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Nonsense Channel Terms Associated with the $O_{3,1}$ Partial Wave Expansion

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Abstract

The matrix element $\mathscr{D}^{M\lambda}_{jm,j'm'}(g)$ of the representation of the group SL_{2C} is easily obtained by the bases that Toller has used, but the M-σ symmetry ($\lambda=\sigma$; 2σ is an integer.) of the representations is conveniently treated by Gel'fand type representatives. We give this representative to Toller's base vector, and prove the M-σ symmetry of the function $d^{M\sigma}_{jmj'}(\zeta)$.

1 Introduction

When the $O_{3,1}$ partial wave expansion[1] of a $t=0$, equal-mass scattering amplitude is continued analytically to the O_4 partial wave expansion, the nonsense channel terms appear, which vanish as a whole according to the M-σ symmetry of $d^{M\sigma}_{jmj'}(\zeta)$.[2]

The M-σ symmetry is somewhat different in its nature from other symmetry properties of $d^{M\sigma}_{jmj'}(\zeta)$, since it is concerned in two inequivalent representations. To obtain the matrix element $d^{M\sigma}_{jmj'}(\zeta)$, we have to use the base vectors of Toller type,[3] but the treatment of Gel'fand et al.[4] is more favorable to understand the group theoretical properties of the representations. We intend to give a formalism which combines these two types of treatments, and prove the M-σ symmetry of the function $d^{M\sigma}_{jmj'}(\zeta)$ by this formulation.*)

2 Representation Spaces of the Group SL_{2C}

Using Toller's notation,[3] we denote by K the subgroup which consists of all the elements of the form

$$k=\begin{pmatrix} p^{-1} & q \\ 0 & p \end{pmatrix}\in K\subset SL_{2C}. \tag{1}$$

*) This symmetry relation was cited in 2) as it was proved by M. Toller, but we could not know its details.

Each right coset of SL_{2C} with respect to K contains a unique element of the form

$$z=\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \tag{2}$$

hence we attach the index z to the coset involving the element z such that $A_Z = Kz$. If we use SL_{2C}/K as a homogeneous space, the representation spaces are constructed as follows.

Since a function $f(A_Z)$ defined on SL_{2C}/K can be regarded as a function of the element z or a complex number z defined by (2), we can put

$$f(A_Z) = \varphi(z).$$

The linear space spanned by these functions $f(A_Z)$ or $\varphi(z)$, which satisfy the conditions of Gel'fand et al.,[4] is the representation space D_χ, and the representation T_ω on this space is given by the equation

$$[T_\omega(g)f](A_Z) = \omega(A_Z,\ g)f(A_Z g), \qquad g \in SL_{2C}, \tag{3}$$

where

$$\omega(A_Z,\ gg') = \omega(A_Z,\ g)\omega(A_Z g,\ g'), \qquad g,\ g' \in SL_{2C}. \tag{4}$$

If we regard $\varphi(z)$ as a function of the element z, it can be extended to the function defined on the whole group SL_{2C} by the equation

$$f(g) = L^{n_1-1,\, n_2-1}(k)\varphi(z), \qquad k = gz^{-1} \in K, \tag{5}$$

where $L^{n_1-1,\, n_2-1}(k) = p^{n_1-1}\,\bar{p}^{n_2-1}$ for the element k of (1), which is the one-dimensional representation of the subgroup K, and n_1, n_2 are arbitrary complex numbers.

In the following, we use the notation

$$\chi = (n_1 | n_2) = (M, \lambda), \quad \text{where} \quad n_2 - n_1 = 2M,\ \ n_2 + n_1 = 2\lambda, \tag{6}$$

and write L^χ or $L^{M\lambda}$ for $L^{n_1-1,\, n_2-1}$. Making use of this one-dimensional representation, we can construct a representation which is induced on SL_{2C} from it, i.e.

$$[\mathscr{D}^\chi(g)f](a) = f(ag), \qquad a \in SL_{2C}, \tag{7}$$

taking into account of (5) which shows

$$f(kg) = L^\chi(k)f(g). \tag{8}$$

This representation, however, is the same as the one given by (3) and (4). In fact, if we write an element g of SL_{2C} as

$$g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\delta^{-1} & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma/\delta & 1\end{pmatrix}, \qquad \alpha\delta-\beta\gamma=1,$$

