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Existence of Some Metrics on Manifolds

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Introduction.

The purpose of this paper is to show the following theorem.

Theorem. *If X is an arcwise connected paracompact (topological) manifold, then there exists a metric ρ by which the topology of X is given and has the following properties.*

(i). *X is complete as a metric space by the metric ρ.*

(ii). *If $x,\ y \in X$, then there exists a curve γ which joins x and y and its length with respect to the metric ρ (cf. [2], [4], [6]) is equal to $\rho(x, y)$. Moreover, such ρ is unique up to the change of parameters if $\rho(x, y)$ is sufficiently small.*

(iii). *The Alexander-Spanier n-cochain $\rho(x_0, x_1)\, \rho(x_0, x_2) \cdots \rho(x_0, x_n)$ ($n=\dim. X$) defines a positive Radon measure $m=m(\rho)$ such that*

$$m(\rho)\ (E) \neq 0,$$

if E is measurable and contains some non-empty open set of X (cf. [4]).

If X is a smooth manifold, then these has been known. In fact, (ii) is a part of the theorem of Hopf-Rinow ([5], [8], [12], [13]), (i) is the theorem of Nomizu-Ozeki ([10]) and (iii) follows from the existence of a Riemannian metric ([17]. For the properties of Radon-measure, we refer [12]).

For the above metric ρ, we can show the followings which are also parts of the theorem of Hopf-Rinow if X is smooth.

Theorem. *ρ also has following properties.*

(i). *A bounded set of X by the metric ρ is relative compact.*

(ii). *If a curve γ, $f : [0, 1]\to X$ or $g : [0, 1)\to X$ satisfies*

(a) *the length of $\gamma_{a,b}$, $f_{a,b} : [0, 1]\to X$ or $g_{a,b} : [0, 1)\to X$ with respect to ρ is $\rho(f(a), f(b))$ (or $\rho(g(a), g(b))$) for any a, b ($0 \leqq a < b \leqq 1$ for $f_{a,b}$ and $0 \leqq a < 1$ for $g_{a,b}$),* where *$f_{a,b}(t)$ is given by $f_{a,b}(t) = f(a+(b-a)t)$ ($g_{a,b}(t) = g(a+(b-a)t)$).*

Then either the length of γ is infinite or there is a curve $\hat{\gamma}$ with infinite length such that $\hat{\gamma}$ satisfies (a) and γ is written as $\hat{\gamma}_{a,b}$ for some a, b.

The outline of this paper is as follows : In § 1, we consider the length of a curve γ with respect to a metric ρ. As in [2], [4] and [6], it is defined to be the limit

$$\lim_{|t_{i+1}-t_i|\to 0} \sum_{i=1}^{m} \rho(f(t_{i+1}),\ f(t_i)),\quad 0=t_1<t_2<\cdots<t_{m+1}=1,$$

if γ is given by $f : \boldsymbol{I}\to X$ ($\boldsymbol{I}$ means the closed interval $[0, 1]$). If we use the notation of [4], then we may write

$$\textit{the length of } \gamma \textit{ with respect to } \rho = \int_\gamma \rho.$$

After treating some elementary properties of $\int_\gamma \rho$, we consider a metric space X with metric ρ which satisfy

(*) *For any* $x,\ y\in X$, *there exists a curvr* $\gamma=f\ (I)$ *such that* $f\ (0)=x,\ f\ (1)=y,\ \int_\gamma \rho<\infty$.

Note. This property has been considered in [2] and [6].

We note although X is an arcwise connected paracompact manifold, (*) is not fulfilled for arbitrary metrics. In fact, if $X\subset \boldsymbol{R}^2$ is given by $(\{x,\ x\sin(1/x))\ |x\neq 0\}\cup(0,0)$ with the metric induced from $\boldsymbol{R}^2$, then it does not satisfy (*).

If X and ρ satisfy (*), then setting

$$\hat{\rho}(x,y)=\inf._{\gamma,\gamma \text{ joins } x \text{ and } y,} \int_\gamma \rho,$$

$\hat{\rho}$ is a metric of X and if X is locally compact, then as in [6], § 20, we can show if $\rho\ (x, y)$ is sufficiently small, then there exists a curve γ which joins x and y and $\int_\gamma \rho=\hat{\rho}\ (x,\ y)$ (Theorem 3. we note that we get

$$\int_\gamma \rho=\int_\gamma \hat{\rho},$$

for any curve γ of X).

§ 2 and § 3 are devoted to show that if X is an arcwise connected paracompact manifold, then there is a metric ρ of X by which the topology of X is given and satisfies (*). In § 2, we show that if the structure group of the tangent microbundle of X is reduced to the group of those (germs of) homeomorphisms of $\boldsymbol{R}^n$ which are expressed by the functions of boundded variations, as an $H_*(n)$-bundle (cf. [3]), then X has a metric ρ which satisfies (*). we note that this can be also proved as follows : Take the locally finite open covering $\{U\}$ and homeomorphisms $\{h_U\}$, $h_U : U\to\boldsymbol{R}^n$ and $\{f_U(x)\}$, $f_U(x) : \boldsymbol{R}^n\to\boldsymbol{R}^n$ $(x\in U)$ such that $f_U(x) h_U h_V{}^{-1} f_V(x)^{-1}$ is a local homeomorphism of $\boldsymbol{R}^n$ which are expressed by the functions of boundded variations for any U and $x\in U$. Then taking a partition of unity $\{e_U(x)\}$ subordinated to $\{U\}$, we set

$$d(x,\ y)$$
$$= \sum_{x,\ y \in U} e_U(x) e_U(y) \| f_U(x) h_U(x) - f_U(y) h_U(y) \|,$$

where $\|\xi-\eta\|$ means the euclidean norm of the vector $\xi-\eta$. Then although $d(x, y)$ may not be a metric on X, $d(x,\ y)$ is defined on $U(\Delta(X))$, a neighborhood of the diagonal $\Delta(X)$ in $X \times X$ and $d(x,\ y) \geqq 0$, $d(x,\ y) = 0$ if and only if $x = y$, $d(x,\ y) = d(y,\ x)$ and there exists a curve γ which joins x and y and

$$\int_\gamma d < \infty.$$

Then to set $\rho(x,\ y) = inf.\ \gamma, \gamma$ *joins* x *and* $y \int_\gamma d$, ρ is a metric of X which satisfies (*).

In § 3, we prove that $H_*(n)$ is defomed to its subset consisted by those homeomorphisms φ such that each φ is represented as
$\varphi(x) = (f_1(x),\ f_2(x),\ \cdots\cdots,\ f_n(x))$ where each $f_i(x)$ satisfies

$f_i(x)$ is a function of bounded variations and
$\{\log 2/\log(2n+2)\}$ —Hölder continuous.

In this proof, we use the axiom of choice and Kolmogorov's representation theorem of continuous function of several variables by finite sums and superposition of continuous functions of one variable with its refinement by Sprecher ([9], [15], cf. [1], [16]), which asserts that there exist $n(2n+1)$ monotonic, $\{\log 2/\log (2n+2)\}$ – Hölder continuous functions $\chi_{ij}(x)$ on unit interval $I = [0,\ 1]$ such that for any continuous functions f on $I^n = \overbrace{I \times \cdots \times I}^{n}$ can be written as

$$f(x_1,\ \cdots\cdots,\ x_n) = \sum_{i=1}^{2n+1} f_i\Big(\sum_{j=1}^{n} \chi_{ij}(x_j)\Big),$$

where each $f_i(x)$ is a continuous function determined by f. we note that the above fact also sugests us the possibility of the existence of $\{\log 2/\log(2n+2)\}$ – Hölder continuous structure on n-dimensional (topological) manifolds (cf. [14]).

