-parameter semi-group of $C(M)$ with the generating operator $(1/a)(U_{X,a}-U_{X,0})$. Hence if $lim._{a\to 0}T_{X,a,t}$ exists, then to set its limit by $T_{X,t}$, $T_{X,t}$ is a 1-parameter semi-group with the generating operator X. But this limit does not exists in general. In fact, there exists an $\mathscr{F}(S^{n-1})$-vector field which does not generate any 1-parameter semi-group of $C(M)$ or $L^p(M)$, $1\leqq p\leqq\infty$.

Example. We assume taht M satisfies

(i). *$H^1(M,\boldsymbol{R})$ vanishes.*

(ii). *M is compact.*

To define a $C(S^{n-1})$-1-form $\varphi(x, y)$ on M by $\varphi(x, y)=\lambda$, an (arbitrary) constant, we get

$$d_p\varphi = 0.$$

Hence by (i), there exists a $C(S^{n-1})$-smooth function n on M such that

$$(d_p h)(x, y) = \varphi(x, y).$$

Let X be the $C(S^{n-1})$-vector field on M given by

$$Xf(x) = \langle m(x), d_p f(x)\rangle, \ \textit{m(x) is the canonical measure on } S_x.$$

Then we have for the above h,

$$Xh = \lambda, \ \textit{the constant function with the value } \lambda \textit{ on } M.$$

For this h, we set $k = \exp.(h) = \sum_m (h)^m/m!$. Then we get

$$Xk = \lambda k.$$

This shows λ is a proper value of X in $C(M)$(or in $L^p(M)$, $1 \leq p \leq \infty$, because $C(M)$ is contained in $L^p(M)$ since M is compact), Since M is compact), $C(M)$ is a Banach space. Then by the theorem of Hille-Yosida ([17], [18]), X can not generate any (equi-continuous) 1-parameter semi-group of $C(M)$ (or $L^p(M)$), because λ is arbitrary.

In general, if an $L^2(S^{n-1})$-vector field X is given by

$$Xf = \langle \xi(x), d_p f(x)\rangle, \ \xi(x) \neq 0 \text{ for any } x \in M,$$

and M is compact, then X does not generate any 1-parameter semi-group of $C(M)$ (or $L^p(M)$, $1 \leq p \leq \infty$). In fact, in this case, we may set

$$L^2(S_x) = (\xi(x))^{\perp} \oplus \boldsymbol{R}\xi(x),$$

and denote the projection to $\boldsymbol{R}\xi(x)$ by $P_{\xi(x)}$. Then a cross-section f of the bundle $\cup_{x \in X} \boldsymbol{R}\xi(x)$ is considered to be a function f of M by setting

$$f^{\natural}(x) = a, \ \textit{if } f(x) = a\frac{\xi(x)}{||\xi(x)||}.$$

(We note that this also shows that a fuction of M always defines a crosssection of $\cup_{x \in X} \boldsymbol{R}\xi(x)$). Then by the befinition of X, we have

$$Xf(x) = ||\xi(x)||(P_{\xi(x)} d_p f)^{\natural}(x).$$

We define $P_\xi d_\rho f$ by $(P_\xi d_\rho f)(x) = P_{\xi(x)} d_{\rho,x} f$. Then $P_\xi d_\rho C\ L^2(S^{n-1})$ is dense in the space of the cross-sections of $\cup_{x\varepsilon X} R\xi(x)$, for any constant fuction λ and $\varepsilon > 0$, there exists an $L^2(S^{n-1})$-smooth function $f_{\lambda,\varepsilon}$ such that

$$||Xf_{\lambda,\varepsilon} - \lambda|| < \varepsilon.$$

This means λ is at least continuous spectre of X, because M is compact. Hence by the theorem of Hille-Yosida, we have the assertion.