$f(g)$ depends only on the variables γ and δ according to the covariance condition (8), i. e. $f(g)=f(\gamma,\delta)$. Moreover $f(\gamma,\ \delta)$ is the homogeneous function of the degree $(n_1-1,\ n_2-1)$, so that (7) can be rewritten as

$$[\mathscr{D}^{\chi}(g)\varphi](z)=(\beta z+\delta)^{n_1-1}(\overline{\beta z+\delta})^{n_2-1}\varphi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right), \tag{9}$$

which is nothing but the equation (3) with

$$\omega(A_Z,\ g)=(\beta z+\delta)^{n_1-1}(\overline{\beta z+\delta})^{n_2-1}.$$

On the other hand the coset A_Z contains a unitary element u which is uniquely determined except the uncertainty of $u_Z(\mu)\in K$, i. e. $A_Z=Ku$. Thus we can put

$$f(A_Z)=L^{\chi}(k_1^{-1})f(u), \qquad k_1z=u, \tag{10}$$

taking account of (5). In order to avoid the uncertainty of the phase in $L^{\chi}(k)$, we restrict hereafter $2M$ to be integers. Then (7) or (9) can be rewritten as

$$[\mathscr{D}^{\chi}(g)f](u)=L^{\chi}(ug(ug)_0^{-1})f((ug)_0), \tag{11}$$

where $(g)_0$ is a unitary element in A_Z containing $g\in SL_{2C}$.

Let u be factorized in the form

$$u=\begin{pmatrix}\overline{a} & -\overline{b}\\ b & a\end{pmatrix}=u_z(\mu)u_y(\theta)u_z(\nu),$$

then we have

$$z=b/a=e^{-i\nu}tan\frac{\theta}{2},$$

and so,

$$L^{n_1-1,n_2-1}(k)=|a|^{2(M+\lambda-1)}a^{-2M}=\left(\frac{1+\cos\theta}{2}\right)^{\lambda-1}e^{-i(\mu+\nu)M}, \tag{12}$$

where $k=uz^{-1}$.

The base vectors of D_{χ} corresponding to $\Phi^{M}_{jm}(u)$ used by Toller have the form in Gel'fand type representatives,

$$\varphi_{jm}^{M\lambda}(z)=(2j+1)^{1/2}e^{i(M-m)\nu}\left(\frac{1+\cos\theta}{2}\right)^{1-\lambda}d_{Mm}^{j}(\theta), \tag{13}$$

where

$$\Phi_{jm}^{M}(u)=(2j+1)^{1/2}R_{Mm}^{j}(u),\quad d_{Mm}^{j}(\theta)=e^{iM\mu}R_{Mm}^{j}(u)e^{im\nu},$$

and R^j is the ordinary $(2j+1)$-dimensional representation of SU_2.

Finally, we give the operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ expressed in terms of μ, θ and ν;

$$\frac{\partial}{\partial z}=-2e^{i\nu}\left(\frac{1+\cos\theta}{2}\right)^{3/2}\left(\frac{1-\cos\theta}{2}\right)^{1/2}\frac{\partial}{\partial\cos\theta}+\frac{i}{2}e^{i\nu}\left(\frac{1+\cos\theta}{2}\right)^{1/2}\left(\frac{1-\cos\theta}{2}\right)^{-1/2}\frac{\partial}{\partial\nu},$$

$$\frac{\partial}{\partial\bar{z}}=-2e^{-i\nu}\left(\frac{1+\cos\theta}{2}\right)^{3/2}\left(\frac{1-\cos\theta}{2}\right)^{1/2}\frac{\partial}{\partial\cos\theta}-\frac{i}{2}e^{-i\nu}\left(\frac{1+\cos\theta}{2}\right)^{1/2}\left(\frac{1-\cos\theta}{2}\right)^{-1/2}\frac{\partial}{\partial\nu}, \tag{14}$$

which are easily obtained by direct calculations from the equation above (12).