Then together with the result of § 2, we get,

Theorem. *If X is an arcwise connected paracompact (topological) manifolb, then there is a metric ρ by which the topology of X is given and has following propeties.*

(i). *ρ satisfies (*) and if $\rho\ (x,\ y)$ is sufficiently small, then there is a curve γ which joins x and y and*

$$\int_\gamma \rho = \rho(x,\ y).$$

Moreover, such curve is unique up to the change of parameters.

(ii). *The measure $m = m(\rho)$ determined by the n-cochain* $\rho(x_0,\ x_1)\rho(x_0,\ x_2) \cdots \rho(x_0,\ x_n)$

is a positive Radon measure on X and

$$m(E) \neq 0,$$

if E containes some non empty open set of X.

Since the most part of the proof of the theorems of Hopf-Rinow uses only the usual properties of metric and the fact that the geodesic distance of a Riemannian manifold satisfies above (i) (cf. [5], [8], [11], [12]), we can show the global results stated in the beginning of this introduction only with a little modification of the proofs of the theorems of Hopf-Rinow and Nomizu-Ozeki (note that the proof of the theorem of Nomizu-Ozeki uses heavily the theorems of Hopf-Rinow, cf. [10]. These are stated in § 4.

I would like to thank Dr. Kano who teach me the theorem of Kolmogorov.

§ 1. Metric space whose any two points can be joined by a curve with finite length.

1. ***Definition*** *(cf. [2], [4], [6]). Let X be a metric space with the metric ρ, $\gamma = f(I)$ a curve in X ($I = [0,\ 1]$, $f : I \to X$ is a cotinuous map), then the length of γ with respect to ρ is defined by*

$$\int_{\gamma} \rho(x,\ y) = \lim_{|a_{i+1} - a_i| \to 0} \sum_{i=0}^{m} \rho(f(a_{i+1}),\ f(a_i)),$$

$$0 = a_0 < a_1 < \cdots\cdots < a_{m-1} < a_m = 1.$$

Note. In this definition, γ may not be given to be a continuous image of I. If γ is given to be a continuous image of a closed interval $[a,\ b]$, then we define the length of γ with respect to ρ by the same way. On the other hand, if γ is given, for example, to be a continuous image of $[0,\ 1)$, then to set

$$\gamma_a = f_a(I),\ \ f_a(t) = f(at),\ \ 0 \leqq a < 1,$$

we define the length of γ with respect to ρ to be the limit

$$\lim_{a \to 1} \int_{\gamma_a} \rho(x,\ y).$$

By definietion, if $\gamma = f(I)$ has finite length, then γ_a $(0 \leqq a \leqq 1)$ also has finite length and we have

$$\int_{\gamma_a} \rho(x,\ y) \leqq \int_{\gamma_b} \rho(x,\ y),\ \ if\ a \leqq b.$$

In general, we know that γ has finite length if and only if there exists a constant M such that

$$\sum_{i=0}^{m} \rho(f(\mathrm{a}_{i+1}),\ f(a_i)) \leqq M,$$

for any partition $0 = a_0 < a_1 < \cdots\cdots < a_m = 1$ of I.

Lemma 1. *The length of γ does not depend on the choice of the parametert, that is, for any (into) homeomorphism $\varphi : I \to R^1$, we have*

$$\int_{f(I)} \rho(x,\ y) = \int_{f(\varphi^{-1}([a,b]))} \rho(x,\ y),$$
$$[a,\ b] = \varphi(I).$$

Proof. Since φ is monotonic, setting

$$a = c_0 < c_1 < \cdots\cdots < c_{m-1} < c_m = b,$$

$0 = \varphi(c_0) < \varphi(c_1) < \cdots < \varphi(c_{m-1}) < \varphi(c_m) = 1$ if φ is orientation preserving, and we have the lemma in this case. On the other hand, since $\rho(f(\varphi(c_{i+1})),\ f(\varphi(c_i))) = \rho(f(\varphi(c_i)),\ f(\varphi(c_{i+1})))$, we have

$$\sum_{i=0}^{m} \rho(f(\varphi(c_{i+1})),\ f(\varphi(c_i)))$$
$$= \sum_{i=0}^{m} (f(\varphi(c_{m-i-1}),\ f(\varphi(c_{m-i}))),$$

and $0 = \varphi(c_m) < \varphi(c_{m-1}) < \cdots < \varphi(c_1) < \varphi(c_0) = 1$ is a partition of I if φ is orientation reversing, we get the lemma for orientation reversing φ.

For the length of curves, we obtain by the triangle inequality

Lemma 2. *For any curve γ, we have*

$$\rho(f(0),\ f(1)) \leqq \int_\gamma \rho(x,\ y). \tag{1}$$

Moreover, setting $f(0) = \alpha$, we know that k_α defined by $(\mathrm{k}_\alpha \rho)\ (x) = \rho(\alpha,\ x)$ is defined on X and we have

$$\rho = \delta k_\alpha \rho + k_\alpha \delta \rho.$$

Then since $\int_\gamma \delta k_\alpha \rho = \rho(\alpha,\ f(1)) - \rho(\alpha, f(0)) = \rho(\alpha, f(1))$, we obtain

Lemma 3. *We have $\rho(x,\ y) = \rho(f(0),\ f(1))$ if and only if*

$$\int_\gamma (k_\alpha \delta \rho)(x,\ y) = 0, \tag{2}$$
$$(k_\alpha \delta \rho)(x,\ y) = \rho(x,\ y) - \rho(\alpha, y) + \rho(\alpha,\ x).$$

2. In the rest of this §, we assume that X and ρ satisfy the following (*). (cf. [2], [6]).

(*). *For any $x,\ y \in X$, there exists a curve $\gamma = f(I)$ such that*

$$f(0)=x,\ \ f(1)=y,\ \ \int_{\gamma}\rho<\infty. \tag{3}$$

Note. Since γ is compact, (*) is equivalent the following (*)$'$.