Note. The generating operator of a 1-parameter semi-group $\{T_t\}$ is an $\mathscr{F}(S^{n-1})$-vector field of M, if and only if $\{T_t\}$ satisfies

$$T_t(f_1 f_2) - (T_t f_1)(T_t f_2) = o(t), \text{ if } f_1, f_2 \in C_{\mathscr{F}(S^{n-1})}(M). \tag{20}$$

7. In this n°, we give some definitions about X, an $\mathscr{F}(S^{n-1})$-vector field on M.

Definition. *X is called to be 0 at a, $a \in M$, if $(Xf)(a) = 0$ for all $\mathscr{F}(S^{n-1})$-smooth functions.*

By definition, if X is given by $Xf = <\xi(x), d_\rho f(x)>$, then X is 0 at a if and only if $\xi(a)=0$ as an element of $\mathscr{F}^*(S_a)$. As usual, we set

$$car. X = \overline{\{x \mid X \text{ is not } 0 \text{ at } x\}}.$$

Definition. *For X, we set*

$$CAR.(X) = \overline{\bigcup_{x\varepsilon M} \text{car.}\, \xi(x)}, \text{ if } (Xf)(x) = <\xi(x), d_\rho f(x)>. \tag{21}$$

By definition, $CAR.\ X$ is a (closed) subset of $s(M)$ and we have

$$\pi(CAR.X) = car.X. \tag{22}$$

We note that if M is smooth and X is a usual vector field on M regarded to be a $C(S^{n-1})$-vector field on M and does not vanish at any point of M, then $CAR.\ X$ is a cross-section of $s(M)$ (cf. n°9).

Definition. *X is called to be positive if X is given by $Xf = <\xi(x), d_\rho f(x)>$ and*

$$\xi(x) \geqq 0 \text{ for any } x \in M.$$

As usual, we call $X \geqq Y$ if $X - Y \geqq 0$. Then since

$$(\sup_\alpha.\ \{X\})f = <\sup_\alpha.\{\xi_\alpha(x)\},\ d_\rho f(x)>,$$

if $\{X_\alpha\}$ is upper (or lower) bounded, then $sup.\ \{X_\alpha\}$ (or $inf.\{X_\alpha\}$) exists to be an

$\mathscr{F}(S^{n-1})$-vector field. Especially, we may define $X^+ = max.\ (X, 0)$ and $X^- = (-X)^+$ for any $\mathscr{F}(S^{n-1})$-vector field X and we have

$$X = X^+ - X^-. \tag{23}$$

We note that if $Xf = <\xi(x),\ d_\rho f(x)>$, then

$$(X^+f)(x) = <(\xi(x))^+,\ d_\rho f(x)>,\ (X^-f)(x) = <(\xi(x))^-,\ d_\rho f(x)>,$$

where $(\xi(x))^+$ is $max.\ (\xi(x),\ 0)$ and $(\xi(x))^-$ is $(\xi(x))^+$.

Note. Since the space of $\mathscr{F}(S^{n-1})$-vector field of M is a vector space, these shows that this space has the structure of (complete) vector lattice. Hence to fix an $\mathscr{F}(S^{n-1})$-vector field Y, $Yf = <\eta(x),\ d_\rho f>$, the Radon-Nykodim partition of any $\mathscr{F}(S^{n-1})$-vector field X, $Xf = <\xi(x),\ d_\rho f>$ with respect to Y is possible. It corresponds to the Radon-Nykodim partition of $\xi(x)$ with respect to $\eta(x)$.

Definition. *If $\mathscr{F}(S^{n-1})$-vector fields X_1 and X_2 are given by $(X_i f)(x) = <\xi_i(x),\ d_\rho f(x)>$, $i = 1,\ 2$, and $Y = [X_1,\ X_2]$ is defined to be an $\mathscr{F}(S^{n-1})$-vector field of M, then we denote*

$$\eta(x) = [\xi_1(x),\ \xi_2(x)]. \tag{24}$$

Here Y is given by $(Yf)(x) = <\eta(x),\ d_\rho f(x)>$.

We note that if x is fixed in (24), then (24) defines the bracket product for some elements of $\mathscr{F}^*(S_x)$. Or, in other word, $\mathscr{F}^*(S_x)$ contains (as a dense subset), a Lie pseudoalgebra.

§ 3. Generalized tangent of a curve.

8. We denote the set of germs of $\mathscr{F}(S^{n-1})$-smooth functions of M at a, $a \in M$, by $C_{\mathscr{F}(S^{n-1}),\ *,\ a}(M)$.

