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Notes on Co-H-spaces

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1. Introduction. Recently, many results on H-spaces have been obtained (*e.g.* [4] and [6)], however, it seems to be known only a few facts about co-H-spaces. In the present paper we shall show the existence of homological co-inverses for appropriate co-H-spaces, which is a partial dual of I. M. James' Theorem (*cf.* [2]).

We work in the category of pointed Hausdorff spaces, and so homotopies are base point preserving homotopies.

NOTATIONS: Let (X, x_0) and (Y, y_0) be given two spaces. Then wedge product $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ with the base point (x_0, y_0);

inclusions $i'_1 : X \longrightarrow X \vee Y,\ i'_2 : Y \longrightarrow X \vee Y$;

projections $p'_1 : X \vee Y \longrightarrow X,\ p'_2 : X \vee Y \longrightarrow Y$;

smash product $X \wedge Y = X \times Y / X \vee Y$ with the identification topology;

folding map ∇: $X \vee X \longrightarrow X$ defined by $\nabla(x, x_0) = x = \nabla(x_0, x)$;

$\{X, Y\}_o = \{f : (X, x_0) \longrightarrow (Y, y_0)$ continuous maps$\}$ with the compact-open topology and the base point $0_{y_0} : X \longrightarrow y_0$;

$$[X, Y]_0 = \{X, Y\}_0 / \{\text{homotopy}\}.$$

2. Preliminaries. We shall begin with

Definition. Given a space (X, e), a *co-multiplication* μ' on X is a continuous map $\mu' : X \longrightarrow X \vee X$ which makes the following diagram homotopy-commutative:

$$\begin{array}{ccccc} & X \vee X & \xrightarrow{1 \vee 0_e} & X \vee X & \\ {}^{\mu'}\nearrow & & & & \searrow{}^{\nabla} \\ X & & \xrightarrow{\quad\quad 1 \quad\quad} & & X. \\ {}_{\mu'}\searrow & & & & \nearrow{}_{\nabla} \\ & X \vee X & \xrightarrow[0_e \vee 1]{} & X \vee X & \end{array}$$

e is the *homotopy co-unit* of μ', and the triple (X, e, μ') is a *co-H-space*.

Let (X, e_X, μ'_X) and (Y, e_Y, μ'_Y) be given two co-H-spaces, a continuous map

$f: X \longrightarrow Y$ is a *co-H-map*, provided that the following diagram is homotopy commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mu'_X \downarrow & & \downarrow \mu'_Y. \\ X \vee X & \xrightarrow[f \vee f]{} & Y \vee Y \end{array}$$

2. 1. **Lemma.** *Let (X, e_X, μ'_X) and (Y, e_Y, μ'_Y) be given two co-H-spaces, then $X \vee Y$ is a co-H-space.*

Proof. Let T': $X \vee Y \longrightarrow Y \vee X$ be the switching map, *i.e.*,

$$T'(x, e_Y) = (e_Y, x) \quad \text{and} \quad T'(e_X, y) = (y, e_X),$$

then the composition

$$\mu' : X \vee Y \xrightarrow{\mu'_X \vee \mu'_Y} X \vee X \vee Y \vee Y \xrightarrow{1 \vee T' \vee 1} X \vee Y \vee X \vee Y$$

is a co-multiplication on $X \vee Y$.

2. 2. **Proposition.** *Let (X, e) be a given space. A necessary and sufficient condition for (X, e) being a co-H-space is that*

(*) *for any (Y, y_0), $\{X, Y\}_0$ has a functorial H-structure having 0_e as a homotopy unit.*

Proof. Necessity. Assume that (X, e) is a co-H-space with the co-multiplication μ'. Define $\mu:\{X, Y\}_0 \times \{X, Y\}_0 \longrightarrow \{X, Y\}_0$ by

$$\mu(f, g) = \nabla (f \vee g)\mu' \quad \text{for all } f, g \in \{X, Y\}_0.$$

If $W(C, U)$ is a sub-basic open set containing $\mu(f, g)$, *i.e.*, C is a compact set in X, and U is an open set in Y, such that $\mu(f, g)(C) \subset U$. Put $\mu'(C) = C_1 \vee C_2$, then $f \in W(C_1, U)$, $g \in W(C_2, U)$ and $\mu(W(C_1, U), W(C_2, U)) \subset W(C, U)$. This implies the continuity of μ. The remainder of proof is easy.

Sufficiency. Put $\mu' = \mu(i_1', i_2')$: $X \longrightarrow X \vee X$. Then,

$$\begin{aligned} \nabla \circ (1 \vee 0_e) \circ \mu' &= \mu \circ (\nabla \circ (1 \vee 0_e) \circ i_1', \ \nabla (1 \vee 0_e) \circ i_2') \\ &= \mu(1, 0_e) \simeq 1, \end{aligned}$$

where μ stands for the H-structures of $\{X, X \vee X\}_0$ and $\{X, X\}_0$.