(*)$'$. *X is arcwise connected and if $\rho\ (x,\ y)$ is sufficiently small, then there exists a curve $\gamma=f(I)$ which satisfies (3) for $x,\ y$.*

Definition. *On X, we set*

$$\hat{\rho}(x,\ y)=\inf_{\substack{\gamma=f(I)\\ f(0)=x,\ (f1)=y}}\int_{\gamma}\rho. \tag{4}$$

Theorem 1. *$\hat{\rho}$ is a metric of X.*

Proof. By (1), we get

$$\hat{\rho}(x,\ y)\geqq\rho(x,\ y)>0,\ \ if\ x\neq y, \tag{5}$$

and $\int_{f(I)}\rho=0$ if f is given by $f(t)=x,\ \ 0\leqq t\leqq 1$, we also get

$$\hat{\rho}(x,\ x)=0.$$

Since we know

$$\int_{f(I)}\rho=\int_{f^{-1}(I)}\rho,\ \ f^{-1}(t)=f(1-t),$$

we have

$$\hat{\rho}(x,\ y)=\hat{\rho}(y,\ x).$$

To show $\hat{\rho}(x,\ y)+\hat{\rho}(y,\ x)\leqq\hat{\rho}(x,\ z)$, we take for given $\varepsilon>0$, two curves $f_1: I\rightarrow X$ and $f_2: I\rightarrow X$ such that

$$f_1(0)=x,\ \ f_1(1)=y,\ \ \int_{f1(I)}\rho\leqq\hat{\rho}(x,\ y)+\frac{\varepsilon}{2},$$

$$f_2(0)=y,\ \ f_2(1)=z,\ \ \int_{f2(I)}\rho\leqq\hat{\rho}(y,\ z)+\frac{\varepsilon}{2},$$

and define $f_3: I\rightarrow X$ *by*

$$f_3(t)=f_1(2t),\ \ 0\leqq t\leqq\frac{1}{2},\quad f_3(t)=f_2(2t-1),\ \ \frac{1}{2}\leqq t\leqq 1.$$

Then we get

$$\int_{f3(I)}\rho=\int_{f1(I)}\rho+\int_{f2(I)}\rho\leqq\hat{\rho}(x,\ y)+\hat{\rho}(y,\ z)+\varepsilon.$$

Hence we obtain the triangle inequality.

Note. Although ρ is not a metric of X, we can define $\hat{\rho}$ and it becomes a metric of X if ρ is an Alexander-Spanier 1-cochain of X such that

(i). *$\rho(x, y) \geqq 0$ and $\rho(x, y)=0$ if and only if $x=y$.*
(ii). *$\rho(x, y)=\rho(y, x)$*
(iii). *ρ satisfies $(*)'$.*

Lemma 4. *For any curve γ, we have*

$$\int_\gamma \rho = \int_\gamma \hat{\rho}. \tag{6}$$

Proof. By (5), we obtain

$$\int_\gamma \rho \leqq \int_\gamma \hat{\rho}.$$

To get the counter inequality, we may assume $\int_\gamma \rho < \infty$, and set

$$f_i(t)=f(a_i+(a_{i+1}-a_i)t), \quad i=0, \ \cdots\cdots, \ m-1,$$
$$\gamma \text{ is given by } f: I-X, \quad 0=a_0<a_1<\cdots\cdots<a_{m-1}<a_m=1.$$

Then by the definition of the integral, we get

$$\int_\gamma \rho(x, y)=\sum_{i=0}^{m-1}\int_{f_i(I)}\rho(x, y).$$

Then since $\int_{f_i(I)}\rho(x, y) \geqq \hat{\rho}(f(a_{i+1}), f(a_i))$, we have

$$\sum_{i=0}^{m-1}\hat{\rho}(f(a_{i+1}), f(a_i)) \leqq \int_\gamma \rho(x, y), \tag{7}$$

for any partition $0=a_0<a_1<\cdots<a_m=1$.

Since $\sum\hat{\rho}(f(a_{i+1}), f(a_i))$ is monotone increasing for the refinement of the partition of I, (7) shows the existence of $\int_\gamma \hat{\rho}(x, y)$ and the inequality

$$\int_\gamma \hat{\rho}(x, y) \leqq \int_\gamma \rho(x, y).$$

Hence we have the lemma.

Corollary. *$\hat{\hat{\rho}}$ is equal to $\hat{\rho}$.*

We consider the following condition (**).

(**). *For any $x \in X$, there exists a compact set $K=K_x$ and a family of curves with finite length $\{\gamma_\xi, \ \xi \in K\}$, where each γ_ξ is parametlized by its arclength and this parameter representation is denoted by*

$$\gamma_\xi=f_\xi([0, a]), \quad a=\int_{\gamma_\xi}\rho, \quad f_\xi(0)=x,$$

which are parametlized continuously by K and for any $\varepsilon>0$,

$$\bigcup_{\xi\in K}\{f_\xi(t) \mid 0 \leqq t \leqq \varepsilon\} \supset U(x).$$

Here $U(x)=U_\varepsilon(x)$ is a neighborhood of x.

Theorem 2. If X satisfies (**), *then the topology of X determined by $\hat{\rho}$ is equivalent to the topology of X determined by ρ.*

Proof. By (5), we only need to show

(8) $$\hat{\rho}(x, y) \leqq L\rho(x, y),\ y \in U(x),\ \textit{a neighborhood of } x,$$

for each x (Here L may depend on x). But since $\int_{\gamma_\xi}\rho = \int_{\gamma_\xi}\hat{\rho}$, we have

(9)' $$\hat{\rho}(x, f_\xi(t)) = \rho(x, f_\xi t)) + o(t),\ t \in [0, a].$$

Because the parameter t is taken to be the arclength. Hence for each $\xi \in K$, there exists $t = t(\xi)$ such that

(9) $$\hat{\rho}(x, f_\xi(t) \leqq 2(x, f_\xi(t)),\ t \leqq t(\xi).$$

Then since K is compact and f_ξ depends continuously on ξ, setting

$$t_0 = \min_{\xi \in K} t(\xi),$$

$t_0 > 0$. Hence by (**), taking $L = 2$, $U(x) = U_{t_0}(x)$, we get (8).

3. As in [6], we can show the following theorem for $\hat{\rho}$.

Theorem 3. If X is locally compact and $\hat{\rho}(x, y)$ is sufficiently small, then there is a curve $\gamma = \gamma_{x,y}$ such thot

(10) $$\gamma_{x,y} \textit{ is given by } f : \boldsymbol{I} \to X,\ f(0) = x,\ f(1) = y,$$

$$\int_{\gamma_{x,y}} \hat{\rho} = \hat{\rho}(x, y)$$

Proof. By assumption, we may assume $\{z \mid \rho(x, z) \leqq 2a\}$ is comapct for given x, where a is a positive number, and assume $\hat{\rho}(x, y) \leqq a$.

By the definition of $\hat{\rho}(x, y)$, there exists a series of curves γ_n such that *each γ_n starts from x and ends at y.*

$$\int_{\gamma_n} \rho(x, y) = c_n \leqq \hat{\rho}(x, y) + \frac{c}{n},\ c = \hat{\rho}(x, y).$$

We assume that each γ_n is parametlized such as

$$\gamma_n \textit{ is given by } f_n : \boldsymbol{I} \to X,\ \int_{f_{n}, t(I)} \hat{\rho} = c_n t,\ 0 \leqq t \leqq 1.$$

Then since $\int_{\gamma_n} \hat{\rho} = c_n \leqq 2c \leqq 2a$, each γ_n is contained in $\{z \mid \rho(x, z) \leqq 2a\}$ and

$$\hat{\rho}(f_n(t), f_n(s)) \leqq |t - s| c_n \leqq 2c|t - s|,$$

because to define $f_n^{t,s}$ by

$$f_n^{t,s}(u) = f_n(t + (s - t)u),\ u \in \boldsymbol{I},\ (t < s),$$

we get by (1) and the definition of the parameter of f_n,

$$\hat{\rho}(f_n(t),\ f_n(s)) \leqq \int_{f_n{}^{t,s}(I)} \hat{\rho} = (s-t)c_n.$$

Hence the family of the functions $\{f_n\}$, $f_n : \boldsymbol{I} \to X$, $f_n(0)=x$, $f_n(1)=y$, is uniformly bounded and equicontinuous. Therefore by the theorem of Ascoli-Arzéla, $\{f_n\}$ contains a series $\{f_{n_\nu}\}$ which converges uniformly to a continuous map $f : \boldsymbol{I} \to X$. Then we obtain

$$f(0) = x,\ \ f(1) = y,$$
$$\int_{f(I)} \hat{\rho} = \lim_{n_\nu \to \infty} \int_{f n_\nu (I)} \hat{\rho} = \hat{\rho}(x,\ y).$$

Hence we have the theorem by taking $\gamma_{x,y} = f(\boldsymbol{I})$.