Lemma 7. *If $\mathscr{F}(S^{n-1})$-smooth functions f_1 andf $_2$ defines same germ inC$_{\mathscr{F}(S^{n-1})}$ (M) and $|f_1(x) - f_1(a)| = o(\rho(x,\ a))$, then $|f_2(x) - f_2(a)|$ is also $o(\rho(x,\ a))$.*

By this lemma, we can say $|f(x) - f(a)|$ is $o(\rho(x, a))$ although f is regarded to be an element of $C_{\mathscr{F}(S^{n-1}),\ *,\ a}(M)$.

Definition. *A linear map $\mathfrak{X}$ from $C_{\mathscr{F}(S^{n-1}),\ *,\ a}(M)$ to R is called an $\mathscr{F}(S^{n-1})$-vector of M at a if it satisfies the following (i), (ii), (iii).*

(i). $\mathfrak{X}(f_1 f_2) = f_1(a)\mathfrak{X}(f_2) + f_2(a)\mathfrak{X}(f_1)$.

(ii). $\mathfrak{X}(f) = 0$, *if* $|f(x) - f(a)| = o(\rho(a,\ x))$.

(iii). $\mathfrak{X}(f) = (Xf)(a)$, *where X is an $\mathscr{F}(S^{n-1})$-vector field of $U(a)$, a neighborhood of a.*

By (iii) and theorem 2, we have

Theorem 2′. *For any $\mathscr{F}(S^{n-1})$-vector $\mathfrak{X}$ of M at a, there exists an element ξ of $\mathscr{F}^*(S_a)$ such that*

$$\mathfrak{X}(f) = <\xi,\ d_{\rho,a}f>,$$

and such ξ is determined uniquely by $\mathfrak{X}$. Conversely, if $\xi \in \mathscr{F}^*(S_a)$, then $<\xi,\ d_{\rho,a}f>$ is an $\mathscr{F}(S^{n-1})$-vector of M at a.

Let γ be a curve of M given by $\varphi : I \to M$ such that

(25) $$\varphi(0) = a,\ \ \varphi(t) \neq a \ \ if\ t > 0.$$

(25)′ $$\rho(a,\ \varphi(t)) = 0(t).$$

Then we set

(26) $$\mathfrak{X}_\varphi(f) = \lim_{s\to 0} \frac{1}{s}[\lim_{h\to 0} \int_h^s \frac{1}{t}\{f(\varphi(t)) - f(a)\}\ dt],$$

where f is an $\mathscr{F}(S^{n-1})$-smooth function at a.

By (25) and (25)′, we have

(26)′ $$\mathfrak{X}_\varphi(f) = \lim_{s\to 0} \frac{1}{s}[\lim_{h\to 0} \int_h^s \frac{\rho(a,\ \varphi(t))}{t}(d_{\rho,a}f)(\varepsilon_{a,\ \varphi(t)})dt].$$

Lemma 8. *If $\mathfrak{X}_\varphi(f)$ exists for all $\mathscr{F}(S^{n-1})$-smooth functions at a, then $\mathfrak{X}_\varphi$ is an $\mathscr{F}(S^{n-1})$-vector of M at a.*

Proof. By (26)′, we only need to show (i). But, since we know

$$\begin{aligned}&(d_{\rho,a}(f_1 f_2))(\varepsilon_{a,\varphi(t)})\\ &= f_1(a)(d_{\rho,a}f_2)(\varepsilon_{a,\varphi(t)}) + f_2(a)(d_{\rho,a}f_1)(\varepsilon_{a,\varphi(t)}),\end{aligned}$$

we have (i) by (26)′.

Definition. *If $\mathfrak{X}_\varphi$ is defined on $C_{\mathscr{F}(S^{n-1}),\ *,\ a}(M)$, then γ is called $\mathscr{F}(S^{n-1})$-smooth at a.*

By theorem 2′ and lemma 8, If $\mathfrak{X}_\varphi$ is defined on the space of $\mathscr{F}(S^{n-1})$-smooth functions at a, then there exists an element $\xi = \xi(\varphi)$ of $\mathscr{F}^*(S_a)$ such that

$$\mathfrak{X}_\varphi(f) = <\xi(\varphi),\ d_{\rho,a}(f)>.$$

We note that since $C^*(S_a)$ contains $L^p(S_a)$ for all p, we may consider ξ to be a Radon-measure on S_a.