3 Intertwining Operators

When a linear mapping A of a representation space D_{χ_1} into another space D_{χ_2} satisfies the relation

$$A\mathscr{D}^{\chi_1}(g)=\mathscr{D}^{\chi_2}(g)A, \tag{15}$$

A is called an intertwining operator. Gel'fand et al.[4] showed that in the case of the representation of SL_{2C} an intertwining operator exists if and only if one of the following conditions is fulfilled:

Case 1) $n_1=-m_1$, $n_2=-m_2$, n_1, n_2 nonnegative integers;

Case 2) $n_1=-m_1=1, 2, 3, \ldots,$ $n_2=m_2$;

Case 3) $n_1=m_1$, $n_2=-m_2=1, 2, 3, \ldots$;

Case 4) $n_1=-m_1$, $n_2=-m_2$, n_1, n_2 not simultaneously nonnegative integers;

Case 5) $n_1=m_1$, $n_2=m_2$,

where $\chi_1=(n_1|n_2)$ and $\chi_2=(m_1|m_2)$. Now we list the intertwining operators in the first three cases, which are concerned with our $M-\sigma$ symmetry:

Case 1) $A=\partial^{n_1+n_2}/\partial z^{n_1}\partial\bar{z}^{n_2}$;

Case 2) $A=\partial^{n_1}/\partial z^{n_1}$;

Case 3) $A=\partial^{n_2}/\partial\bar{z}^{n_2}$.

Let us calculate the matrix element of A in *Case* 2). As $n_1=\sigma\text{-}M$ is a nonnegative integers, we can differentiate $\varphi_{jm}^{M\sigma}(z)$ n_1 times with respect to z, and we have then

$$A\varphi_{jm}^{M\sigma}(z)=\frac{\partial^{n_1}}{\partial z^{n_1}}\varphi_{jm}^{M\sigma}(z)=(-1)^{\sigma-M}\left[\frac{\Gamma(j-M+1)\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)\Gamma(j+M+1)}\right]^{1/2}\varphi_{jm}^{\sigma M}(z), \tag{16}$$

and, for the matrix element,

$$A_{jm,j'm'}=(-1)^{\sigma-M}\left[\frac{\Gamma(j-M+1)\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)\Gamma(j+M+1)}\right]^{1/2}\delta_{jj'}\delta_{mm'}, \tag{17}$$

where we used the relation

$$\frac{\partial^r}{\partial z^r}\varphi_{jm}^{M\sigma}(z)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{\Gamma(\sigma-M-k)}{\Gamma(\sigma-M-r)}\left(\frac{\bar{z}}{1+|z|^2}\right)^{r-k}\left[\frac{\Gamma(j-M+1)\Gamma(j+M+k+1)}{\Gamma(j-M-k+1)\Gamma(j+M+1)}\right]^{1/2} \times\varphi_{jm}^{M+k,\,\sigma-k}(z)$$

obtained from (13) and (14).

A similar method can be used for *Case* 3), but the differentiation must be made with respect to $\bar{z}$ for z in *Case* 2), and we have then

$$A\varphi_{jm}^{M\sigma}(z)=\left[\frac{\Gamma(j+M+1)\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)\Gamma(j-M+1)}\right]^{1/2}\varphi_{jm}^{-\sigma,-M}(z),$$

and

$$A_{jm,j'm'}=\left[\frac{\Gamma(j+M+1)\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)\Gamma(j-M+1)}\right]^{1/2}\delta_{jj'}\delta_{mm'}. \tag{18}$$

Finally, *Case* 1) is the combination of *Cases* 2) and 3), that is,

$$A=A_{Case 3)}\cdot A_{Case 2)},$$

i. e.

$$A_{jm,j'm'}=(-1)^{\sigma-M}\frac{\Gamma(j+\sigma+1)}{\Gamma(j-\sigma+1)}\delta_{jj'}\delta_{mm'}. \tag{19}$$

Using these results, for instance, we can construct the special invariant subspaces denoted by E_{χ_1} and F_{χ_2} in Gel'fand's book.[4] If $\sigma+M\geq 0$ in *Case* 2), then $A\varphi_{jm}^{M\sigma}(z)=0$ for j such that $\sigma>j\geq|M|$. Hence the subspace spanned by these $\varphi_{jm}^{M\sigma}(z)$ is invariant and is just E_{χ_1}. If $\sigma+M<0$, the indices j appearing in the image of A are not smaller than $|M|$, and therefore the image is an invariant subspace of D_{χ_2}, which is just F_{χ_2}.