Note. Similarly, if X is compact, then we can show the existence of a curve $\gamma = \gamma_{x,y}$ such that γ joins x and y and its length is equal to $\hat{\rho}(x,\ y)$ for any x, $y \in X$ (cf. [6]).

§ 2. Metrics on manifolds.

4. Let $X = \{U,\ h_U\}$ is a (paracompact arcwise connected) topological manifold, where $\{U\}$ is a (locally finite) open covering of X, $\{h_U : U \to \boldsymbol{R}^n\}$ are the homeomorphisms by which the manifold structure of X is given, ρ a metric of X, then setting

$$\rho^U{}_x(\xi, \eta) = \rho(h_{U,x}{}^{-1}(\xi),\ h_{U,x}{}^{-1}(\eta)),$$
$$h_{U,x}(y) = h_U(y) - h_U(x),\ \ x,\ \ y \in U,$$

$\rho^U{}_x$ is a metric of $\boldsymbol{R}^n$ and we have

(11) $$g_{UV}(x)^* \rho^U{}_x = \rho^V{}_x,$$
$$g_{UV}(x) = h_{U,x} h_{V,x}{}^{-1},$$

(11)′ $$k_{U,x,y}{}^* \rho^U{}_x = \rho^U{}_y$$
$$k_{U,x,y} = h_{U,x} h_{U,y}{}^{-1},$$

where $g_{UV}(x)^*\rho^U{}_x$ and $k_{U,x,y}{}^*\rho^U{}_x$ are given by

$$g_{UV}(x)^*\rho^U{}_x(\xi, \eta) = \rho U_x(g_{UV}(x)\xi,\ g_{UV}(x)\eta),$$
$$k_{U,x,y}{}^*\rho^U{}_x(\xi, \eta) = \rho^U{}_x(k_{U,x,y}\ \xi,\ k_{U,x,y}\ \eta).$$

Conversely, if there is a collection $\{\rho^U{}_x\}$, $\rho^U{}_x = \rho^U(x)$, $\rho^U : U \to \{$*the space of (germs of) metrics of* $\boldsymbol{R}^n\}$, such that

(12) $$g_{UV}(x)'^* \rho^U{}_x = \rho^V{}_x,$$

(12)′ $$k_{U,x,y}{}'^* \rho^U{}_x = \rho^U{}_y,$$

where $g_{UV}(x)' = f_U(x)g_{UV}(x)f_V(x)^{-1}$ and $k_{U,x,y}' = f_U(x)h_{U,x}h_{U,y}^{-1}f_U(y)^{-1}$, $f_U(x)$ is a continuous map from U into $H_*(n)$, then setting

$$\rho_{U,x}(p,\ q) = \rho^U{}_x(f_U(x)h_{U,x}(p),\ f_U(x)h_{U,x}(q)),$$

$\rho_{U,x}$ is a metric on U and we obtain

$$\rho_{U,x}(p,\ q) = \rho_{V,x}(p,\ q),$$
$$\rho_{U,x}(p,\ q) = \rho_{U,y}(p,\ q),$$

by (12) and (12)$'$. Hence we get a (local) metric of X.

***Lemma* 5.** *If* $\{\rho^U{}_x\}$ *satisfies* (12) *and*

$$(12)'' \qquad \varphi_{U,x,y}{}^*\rho^U{}_x = \rho^U{}_y,$$

where $\varphi_{U,x,y}$ *is a local homeomorphism of* $\boldsymbol{R}^n$, *then we can take* $\{\varphi_U\}$ *to satisfy*

$$(13) \qquad g_{VU}(x)'\varphi_{U,x,y}g_{UV}(y)' = \varphi_{V,x,y}.$$

we note that (13) shows that if φ_U fix the origin, then $\{\varphi_U\}$ is a connection of $\{g_{UV}'\}$ (cf. [3]).

We also know that if $\rho^U{}_y$, $y \in U$ is given by (12)$''$ for each U *and* $\varphi_{U,x,y}$ satisfies (13), then $\rho^U{}_y$ and $\rho^V{}_y$ satisfies (12) if $\rho^U{}_x$ and $\rho^V{}_x$ satisfy (12).

Note. By definition, we can also take $\{\varphi_{U,x,y}\}$ to satisfy

$$(14) \qquad \varphi_{U,x,y}\varphi_{U,y,z} = \varphi_{U,x,z}, \quad x, y, z \in U.$$

5. ***Lemma* 6.** *If* X *is paracompact and a collection* $\{\rho^U{}_x\}$, $\rho^U{}_x = \rho^U(x)$, $\rho^U : U \to$ $\{$*the space of* (*germs of*) *metrics of* $\boldsymbol{R}^n\}$, *satisfies* (12) *and* (12)$''$, *then there is a collection* $\rho^U{}_x$, $\rho^U{}_x = \rho^U(x)$, $\rho^U : U \to \{$*the space of* (*germs of*) *metrics of* $\boldsymbol{R}^n\}$, *which satisfies* (12) *and* (12)$'$.

Proof. Since X satisfies second axiom of countability, we may assume the manifold structure of X is given by countable open covering $\{U_\alpha\}$ of X. We denote $\rho^\alpha{}_x$, $k_{\alpha,x,y}'$, etc. instead of $\rho^{U_\alpha}{}_x$, $k_{U_\alpha,x,y}'$, etc..

We assume X to be connected. We take $x_1 \in X$ and set

$$V_1 = \bigcup_{X_1 \in U_\alpha} U_\alpha.$$

If $V_1 \neq X$, then we take $x_2 \in V_1$to satisfy there exists some U_β such that $x_1 \notin U_\beta$, $x_2 \in U_\beta$ and set

$$V_2 = \bigcup_{x_2 \in U_\beta,\ x_1 \notin U_\beta} U_\beta.$$

Repeating this, if $x_1, \cdots\cdots, x_n$ have been taken, $V_1, \cdots\cdots,\ V_n$ have been defined and if $\bigcup_{i=1}^{n} V_i \neq X$, then we take $x_{n+1} \in V_n$ to satisfy there exists some U_r such that

$$x_1, \cdots\cdots, x_n \notin U_r, \quad x_{n+1} \in U_r,$$

and set

$$V_{n+1} = \bigcup_{x_{n+1} \in U_r, x_1, \cdots\cdots, x_n \notin U_r} U_r.$$

Then by assumption, $\{V_m\}$ is an open covering of X.