Definition. *$\xi(\varphi)$ is called the generalized tangent of γ at a.*

Note. If M is smooth, real analytic or real algebraic, then to take $C^\infty(S_a)$, $C^\omega(S_a)$ or $C^{alg.}(S_a)$ as $\mathscr{F}(S_a)$, we may define the generalized tangent for wider class of curves. Here $C^{alg.}(S^{n-1})$, the model of $C^{alg.}(S_a)$, is given by

$$C^{alg.}(S^{n-1}) = \boldsymbol{R}[x_1, \cdots\cdots, x_n]/(x_1^2 + \cdots\cdots + x_n^2 - 1),$$

which is dense in $C(S^{n-1})$ or in $L^p(S^{n-1})$ (cf. [5], [11]).

9. In this n°, we give some examples of the generalized tangent.

Example 1. We assume γ is smooth at a, that is

$$\lim_{t\to 0} \varepsilon_{a,\varphi(t)} = y, \quad y \in S_a,$$

$$\lim_{t\to 0} \frac{\rho(a, \varphi(t))}{t} = c, \quad c \text{ is a (positive) real number,}$$

both exists and f is $C(S^{n-1})$-smooth at a, then we have by the mean value theorem

$$\int_h^s \frac{(a, \varphi(t))}{t} (d_{\rho,a} f)(\varepsilon_{a,\varphi(t)}) dt$$

$$= \frac{\rho(a, \varphi(s_0))}{s_0} (d_{\rho,a} f)(\varepsilon_{a,\varphi(s_0)})(s-h), \quad h < s_0 < s.$$

Hence we have

$$\mathfrak{X}_\varphi(f) = c(d_{\rho,a} f)(y).$$

Therefore, denoting the Dirac measure of S_a concentrated at y by δ_y, we get

(27) $$\mathfrak{X}_\varphi(f) = \langle c\delta_y, \ d_{\rho,a} f\rangle.$$

We note that if f is smooth at a, then $\mathfrak{X}_\varphi(f)$ coincide to the usual definition of the (one-sided) derivation of f along γ.

Note. If M is smooth and X is a usual vector field of M which does not vanish at any point of M, then at any point a of M, X has a smooth integral curve γ_a given by φ_a: $\boldsymbol{I} \to M$, $\varphi_a(0) = a$, and

$$(Xf)(a) = \mathfrak{X}_{\varphi a}(f).$$

Hence we have by (27)

$$(Xf)(a) = \langle c(a)\delta_{y(a)}, \ d_{\rho,a} f\rangle.$$

Hence we have

(28) $$CAR.\, X = \bigcup_{a \in M} y(a).$$

Since y(a) depends continuously on a, $CAR.\, X$ is a (continuous) cross-section of $s(M)$.

In the following two examples, we need the following

***Lemma* 9.** *If $g(t)$ is a continuous periodic function on $\boldsymbol{R}^1$ with the period T, then*

$$\lim_{s\to\infty} s\int_s^\infty \frac{g(t)}{t^2}dt=\frac{1}{T}\int_0^T g(t)dt. \tag{29}$$

Proof. We define a periodic function $e[_{a,b}]t)$, $0\leqq a<b\leqq T$, with the period T by

$$\begin{aligned} e[_{a,b}](t) &= 1,\ \ t\in[a+nT,\ b+nT],\ \ \textit{for some integer } n,\\ &= 0,\ \ \textit{otherwise}. \end{aligned}$$

Then for $0\leqq a'\leqq a<b\leqq b'\leqq T$, to set

$$v^{a,b}_{m,a',b'}=\frac{b'-a'}{b-a}(t-(mT+a))+mT+a',\quad mT\leqq v^{a,b}_{m,a,'b'}\leqq(m+1)T,$$

we have

$$e\,[_{a,b}]\,(v^{a,b}_{m,a',b'})=e[_{a',\,b'}](t),\ \ mT\leqq v^{a,b}_{m,a',\,b'}\leqq(m+1)T.$$

Hence we get

$$\int_{mT}^\infty \frac{e[_{a,b}](t)}{t^2}dt=\frac{b-a}{b'-a'}\int_{mT}^\infty \frac{e[_{a',\,b'}](t)}{t^2}dt.$$