Remark. For co-H-groups, i. e., co-associative co-H-spaces with co-inverses, Proposition 2. 2 holds without restriction on homotopy units. (*cf.* [1]).

Remark. If (X, e, μ') is a co- H-space, we have $\mu(i_1', i_2') = \mu'$, and $\mu(p_1', p_2') \simeq \nabla$.

2. 3. **Proposition. (Square lemma)** *Let (X, e_X, μ') be a co-H-space and (Y, e_Y, μ) an H-space. Then, two H-structures on $\{X, Y\}_0$, induced by μ' and μ, are homotopic.*

Moreover, $\{X, Y\}_0$ *is homotopy-commutative.*

Proof. See [1].

3. Additive systems.

Definition. A non-empty set A is called an *additive system*, if there exists a binary operation, called addition, $+: A \times A \longrightarrow A$, satisfying the following two axioms

commutativity: for any $a, b \in A$, $a+b=b+a$;

existence of zero: there exists an element $0 \in A$, such that

$$0+a=0 \qquad \text{for all } a \in A.$$

(Obviously, such an element is unique.)

Additive sub-systems, homomorphisms, isomorphisms and so on, are defined similarly as for groups.

Examples 1. Any monoid is an additive system.

2. Let X be a co-H-space, Y an H-space, then $[X, Y]_0$ is an additive system.

Definitin. Let A_1 and A_2 be given two additive systems. An additive system A is called the *direct sum* of A_1 and A_2 if there exists an isomorphism f: $A \longrightarrow A_1 \oplus A_2$, where $A_1 \oplus A_2$ is an additive system defined as follows:

$$A_1 \oplus A_2 = \{(a_1, a_2) | a_1 \in A_1, a_2 \in A_2\},$$
$$(a_1, a_2) + (a_1', a_2') = (a_1 + a_1', a_2 + a_2').$$

3.1. Proposition. *Let A, A_1 and A_2 be given three additivc systems. A is the direct sum of A_1 and A_2 if and only if there exist homomorphisms*

$$i_1: A_1 \longrightarrow A,\ i_2: A_2 \longrightarrow A \text{ and } p_1: A \longrightarrow A_1,\ p_2: A \longrightarrow A_2,$$

such that

$$p_1 \circ i_1 = 1,\ p_2 \circ i_2 = 1,\ p_1 \circ i_2 = 0,\ p_2 \circ i_1 = 0,$$

and

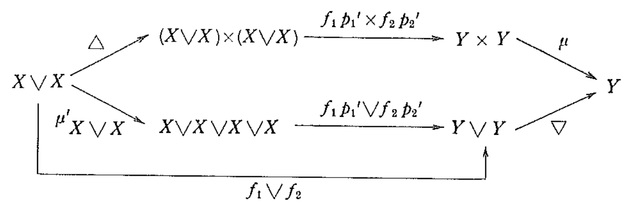
$$i_1 \circ p_1 + i_2 \circ p_2 = 1.$$

Proof is easy, and omitted.

3.2. Lemma. *Let (X, e_X, μ') be a co-H-space, (Y, e_Y, μ) an H-space and $[f_1]$, $[f_2] \in [X, Y]_0$, then $\nabla(f_1 \vee f_2): X \vee X \longrightarrow Y \vee Y \longrightarrow Y$ represents*

$$p_1'^*[f_1] + p_2'^*[f_2] \in [X \vee X, Y]_0.$$

Proof. By Proposition 2.3, we have the following homotopy commutative diagram



This concludes our result.

3.3. Corollary. *Let* (X, e_X, μ') *be a co-H-space,* (Y, e_Y, μ) *an H-space, then we have* $[X\vee X,\ Y]_0\approx[X,\ Y]_0\oplus[X,\ Y]_0$.

Proof. As easily seen, $i_h'^*p_k'^*=\delta_{hk}$, $h, k=1, 2$. Let f be an arbitrary element of $[X\vee X,\ Y]_0$, then Lemma 3.2 implies that $\nabla[(fi_1'\vee fi_2')]$ represents $[fi_1'p_1']+[fi_2'p_2']=p_1'^*i_1'^*[f]+p_2'^*i_2'^*[f]$. However,

$$\begin{aligned}\nabla(fi_1'\vee fi_2')&=\nabla(f\vee f)(i_1'\vee i_2')\\&=f\nabla_{X\vee X}(i_1'\vee i_2')\\&=f.\end{aligned}$$

Therefore, we have $p_1'^*i_1'^*[f]+p_2'^*i_2'^*[f]=[f]$. By Proposition 3.1, we have the desired result.

4. Main Theorem. To state our Theorem, we shall give the following

Definition. Let (X, e, μ') be a given co-H-space. A continuou map ν_R': $X\longrightarrow X$ is a *right homological co-inverse* of μ', provided that for any H-space $(Y,\ e_Y)$ we have

$$[f]+[f\nu_R']=0 \qquad \text{for all } [f]\in[X,\ Y]_0.$$

Similarly, a *left homological co-inverse* ν_L' of μ' is defined.