On V_1, we set

$$\tilde{\rho}^{\alpha}{}_{y} = k_{\alpha, x_1, y}{}'^{*} \rho^{\alpha}{}_{x_1}, \quad y \in U_{\alpha}.$$

Then by $(12)''$, we get

$$\tilde{\rho}^{\alpha}{}_{y} = k_{\alpha, x_1, y}{}^{-1*} k_{\alpha, x_1, y}{}'^{*} \rho^{\alpha}{}_{x_1}.$$

Hence by (13), if $U\alpha_1$ and $U\alpha_2$ both contains x_1, we have

$$g_{\alpha_1, \alpha_2}(y)'^{*} \rho^{\alpha_1}{}_{y} = \tilde{\rho}^{\alpha_2}{}_{y}, \quad y \in U_{\alpha_1} \cup U_{\alpha_2}. \tag{15}$$

At X_2, we set

$$\tilde{\rho}^{\beta}{}_{x_2} = g_{\alpha, \beta}(x_2)'^{*} \tilde{\rho}^{\alpha}{}_{x_2},$$
$$x_1 \notin U_{\beta}, \quad x_1 \in U_{\alpha}, \quad x_2 \in U_{\alpha} \cap U_{\beta}.$$

Then by (15), $\tilde{\rho}^{\beta}{}_{x_2}$ is determined by x_2 and does not depend on the choice of α. Using this $\tilde{\rho}^{\beta}{}_{x_2}$, we set

$$\tilde{\rho}^{\beta} = k_{\beta, x_2, y}{}'^{*} \tilde{\rho}^{\beta}{}_{x_2}, \quad y \in U_{\beta}.$$

Then by (13), we get

$$\tilde{\rho}^{\beta_2}{}_{y} = g_{\beta_1, \beta_2}(y)'^{*} \tilde{\rho}^{\beta_1}{}_{y}, \quad y \in U_{\beta_1} \cap U_{\beta_2},$$

if U_{β_1} and U_{β_2} contains either x_1 or x_2.

Repeating this, if $\tilde{\rho}^{\alpha}$ has been defined for those U_{α} that contains either of $x_1, \cdots\cdots, x_n$ and U_r does not contain neither of $x_1, \cdots\cdots, x_n$ and $x_{n+1} \in U_r$, then we set

$$\tilde{\rho}^{r}{}_{x_{n+1}} = g_{\alpha, r}(x_{n+1})'^{*} \tilde{\rho}^{\alpha}{}_{x_{n+1}},$$
$$U_{\alpha} \textit{ contains some of } x_1, \cdots\cdots, x_n, \quad x_{n+1} \in U_{\alpha} \cap U_r.$$

Since $\{\tilde{\rho}^{\alpha}{}_{x}\}$ satisfies (12), this definition does not depend on the choice of α, and setting

$$\tilde{\rho}^{\alpha}{}_{y} = k_{r, x_{n+1}, y}{}'^{*} \tilde{\rho}^{r}{}_{x_{n+1}}, \quad y \in U_r,$$

we get by (13),

$$\tilde{\rho}^{\alpha_2}{}_y = g_{\alpha_1, r_2}(y)'^* \rho^{r_1}{}_y, \quad y \in U_{r_1} \cap U_{r_2},$$

if U_{r_1} and U_{r_2} contains either of $x_1, \cdots\cdots, x_{n+1}$. Hence we can define $\tilde{\rho}^\alpha{}_y$ for all U_α because $\{V_n\}$ is a covering of X, and $\{\tilde{\rho}^\alpha{}_x\}$ satisfies (12) and (12)$'$ by their definition and (14).

we also obtain by this proof,

Lemma 6$'$. *If $\{x_1, x_2, \cdots\cdots\}$ is a countable set of points of X such that there exists a coordinate neighborhood system $\{U_\alpha\}$ of X which satisfies*

(i). *Each U_α contains some of $\{x_1, x_2, \cdots\cdots\}$.*

(ii). *For each α, $\{\rho^\alpha{}_{x_i}\}$ is defined if x_i belongs in U_α* and $\{\rho^\alpha{}_{x_i}\}$ satisfies (12) *and* (12)$''$ *for x_i, $x_i \in U_\alpha$.*

Then there is a collection $\{\tilde{\rho}^\alpha{}_x\}$, $\rho^\alpha{}_x = \rho^\alpha(x)$, $\rho^\alpha : U_\alpha \to \{$the space of (germs of) metrics of $\boldsymbol{R}^n\}$, which satisfies (12) *and* (12)$'$.

6. If ρ is a (local) metric of $\boldsymbol{R}^n$, then to define $\varphi^*\rho$ by

$$\varphi^*\rho(\xi, \eta) = \rho(\varphi(\xi), \varphi(\eta)),$$

$\varphi^*\rho$ is a (germ of) metric of $\boldsymbol{R}^n$ if φ is a (germ of) homeomorphism of $\boldsymbol{R}^n$. Hence we can construct a (local) metric of a paracompact manifold by lemma 6$'$, because $\{x_1, x_2, \cdots\cdots\}$ is a discrete set.

In general, we get

Lemma 7$'$. *If the structure group of the tangent microbundle of X is reduced to G, a subgroup of $H_*(n)$, as an $H_*(n)$ -bundle, then X has a (local) metric ρ such that*

$$\rho^U{}_x(\xi, \eta) = ||\varphi(\xi) - \varphi(\eta)||, \quad \varphi \in G,$$

where $\rho^U{}_x(\xi, \eta)$ is given by

$$\rho^U{}_x(\xi, \eta) = \rho(h_{U,x}{}^{-1} f_U(x)^{-1}(\xi), \ h_{U,x}{}^{-1} f_U(x)^{-1}(\eta)),$$

and $||\xi - \eta||$ is the euclidean norm of the vector $\xi - \eta$.

On the other hand, we know that

$$\int_{f(I)} \varphi^*\rho(x, y) = \int_{\varphi(f(I))} \rho(x, y), \tag{16}$$

Hence we have

Lemma 8. *If $\rho(\xi, \eta)$ is given by*

$$\rho(\xi, \eta) = ||\varphi(\xi) - \varphi(\eta)||,$$

φ is a (local) homeomorphism of $\boldsymbol{R}^n$ represented by the functions with bounded variation,

then $\int_\gamma \rho(\xi, \eta)$ *is finite if and only if* γ *has finite length by the euclidean metric.*

Since we can extend a local metric of X to a metric of X if it satisfies (*)' (cf. note in n°2), we obtain by lemma 8 and lemma 7',

Lemma 7. *A paracompact manifold* X *has a metric which satisfies* (*)' *if the structure group of the tangent microbundle of* X *is reduced to the subgroup* G *of* $H_*(n)$ *which is consisted those (germs of) homeomorphisms which are represented by the functions of bounded variations.*

we note that in this case, the metric consisted by lemma 6' and X satisfy (**) because $\boldsymbol{R}^n$ and its euclidean metric satisfies (**). (For example, we can take $K_\xi = \{\xi \mid |\xi-\eta| = 1\}$ and $\gamma_\xi = \{\xi + t(\eta-\xi) \mid 0 \leqq t \leqq 1,\ \eta \in K\xi\}$ for any $\xi \in \boldsymbol{R}^n$).