Then, since we know

$$\lim_{\substack{a'\to 0\\ b'\to T}} s\int_s^\infty \frac{e[_{a',\,b'}](t)}{t^2}dt = s\int_s^\infty \frac{dt}{t^2},$$

we obtain

$$\lim_{s\to\infty} s\int_s^\infty \frac{e[_{a,b}](t)}{t^2}dt=\frac{|b-a|}{T}.$$

Then, since $g(t)$ is bounded and (uniformly) continuous, we have

$$\begin{aligned} &\lim_{s\to\infty} s\int_s^\infty \frac{g(t)}{t^2}dt\\ &=\lim_{s\to\infty}[\lim_{|a_{i+1}-a_i|\to 0}\sum_i s\int_s^\infty g(a_i)\frac{e[_{a_i,\,a_{i+1}}]\,(t)}{t^2}dt]\\ &=\lim_{|a_{i+1}-a_i|\to 0}\sum_i g(a_i)[\lim_{s\to\infty} s\int_s^\infty \frac{[e_{a_i,\,a_{i+1}}(t)]}{t^2}dt] \end{aligned}$$

$$= \lim_{|a_{i+1}-a_i|\to 0} \sum_i g((a_i) \frac{|a_{i+1}-a_i|}{T}$$

$$= \frac{1}{T}\int_0^T g(t)dt.$$

Here, $0=a_0 < a_1 < \cdots\cdots < a_m < a_{m+1} = T$ is a partition of $[0, T]$.

Example 2. Let M be R^2 with the euclidean metric, a the origin $0(=(0, 0))$ of $\boldsymbol{R}^2$ and γ is given by φ: $\boldsymbol{I} \to \boldsymbol{R}^2$, where φ is given by

$$\varphi(t) = (t \cos(\frac{1}{t}), \quad t \sin(\frac{1}{t})), \quad t > 0,$$

$$\varphi(0) = 0.$$

Hence, if we use the polar coordinate (r, θ) of $\boldsymbol{R}^2$, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$, then γ is given by

$$r\theta = 1, \quad r > 0.$$

Then, if $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ is parametrized by θ and g is continuous on S^1, we get

$$\lim_{s\to 0} \frac{1}{s}[\lim_{h\to 0} \int_h^s \frac{\rho(0, \varphi(t))}{t} g(\varepsilon_{0,\varphi t})\, dt]$$

$$= \lim_{s\to 0} \frac{1}{s}[\lim_{h\to 0} \int_h^s g(\frac{1}{t})dt] = \lim_{u\to\infty} u\int_u^\infty \frac{g(v)}{v^2} dv,$$

Hence by lemma 9, we have

$$\mathfrak{X}_\varphi(f) = \frac{1}{2\pi}\int_0^{2\pi} (d_{\rho,0} f)(\theta) d\theta. \tag{30}$$

Or, in other word, the generalized tangent of the curve $r\theta = 1$ at 0 is the standard measure of S^1.

Example 3. We take M and ρ same as above and take φ to be

$$\varphi(t) = (t, \quad t \sin(\frac{1}{t})), \quad t > 0, \quad \varphi(0) = 0, \quad \textit{the origin of } \boldsymbol{R}^2.$$

By befinition, we have

$$\frac{\rho(0, \varphi(t))}{t} = \sqrt{1 + \sin^2(\frac{1}{t})}, \quad \varepsilon_{0,\varphi(t)} = \tan^{-1}(\sin(\frac{1}{t})).$$

Hence we have by lemma 9,

$$\lim_{s\to 0}\frac{1}{s}[\lim_{h\to 0}\int_h^s \frac{\rho(0,\ \varphi(t))}{t} g(\varepsilon_{0,\varphi(t)})\,dt]$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+\sin^2 v}\,g(\tan^{-1}(\sin(v)))dv$$

$$=\frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} g(\theta)\frac{1}{\cos^2\theta\sqrt{\cos(2\theta)}}d\theta.$$

Therefore, the generalized tangent of the curve $x\sin(1/x)$ at the origin is the measure on S^1 concentrated on $-(\pi/4)\leqq\theta\leqq\pi/4$ with the weight $(1/\pi)(1/\cos^2\theta\sqrt{\cos(2\theta)})$.

Note. If γ is given by $(-t,\ t\sin(1/t))$, $t>0$, then the generalized tangent of γ at the origin is similar as above but has carrier on $3\pi/4\leqq\theta\leqq 5\pi/4$.