4 M-σ Symmetry of $d_{jmj'}^{M\sigma}(\zeta)$

For the element

$$a_z(\zeta)=e^{\frac{1}{2}\sigma_z\zeta}=\begin{pmatrix} e^{\zeta/2} & 0 \\ 0 & e^{-\zeta/2}\end{pmatrix},$$

it has been known that $\mathscr{D}^{\chi}(a_z(\zeta))$ has the form[3)]

$$\mathscr{D}^{\chi}_{jm,j'm'}(a_z(\zeta)) = \delta_{mm'} \; d^{M\lambda}_{jmj'}(\zeta), \quad \chi = (M, \lambda),$$

which is the definition of the function $d^{M\lambda}_{jmj'}(\zeta)$. The symmetry relation to be proved is

$$d^{M\sigma}_{jmj'}(\zeta) = \left[\frac{\Gamma(j+M+1))\Gamma(j-\sigma+1)\Gamma(j'-M+1)\Gamma(j'+\sigma+1)}{\Gamma(j+\sigma+1)\Gamma(j-M+1)\Gamma(j'-\sigma+1)\Gamma(j'+M+1)} \right]^{1/2} d^{\sigma M}_{jmj'}(\zeta),$$
$$\text{where } j, j' \geq \text{Max}\,(|M|, |\sigma|). \tag{20}$$

For convenience of verification, we divide the (M, σ) region into four parts.

Region A; $\sigma-M \geq 0$, $\sigma + M \geq 0$. In this case (20) is nothing but the partial equivalence of the representations on D_{χ_1} and D_{χ_2}, i.e. the equivalence of the representations on D_{χ_1}/E_{χ_1} and D_{χ_2}, and therefore it can be derived directly from (16) or (17).

Region B; $\sigma - M \leq 0$, $\sigma + \geq 0$. If we interchange the roles of M and σ, this region becomes *Region A*, and (20) is also valid, but in this case

$$j, j' \geq |M|.$$

Region C; $\sigma - M \leq 0$, $\sigma + M \leq 0$. This is the case in which roles of M and σ are interchanged in *Region D*, and (20) is also valid.

Region D; $\sigma - M \geq 0$, $\sigma + M \leq 0$. In this case $|\sigma| \leq |M|$, and then A maps D_{χ_1} onto the subspace F_{χ_2} which is invariant under the transformation $\mathscr{D}^{\chi_2}(g)$, so that (20) is also derived from (17) and

$$j, j' \geq |M|.$$

Thus, the symmetry relation (20) is verified. Of course, M and σ are both integers or half-odd integers, and m must be satisfied with the condition

$$-\text{Min}\,(j, j') \leq m \leq \text{Min}\,(j, j').$$

Above results are all derived by making use of *Case* 2) in § 3, but a similar result can be obtained from *Case* 3), i.e.

$$d^{M\sigma}_{jmj'}(\zeta) = \left[\frac{\Gamma(j-\sigma+1)\Gamma(j-M+1)\Gamma(j'+M+1)\Gamma(j'+\sigma+1)}{\Gamma(j+M+1)\Gamma(j+\sigma+1)\Gamma(j'-\sigma+1)\Gamma(j'-M+1)} \right]^{1/2} d^{-\sigma,-M}_{jmj'}(\zeta),$$
$$\text{where } j, j' \geq \text{Max}\,(|M|, |\sigma|). \tag{21}$$

For arbitrary λ, however, $d^{M\lambda}_{jmj'}(\zeta)$ satisfies the relation[2)]

$$d^{-M,-\lambda}_{jmj'}(\zeta) = \frac{\Gamma(j+\lambda+1)\Gamma(j'-\lambda+1)}{\Gamma(j-\lambda+1)\Gamma(j'+\lambda+1)} d^{M\lambda}_{jmj'}(\zeta),$$

and therefore (21) is equivalent to (20).

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