Note. Since the local metric ρ of X constructed by lemma 7' for the group G consisted by the homeomorphisms which are represented by the functions of bounded variations satisfies

(i). *If* $\rho(x, y)$ *is sufficiently small, then there exists a curve* γ *which joins* x *and* y *and*

$$\int_\gamma \rho = \rho(x, y),$$

(cf. n°9). Hence denoting $\hat{\rho}$ the metric constructed from ρ by the method of n°2, the Alexander-Spanier *1*-cochain with representation $\hat{\rho}$ is equal to the Alexander-Spanier *1*-cochain with representation ρ.

§ 3. Deformation of $\mathrm{H}_*(\mathrm{n})$.

7. If φ is a (local) homeomorphism of $\boldsymbol{R}^n$ which fix the origin, then we can set

$$\varphi(x_1, \cdots\cdots, x_n) = (f_1(x_1, \cdots\cdots, x_n), \cdots\cdots, f_n(x_1, \cdots\cdots, x_n)),$$
$$f_1(0, \cdots\cdots, 0) = \cdots\cdots = f_n(0, \cdots\cdots, 0) = 0,$$

where each $f_1(x_1, \cdots\cdots, x_n)$ is a continuous function of $x_1, \cdots\cdots, x_n$ and defined on some neighborhood of the origin.

We assume each f_i is defined on $\{(x_1, \cdots\cdots, x_n) \mid |x_i| \leqq a\}$, where a is a positive constant. Then by Kolmogorov's theorem ([9], cf. [1], [15], [16]), we can set

$$f_i(x_1, \cdots\cdots, x_n) = \sum_{j=1}^{2n+1} f_{ij}\Big(\sum_{k=1}^{n} \chi_{jk}(x_k)\Big), \quad i = 1, \cdots\cdots, n, \tag{17}$$

where each χ_{jk} is a $\{\log(2)/\log(2_n+2)\}$-Hölder continuous monotonic function and does not depend on f_i ([15]). We may assume that each χ_{jk} satisfies $\chi_{jk}(0) = 0$.

For each f_{ij}, we take a continuous function g_{ij} with compact carrier (therefore defined on $\boldsymbol{R}^1$) such that its germ at the origin is equal to that of f_{ij}. Then we set

$$g_{ij,t}(x) = g_{ij}*e_t(x) = \int_{-\infty}^{\infty} g_{ij}(y)e_t(x-y)dy, \quad 0 < t \leqq 1, \tag{18}$$

Where e_t is given by

$$e_t(x) = e(\frac{x}{t}) / \int_{-\infty}^{\infty} e(\frac{x}{t})dx,$$

$$e(x) = \exp\left(\frac{1}{x^2-1}\right), \quad -1 < x < 1, \quad e(x) = 0, \quad x \leqq -1, \ or \ x \geqq 1.$$

Then if t is sufficiently small, the germ of $g_{ij,t}$ at the origin is determined by f_{ij} and does not depend on the choice of g_{ij}. For example, if $f_{ij}(x)$ is defined on $|x| < b$, $b > 2$, and $f_{ij}(x) = g_{ij}(x)$ if $|x| \leqq 2$, then $g_{ij,t}(x)$ is determined by $f_{ij}(x)$ for all $t \leqq 1$ if $|x| \leqq 1$.

It is known that $g_{ij,t}(x)$ is smooth for all $t > 0$ and

$$\lim_{t \to 0} g_{ij,t}(x) = g_{ij}(x),$$

where the convergence is uniform.

Since each $\chi_{jk}(x)$ is a $\{\log(2)/\log(2n+2)\}$-Hölder continuous monotonic function, we obtain

Lemma 9. *To set*

$$f_{i,t}(x_1, \cdots\cdots, x_n) = \sum_{j=1}^{2n+1} g_{ij,t}(\sum_{k=1}^{n} \chi_{jk}(x_k)), \quad i = 1, \cdots\cdots, n, \tag{19}$$

each $f_{i,t}(x_1, \cdots\cdots, x_n)$ is a fuuction of bounded variations and $\{\log(2)/\log(2n+2)\}$-Hölder continuous. Moreover, the germ of $f_{i,t}$ at the origin is determined by f_i and does not depend on the choice of g_{ij} if t is sufficiently small and have

$$\lim_{t \to 0} f_{i,t}(x_1, \cdots\cdots, x_n) = f_i, (x_1, \cdots\cdots, x_n),$$

if $|x_i|$, $i = 1, \cdots\cdots, n$ are sufficiently small.

8. We set

$$\varphi_t(x_1, \cdots\cdots, x_n) = (f_{1,t}(x_1, \cdots\cdots, x_n) - f_{1,t}(0, \cdots\cdots, 0), \cdots\cdots, f_{n,t}(x_1, \cdots\cdots, x_n) - f_{n,t}(0, \cdots\cdots, 0)). \tag{20)'$$

Then $\varphi_t(x)$ is a continuous map from a neighborhood of the origin of $\boldsymbol{R}^n$ to a neighborhood of the origin of $\boldsymbol{R}^n$. Hence by the theorem of Radon-Nykodim (cf.

[7]), there is a measurable function $\sigma(\varphi_t)$ such that

$$\int_{\varphi_t(E)} dx = \int_E \sigma(\varphi_t)dx.$$

Moreover, to define $\sigma(\varphi)$ for φ similarly, we get

$$(21)' \qquad \lim_{t\to 0} \sigma(\varphi_t) = \sigma(\varphi).$$

Note. Since φ is a homeomorphism, the Lebesgue measure of the set $\{x\,|\,\sigma(\varphi)(x)=0\}$ is equal to 0. we also know that $\sigma(\varphi)\geqq 0$ if φ is orientation preserving and $\sigma(\varphi)\leqq 0$ if φ is orientation reversing.

On the other hand, to set

$$f_{i,t,s}(x_1, \cdots\cdots, x_n) = \sum_{j=1}^{2n+1} g_{ij,t}(\sum_{k=1}^{n}(e_s*\chi_{jk})(x_k)), \quad 0 < s \leqq 1,$$

$f_{i,t,s}$ is a smooth function for $s>0$ and we have

$$\lim_{s\to 0} f_{i,t,s} = f_{i,t}, \quad i=1, \cdots\cdots, n, \quad t>0.$$

Using these $f_{i,t,s}$' we define $\varphi_{t,s}$ and $\sigma(\varphi_{t,s})$ similarly. Then we have

$$(22) \quad \lim_{s\to 0} \sigma(\varphi_{t,s}) = \sigma(\varphi_t), \quad t>0.$$

Since $\varphi_{t,s}$ is a differentiable map if $t>0$, $s>0$, $\sigma(\varphi_{t,s})$ is given by

$$\sigma(\varphi_{t,s}) = \frac{\partial(f_{1,t,s}, \cdots\cdots, f_{n,t,s})}{\partial(x_1, \cdots\cdots, x_n)}$$

$$= \begin{vmatrix} \sum_{j=1}^{2n+1} g_{1j,t}'(\sum_{l=1}^{n}\chi_{jl,s}(x_l))\chi_{j1,s}'(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{1j,t}'(\sum_{l=1}^{n}\chi_{jl,s}(x_l))\chi_{jn,s}'(x_n) \\ & ,\cdots\cdots\cdots\cdots, & \\ & ,\cdots\cdots\cdots\cdots, & \\ \sum_{j=1}^{2n+1} g_{nj,t}'(\sum_{l=1}^{n}\chi_{jl,s}(x_l))\chi_{j1,s}'(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{nj,t}'(\sum_{l=1}^{n}\chi_{jl,s}(x_l))\chi_{jn,s}'(x_n) \end{vmatrix},$$

where $\chi_{jk,s}(x)$ means $(e_s*\chi_{jk})(x)$ and $g_{ij,t}'$ and $\chi_{jk,s}'$ are mean the derivatives of $g_{ij,t}$ and $\chi_{jk,s}$.