10. *Lemma 10. The generalized tangent of a curve at a is a positive measure on S_a.*

Proof. If ξ is the generalized tangent of φ: $I\to M$, then we have

$$\int_{Sa} g(y)d\xi=\lim_{s\to 0}\frac{1}{s}[\lim_{h\to 0}\int_h^s\frac{\rho(a,\ \varphi(t))}{t}g(\varepsilon_{a,\varphi(t)})\,dt].$$

Hence if $g\geqq 0$ on S_a, then $\int_{S_a} g(y)d\xi\geqq 0$. Therefore ξ is a positive measure.

Lemma 11. If the parameter of γ is changed to ct instead of t, c is a constant, then the generalized tangent ξ of γ at a is changed to $c\xi$. In general, if the parameter of γ is changed to $\alpha(t)$ and

$$\lim_{t\to 0}\frac{\alpha(t)}{t}=c,$$

then the generalized tangent ξ of γ at a changes to $c\xi$.

By this lemma, we may assume the generalized tangent ξ of γ at a satisifes

$$\xi(S_a)=1. \tag{31}$$

Theorem 3. If ξ is a positive measure on S_a, then there exists a curve of M starts from a such that whose generalized tangent at a is ξ.

Proof. Since the proof for $n=1$ is similar, we assume $n\geqq 2$.

First we note that the problem is local, we may assume $M=\boldsymbol{R}^n$ with the euclidean metric and a is the origin $0(=(0,\ \cdots\cdots,\ 0))$ of $\boldsymbol{R}^n$. Hence S_a is the unit $(n-1)$-sphere S^{n-1}.

We take a positive measure ξ of S^{n-1} such that $\xi(S^{n-1})=1$. By lemma 11, this is not restrictive.

We choose a countable dense subset $\{y_p\}$ of S^{n-1} such that

(32) $$y_p \neq \pm y_q, \ \ if \ p \neq q.$$

For this $\{y_p\}$, we divide S^{n-1} by Borel sets $\{E_p{}^q\}$ as follows:

(33) $$S^{n-1} = \bigcup_{p \leqq q} E_p{}^q, \ \ E_{p'}{}^q \cap E_{p''}{}^q = \phi, \ \ if \ p' \neq p'', \ \ y_p \in E^q{}_p.$$

(33)′ $$\lim_{q \to \infty} . \ dia. \ (E_p{}^q) = 0.$$

Here $dia.$ $(E_p{}^q)$ means the diameter of $E_p{}^q$. Hence, if $g(y)$ is a continuous function of S^{n-1}, then

(34) $$\int_{S^{n-1}} g(y)d\xi = \lim_{q \to \infty} . \sum_{p \leqq q} g(y_p)\xi E_p{}^q).$$

On the other hand, for the above $\{E_p{}^q\}$ and ξ, we take a series of (positive) real numbers $\{t_{q,p}\}$, $p \leqq q$, as follows:

(35) $$t_{q,p} > t_{q,p+1}, \ \ if \ p+1 \leqq q, \ \ t_{q,q} > t_{q+1,1},$$

(35)′ $$\lim_{q \to \infty} . \ t_{q,p} = 0,$$

(35)″ $$\sum_{q; t_{q,\,p} \leqq s} \frac{1}{s} |(t_{q,p} - t_{q,p+1}) - \xi(E_p{}^q)| \leqq \frac{s}{2^p}, \ \ s > 0.$$

This is possible because $\xi(S^{n-1}) = 1$ and $\sum_p \sum_{q; t_{q,p} \leqq s} (1/s) | t_{q,p} - t_{q,p+1}| = 1 - (s - t_{q_0,\,p_0})/s$ is sufficiently near to 1. Here, $t_{q_0,\,p_0}$ is the largest $t_{q,p}$ which is smaller than s.