4.1. Theorem. *Let* (X, e, μ') *be a co-H-space. If X is a simply connected* CW-*complex,* (X, e, μ') *has a right homological co-inverse* ν_R' *and a left homological co-inverse* ν_L'.

In order to prove Theorem 4.1, we recall the following two theorems (cf. [3]):

4.2. Theorem of J.H.C. Whitehead. *Let X and Y be simply connected spaces, and let* f: $X\longrightarrow Y$ *be a map. Then, the following three statements are equivalent:*

(i) *for every n,* f_*: $H_n(X)\longrightarrow H_n(Y)$ *is an isomorphism,*

(ii) *for every n and every coefficient group G,* f^*: $H^n(Y; G)\longrightarrow H^n(X; G)$ *is an isomorphism,*

(iii) *for every n,* f_*: $\pi_n(X)\longrightarrow\pi_n(Y)$ *is an isomorphism.*

4.3. Theorem. *In the category of CW-complexes, a weak homotopy equivalence is a homotopy equivalence.*

Proof of Theorem 4.1. Define $\varphi\colon X\vee X\longrightarrow X\vee X$ by the composition

$$(\nabla\vee 1)(1\vee\mu')\colon X\vee X\longrightarrow X\vee X\vee X\longrightarrow X\vee X.$$

For any integer $n\geqq 1$, and any abelian group G, we have

$$H^n(X\vee X;\ G)\approx p_1'^*H^n(X;\ G)\oplus p_2'^*H^n(X;\ G).$$

Let $f_1, f_2\colon X\longrightarrow K(G,n)$ be representatives of $\alpha_1,\ \alpha_2\in H^n(X;\ G)=[X,\ K(G,\ n)]_0$, respectively. Then,

$$\nabla(f_1\vee f_2)(1\vee\nabla)(\mu'\vee 1)\colon X\vee X\longrightarrow K(G,\ n)$$

represents $\varphi^*(p_1'^*\alpha_1+p_2'^*\alpha_2)$. However, we have

$$\begin{aligned}\nabla(f_1\vee f_2)(1\vee\nabla)(\mu'\vee 1)&=\nabla(\nabla\vee 1)(f_1\vee f_2\vee f_2)(\mu'\vee 1)\\&=\nabla((f_1+f_2)\vee f_2).\end{aligned}$$

By Lemma 3.2, it follows $\varphi^*(p_1'^*\alpha_1+p_2'^*\alpha_2)=p_1'^*(\alpha_1+\alpha_2)+p_2'^*\alpha_2$. This implies that φ^* is an isomorphism. By virtue of Theorems 4.2 and 4.3, we conclude that φ is a homotopy equivalence.

Now, let (Y, e_Y) be an H-space, and $\alpha\in[X, Y]_0$. Define $\nu_R'\colon X\longrightarrow X$ by the following composition

$$p_2'\circ\psi\circ i_1'\colon X\longrightarrow X\vee X\longrightarrow X,$$

where ψ is a homotopy inverse of φ.

Using $[X\vee X, Y]_0\approx[X,\ Y]_0+[X,\ Y]_0$ and $\nabla\simeq p_1'+p_2'$, we have

$$\nabla^*\alpha=(\alpha,\ \alpha)\qquad\text{for all } \alpha\in[X,\ Y]_0. \tag{4.4}$$

Next, put $\psi^*(0,\ \alpha)=(\xi,\ \eta)$, that is $\varphi^*(\xi,\ \eta)=(0,\ \alpha)$, which implies $\xi+\eta=0$ and $\eta=\alpha$, hence $\xi+\alpha=0$. Accordingly, we have

$$\psi^*(0,\ \alpha)=(\xi,\ \alpha)\ \text{ implies }\ \xi+\alpha=0. \tag{4.5}$$

Lastly, let f be a representative α, then we have

$$\begin{aligned}[f]+[f\circ\nu_R']&=\mu'^*(1\vee i_1')^*(1\vee\psi)^*(1\vee p_2')^*\nabla^*(\alpha)\\&=\mu'^*(1\vee i_1'(^*(1\vee\psi)^*(1\vee p_2')^*(\alpha,\alpha)\qquad\text{by (4.4)}\\&=\mu'^*(1\vee i_1')^*(1\vee\psi)^*(\alpha,\ (0,\ \alpha))\end{aligned}$$

$$= \mu'^*(1 \vee i_1')^*(\alpha, (\xi, \eta)) \qquad \text{putting } \phi^*(0, \alpha) = (\xi, \eta)$$
$$= \alpha + \xi$$
$$= 0 \qquad \text{by (4. 5) and commutativity.}$$

Consequently, ν_R' is a right homological co-inverse of μ'. Similanly, we may show the existence of a left homological co-inverse.

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