Then, since each χ_{jk} is a monotonic function, we can define $\sigma(\chi_{jk})$ for each χ_{jk} and we have

$$\lim_{s\to 0} \chi_{jk,s}' = \sigma(\chi_{jk}),$$

for each χ_{jk}. Hence we obtain by (22)

$$\sigma(\varphi_t) \tag{23}$$
$$= \begin{vmatrix} \sum_{j=1}^{2n+1} g_{1j,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{j1})(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{1j,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{jn})(x_n) \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ \sum_{j=1}^{2n+1} g_{nj,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{j1})(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{nj,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{jn})(x_n) \end{vmatrix}.$$

For the right hand side of (23), we note

$$\begin{vmatrix} \sum_{j=1}^{2n+1} g_{1j,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{j1})(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{1j,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{jn})(x_n) \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ \sum_{j=1}^{2n+1} g_{nj,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{jl})(x_1), & \cdots\cdots, & \sum_{j=1}^{2n+1} g_{nj,t}'\left(\sum_{l=1}^{n} \chi_{jl}(x_l)\right)\sigma(\chi_{jn})(x_n) \end{vmatrix} \tag{24}$$
$$= \left| \begin{pmatrix} g_{11,t}'\left(\sum_{l=1}^{n} \chi_{1l}(x_l)\right), & \cdots\cdots\cdots\cdots, & g_{1,2n+1,t}'\left(\sum_{l=1}^{n} \chi_{2n+1,l}(x_l)\right) \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots\cdots\cdots, \\ g_{n1,t}'\left(\sum_{l=1}^{n} \chi_{1l}(x_l)\right), & \cdots\cdots\cdots\cdots, & g_{n,2n+1,t}'\left(\sum_{l=1}^{n} \chi_{2n+1,l}(x_l)\right) \end{pmatrix} \times \times \begin{pmatrix} \sigma(\chi_{11})(x_1), & \cdots\cdots\cdots\cdots, & \sigma(\gamma_{1n})(x_n) \\ ,\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots, \\ ,\cdots\cdots\cdots\cdots, \\ \sigma(\chi_{2n+1,1})(x_1), & \cdots\cdots\cdots, & \sigma(\chi_{2n+1,n})(x_n) \end{pmatrix} \right|,$$

where in the right hand side, the first factor is an $(n,\ 2n+1)$-matrix and the second factor is a $(2n+1,\ n)$-matrix. Moreover, the rank of the second factor is n almost everywhere because χ_{ij}, $i=1, \cdots\cdots, 2n+1$, $j=1, \cdots\cdots, n$, are taken independently to f.

We denote by $C_*(\boldsymbol{R}^1)$ and $C^\infty{}_*(\boldsymbol{R}^1)$ the $\boldsymbol{R}$-vector spaces consisted by the germs of continuous (resp. C^∞-class) real valued functions of $\boldsymbol{R}^1$ at the origin. Then we know

$$dim_R\ C_*(\boldsymbol{R}^1) / C_*^\infty(\boldsymbol{R}^1) = \infty.$$

Hence we can take (using axiom of choice) $n(2n+1)$-continuous functions $h_{ij}(x)$ on $|x| \leqq a$, $i = 1, \cdots\cdots, n$, $j = 1, \cdots\cdots, 2n+1$ such that

(i). $\sum_{i,j} c_{ij} h_{ij}(x)$ *is smooth if and only if each* c_{ij} *is equal to* 0.

(ii). *The* $(2n+1)$*-vectors*

$$\begin{pmatrix} (h_{11}(\sum_{l=1}^{n} \chi_{1l}(x_l)) \\ \vdots \\ h_{n1}(\sum_{l=1}^{n} \chi_{1l}(x_l)) \end{pmatrix}, \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \begin{pmatrix} h_{1,2n+1}(\sum_{l=1}^{n} \chi_{2n+1,l}(x_l)) \\ \vdots \\ h_{n,2n+1}(\sum_{l=1}^{n} \chi_{2n+1,l}(x_l)) \end{pmatrix},$$

are linear independent over $\boldsymbol{R}$.

For these h_{ij}, we set

$$H_{ij}(x) = \int_0^x h_{ij}(t)dt,$$

and define continuous functions on some neighborhood of the origin of $\boldsymbol{R}^n$, $h_1(x_1, \cdots\cdots, x_n)$, $\cdots\cdots$, $h_n(x_1, \cdots\cdots, x_n)$ by

$$h_i(x_1, \cdots\cdots, x_n) = \sum_{j=1}^{2n+1} H_{ij}(\sum_{k=1}^{n} \chi_{jk}(x_k)), \quad i = 1, \cdots\cdots, n.$$

By definition, each h_i is a $\{\log(2) / \log(2n+2)$-Hölder continuous bounded variation function and does not depend on φ.

Using these $\mathrm{h}_1, \cdots\cdots, \mathrm{h}_n$, we set

(20) $$\varphi_t'(x_1, \cdots\cdots, x_n) = (f_{1,t}(x_1, \cdots\cdots, x_n) - f_{1,t}(0, \cdots\cdots, 0) + th_1(x_1, \cdots\cdots, x_n), \cdots\cdots, f_{n,t}(x_1, \cdots\cdots, x_n) - f_{n,t}(0, \cdots\cdots, 0) + th_n(x_1, \cdots\cdots, x_n))$$

Then we have

(25) $$\lim_{t \to 0} \varphi_t'(x) = \varphi(x),$$

and since we get

$\sigma(\varphi_t')$

$$= \begin{vmatrix} \sum_{j=1}^{2n+1} (g_{1j,t}' (\sum_{l=1}^{n} \chi_{j,l}(x_l)) + th_{1j} (\sum_{l=1}^{n} \chi_{j,l}(x_l)))\sigma(\chi_{j1})(x_1), \cdots\cdots \\ \cdots\cdots, \sum_{j=1}^{2n+1} (g_{1j,t}' (\sum_{l=1}^{n} \chi_{j,l}(x_l)) + th_{lj} (\sum_{l=1}^{n} \chi_{j,l}(x_l)))\sigma(\chi_{j,n})(x_n) \\ , \cdots\cdots, \\ , \cdots\cdots, \\ , \cdots\cdots, \\ \sum_{j=1}^{2n+1} (g_{nj,t}' (\sum_{l=1}^{n} \chi_{j,l}(x_l)) + th_{nj} (\sum_{l=1}^{n} \chi_{j,l}(x_l)))\sigma(\chi_{j1})(x_1), \cdots\cdots \\ \cdots\cdots, \sum_{j=1}^{2n+1} (g_{nj,t}' (\sum_{l=1}^{n} \chi_{j,l}(x_l)) + th_{nj} (\sum_{l=1}^{n} \chi_{j,l}(x_l)))\sigma(\chi_{j,n})(x_n) \end{vmatrix}$$