Using this $\{t_{q,p}\}$, we set

$$\Psi(t_{q,p}) = t_{q,p} y_p,$$

$$\Psi(t) = \frac{t_{q,p} - t}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p+1}) + \frac{t - t_{q,p+1}}{t_{q,p} - t_{q,p+1}} \Psi(t_{q,p}),$$

$$if \ t_{q,p} > t > t_{q,p+1},$$

$$\Psi(t) = \frac{t_{q,q} - t}{t_{q,q} - t_{q+1,1}} \Psi(t_{q+1,1}) + \frac{t - t_{q+1,1}}{t_{q,q} - t_{q+1,1}} \Psi(t_{q,q}),$$

$$if \ t_{q,q} > t > t_{q+1,1},$$

$$\Psi(0) = 0.$$

Then since $||y_p|| = 1$, we have the definition of $\Psi(t)$ and (32),

(36)″ $$||\Psi(t)|| \leqq |t|,$$

(36)′ $$\Psi(t) \neq 0, \ \ if \ t \neq 0.$$

We also note that by the definition of $\Psi(t)$, $\Psi(t)$ is continuous for all t, $0 \leqq t \leqq 1$.

By (36)′, to define $\varphi(t)$ by

$$\varphi(t)=\frac{\Psi(t)}{||\Psi(t)||}t,\ \ t>0,\ \ \Psi(0)=0, \tag{37}$$

$\varphi(t)$ is also continuous in t and satifises similar conditions as (36)′ and.

$$||\varphi(t)||=t. \tag{36}$$

By (36) and the mean value theorem, if $\{y_p\}$ satisfies

$$\lim_{p\to\infty}.\ ||y_{p+1}-y_p||=0, \tag{32$'$}$$

then we have for this $\varphi(t)$,

$$\int_{t_{q,\,p+1}}^{t_{q,\,p}}\frac{||\varphi(t)||}{t}g(\varepsilon_{0,\,\varphi(t)})dt$$

$$=g(y_p)(t_{q,\,p}-t_{q,\,p+1})+o(|t_{q,\,p}-t_{q,\,p+1}|).$$

Hence we have

$$\lim_{s\to 0}.\frac{1}{s}[\lim_{h\to 0}.\int_h^s\frac{||\varphi(t)||}{t}g(\varepsilon_{0,\,\varphi(t)})dt] \tag{38}$$

$$=\lim_{s\to 0}.\frac{1}{s}\sum_p g(y_p)(\sum_{q;t_{q,\,p}\leqq s}(t_{q,\,p}-t_{q,\,p+1})).$$

On the other hand, by (35)′′, we obtain

$$|\sum_{p\leqq q,}\sum_{t_{q,\,p}\leqq s}g(y_p)\xi(E_p{}^q)-\frac{1}{s}\sum_p g(y_p)(\sum_{q;t,_{q}\,p\leqq s}(t_{q,\,p}-t_{q,\,p+1}))|$$

$$\leqq\sum_p\frac{s}{2^p}=s.$$

Then, by (34) and (38), we get

$$\int_{S^{n-1}}g(y)d\xi$$

$$=\lim_{s\to 0}.\frac{1}{s}[\lim_{h\to 0}.\int_h^s\frac{||\varphi(t)||}{i}g(\varepsilon_{0,\,\varphi(t)})dt],$$

for this $\varphi(t)$. Therefore the curve γ given by φ: $I\to M$, has the generalized tangent at the origin and it is equal to ξ. Hence we have the theorem.

Note. Since $C^*(S^{n-1})$ contains $L^p(S^{n-1})$, a positive linear fuctional of $L^p(S^{n-1})$ always expressed as the generalized tangent of some curve.

Example 1. If ξ is the Dirac measure of S^{n-1} concentrated at y_1, $y_1 \in S^{n-1}$, then $\{t_{q,p}\}$ is given by

$$t_{q,1} = \frac{1}{2^q}, \quad t_{q,p} = \frac{1}{2^q} - (1 - \frac{1}{2^{p-1}})\frac{1}{8^q}, \quad 2 \leqq p \leqq q.$$

Example 2. If ξ is the standard measure of S^{n-1}, then we take $E_p{}^q$ to satisfy $\xi(E_p{}^q) = 1/q$, Then we can take $\{t_{q,p}\}$ to be

$$t_{q,p} = \frac{1}{q+1} + \frac{q+1-p}{p+1}\left(\frac{1}{q(q+1)}\right).$$

We note that although the curve $\varphi(t) = y_1 t$ has the generalized tangent δy_1, it is not given by the above method.

11. We denote by $H^+(I)$ the group of orientation preserving homeomorphisms of $I = [0, 1]$. The subgroup of $H^+(I)$ consisted by those homeomorphisms that are the identity map on $[0, \varepsilon]$ for some $\varepsilon > 0$, is denoted by $H_e(I)$. Then we set

$$H_*{}^+(I) = H^+(I)/H_e(I).$$

$H_*{}^+(I)$ is the group of germs of the (orientation preserving) homeomorphisms of I (cf. [2]).

If $\alpha \in H^+(I)$, then by the theorem of Radon-Nykodim, there exists a (positive) measurable function m_α on I which does not vanish almost everywhere on I, such that

$$\int_a^b \mu(\alpha(t))dt = \int_{\alpha(a)}^{\alpha(b)} \mu(u) m_\alpha(u) du, \tag{39}$$

where $\mu(t)$ is an (arbitrary) measurable function on I. We note that this $m_\alpha(t)$ also satisfies

$$\int_0^1 m_\alpha(t)dt = 1. \tag{40}$$

Conversely, if $m(t)$ is a positive measurable function on I such that to satisfy (40) and does not vanish almost everywhere on I, then $\int_0^t m(u)du$ is an element of $H^+(I)$. Moreover, we know that

(i). *If α_1, $\alpha_2 \in H^+(I)$ and $\alpha_1(\alpha_2)$ is the composition of α_1 and α_2 in $H^+(I)$, then*

$$m_{\alpha_1(\alpha_2)} = \alpha_2{}^*(m_{\alpha_1}) m_{\alpha_2}, \quad \alpha^* m(t) \text{ means } m(\alpha(t)). \tag{41}$$

(ii). *α belongs in $H_e(I)$ if and only if $m_\alpha(t) = 1$, $0 \leqq t < \varepsilon$, for some $\varepsilon > 0$.*

Hence to denote the set of all positive measurable functions on I which do not vanish almost everywhere on I and satisfy (40) by $\mathscr{M}^{+}(I)$ and to define a multiplication $m_{1*}m_2$ for $m_1, m_2 \in \mathscr{M}^{+}(I)$ by

$$m_{1*}m_2 = \alpha_2{}^*(m_1)\mathrm{m}_2, \qquad \alpha_2(t) = \int_0^t m_2(u)du, \tag{42}$$

$\mathscr{M}^{+}(I)$ is isomorphic to $H^{+}(I)$ and to set

$$\mathscr{M}_e(I) = \{m \mid m \in \mathscr{M}^{+}(I),\ m(t) = 1,\ 0 \leqq t < \varepsilon,\ \textit{for some}\ \varepsilon > 0\},$$

we have

$$\mathscr{M}_*(I) \cong H_*{}^{+}(I),\quad \mathscr{M}_*(I) = {}^{+}(I) / \mathscr{M}_e(I). \tag{43}$$

For φ: $I \to M$, and $\alpha \in H^{+}(I)$, we set

$$\alpha^*(\varphi)(t) = \varphi(\alpha(t)).$$

Then the image of φ and $\alpha^*(\varphi)$ is same. Moreover, we know if $\alpha \in H_e(I)$, then φ has the generalized tangent at its starting point if and only if $\alpha^*(\varphi)$ has the generalized tangent at its starting point and we have by lemma 10,

$$\mathfrak{X}_\varphi(f) = \mathfrak{X}_{\alpha*(\varphi)}(f). \tag{44}$$

By (44), we have

$$\mathfrak{X}_{\alpha*(\varphi)} = \mathfrak{X}_{\beta*(\varphi)}, \quad \textit{if} \quad \alpha \equiv \beta\ \textit{mod}.\, H_e(I). \tag{44$'$}$$

By (43), (44)$'$ and theorem 3, we can define an operation of the element m of $\mathscr{M}_*(I)$ to $\mathscr{D}^*{}_+(S^{n-1})$, the set of positive linear functionals of $\mathscr{D}(S^{n-1})$ by

$$<m(\xi),\ g> = \mathfrak{X}_{\alpha*(\varphi)}(f), \tag{45}$$

where, assuming the starting point of φ is a, $d_{\rho,a}f = g$, $\mathfrak{X}_\varphi(f) = <\xi,\ g>$ and the class of m in $\mathscr{M}_*(I)$ is m. Then, since the change of parameter of γ corresponds to the operation of $\mathscr{M}_*(I)$, we may consider the generalized tangent of γ to be an element of $\mathscr{F}^*{}_+(S^{n-1})/\mathscr{M}_*(I)$.

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