$$= \left| \begin{pmatrix} (g_{11,t}' (\sum_{l=1}^{n}\chi_{1l}(x_l)) + th_{11}(\sum_{l=1}^{n}\chi_{1l}(x_l))), \cdots, (g_{1,2n+1,t}'(\sum_{l=1}^{n}\chi_{2n+1,l}(x_l)) + th_{1,2n+1}(\sum_{l=1}^{n}\chi_{2n+1,l}(x_l))) \\ , \cdots\cdots, \\ , \cdots\cdots, \\ (g_{n1,t}' (\sum_{l=1}^{n}\chi_{1l}(x_l)) + th_{n1}(\sum_{l=1}^{n}\chi_{1l}(x_l))), \cdots, (g_{n,2n+1,t}'(\sum_{l=1}^{n}\chi_{2+1,l}(x_l)) + th_{n,2n+1}(\sum_{l=1}^{n}\chi_{2n+1,l}(x_l))) \end{pmatrix} \times \begin{pmatrix} \sigma(\chi_{11})(x_1), \cdots\cdots, \sigma(\chi_{1n})(x_n) \\ , \cdots\cdots, \\ , \cdots\cdots, \\ , \cdots\cdots, \\ , \cdots\cdots, \\ , \cdots\cdots, \\ , \cdots\cdots, \\ \sigma(\chi_{2n+1,1})(x_1), \cdots\cdots, \sigma(\chi_{n,2n+1})(x_n) \end{pmatrix} \right|,$$

we have

(26) $\sigma(\varphi_t') \neq 0$, *except the set of measure* 0 *near the origin,*

by (i) and (ii). Here $\sigma(\varphi_t')$ is defined similarly as $\sigma(\varphi_t)$.

9. we set $\alpha = \log(2) / \log(2_n + 2)$. Then since each $\chi_{jk}(x)$ is monotonic and α–Hölder continuous,

$$\lim_{h \to +0} \chi_{jk}(h^{\frac{1}{\alpha}}) = c_{j,k}^{+},$$

$$\lim_{h \to -0} \chi_{jk}(-(|h|^{\frac{1}{\alpha}}) = c_{jk}^{-},$$

both exist and positive for any j, k. Then to define a homeomorphism θ of $\boldsymbol{R}^n$ by

$$\theta(x_i)=\mathrm{sgn}(x_i)(|x_i|^{\frac{1}{\alpha}}),\ i=1,\ \cdots\cdots,\ n,\ \ \mathrm{sgn}(x)=\frac{x}{|x|},\ \ x\neq 0,$$
$$\mathrm{sgn}(0)=0,$$

we have by (26),

$$(\varphi_t'\theta)(x)=A^t{}_{\varepsilon_1,\cdots,\varepsilon_n}+0(||x||),$$
$$x\in \boldsymbol{R}_{\varepsilon_1,\cdots,\varepsilon_n}=\{(x_1,\ \cdots\cdots,\ x_n)|\,\mathrm{sgn}(x_1)=\varepsilon_1\cdots\cdots\ \mathrm{sgn}(x_n)=\varepsilon_n\},$$
$$A^t\ :\ A^t(x)=A^t{}_{\varepsilon_1,\cdots,\varepsilon_n}(x),\ \ x\in \boldsymbol{R}_{\varepsilon_1,\cdots,\varepsilon_n},\ \ A^t(0)=0 \textit{ is a homeomorphism},$$

where $\varepsilon_1,\ \cdots\cdots, \varepsilon_n$ are ± 1, $A^t{}_1\cdots\cdots{}_n$ are regular constant matrices and $||x||$ means the euclidean norm of x, if $||x||$ is sufficiently small.

Hence (using same method as in the proof of implicit function theorem), φ_t' is a homeomorphism if $||x||\leqq\beta=\beta(t)$. Here we may assume $\beta(t)$ to be an increasing function of t.

On the other hand, since we know

(21) $$\lim_{t\to 0}\ \sigma(\varphi_t')=\sigma(\varphi),$$

φ_t' is a homeomorphism if $||x||\leqq\beta'$ for some β' if $t\leqq t_0$, $t_0>0$. Moreover, since to define $\hat{\varphi}_t'\ :\ U\times \boldsymbol{I}\rightarrow \boldsymbol{R}^n\times \boldsymbol{I}$ by

$$\hat{\varphi}_t'(x,\ t)=(\varphi_t'(x),\ t),$$

$\hat{\varphi}'$ is a continuous map, we obtain

Lemma 10. *φ_t' is a homeomorphism if $||x||\leqq\beta$ and its image contains the ball $\{x|\,||x||\leqq\delta\}$, where β and δ are positive constants determined by φ and does not depend on t.*

By lemma 10 and (25), We have

Theorem 4. *If X is paracompact, then the structure group of the tangent microbundle of X is reduced to the group that are consisted by those germs of homeomorphisms which are represented by the functions of bounded variations as an $H_*(n)$-bundle.*

Note. Lemma 10 shows that $H_*(n)$ is deformed to its subset consisted by those germs of homeomorphisms which are represented by those functions such that $\log(2)/\log(2n+2)$-Hölder continuous and has finite variations. But this set is not a subgroup of $H_*(\mathrm{n})$, because if f is α-Hölder continuous and g is β-Hölder continuous, then the composed function $f(g)$ is only $\alpha\beta$-Hölder continuous and $\{\log 2/\log(2n+2)\}<1$.

By theorem 4 and lemma 7, we obtain

Theorem 5. *If X is a paracompact manifold, then X has a metric ρ which* *satisfies* (*) *and*(**).

We note that by (16) and (1),

$$\int_{f(I)} ||\varphi(\xi)-\varphi(\eta)|| = \int_{\varphi(f(I))} ||\xi-\eta|| \geqq ||\varphi(f(0))-\varphi(f(1))||,$$

and the equality is hold if and only if φ $(f(\boldsymbol{I}))$ is given by

$$\varphi(f(t)) = t\varphi(f(0)) + (1-t)\varphi(f(1)), \quad 0 \leqq t \leqq 1.$$

Hence we obtain

Theorem 6. *If X is a paracompact (topological) manifold, then there exists a metric ρ of X by which the topology of X is given and if ρ (x, y) is sufficiently small, then there exists a curve γ starts from x, ends at y, and*

$$\int_{\gamma} \rho = \rho(x, y).$$

Moreover, such curve is unique up to the change of parameters.

We also note that if φ is a homeomorphism of $\boldsymbol{R}^n$ which is represented by the functions of bounded variations, then the Alexander-Spanier n-cochain $||\varphi(\xi_0)-\varphi(\xi_1)||\cdots\cdots||\varphi(\xi_0)-\varphi(\xi_1)||$ defines a non-trivial Radon measure $m(\varphi)$ given by

$$m(\varphi)(E) = \int_E \sqrt{\sum_{i_1\cdots\cdots i_n} (\sigma_1 f_{i_1})^2 \cdots\cdots (\sigma_n f_{i_n})^2} dx,$$

where dx means the integration by the Lebesgue measure, $\varphi=(f_1, \cdots\cdots, f_n)$ and $\sigma_i f_j$ is the measure defined from $f_j(x_1, \cdots\cdots, x_n)$ regarding $(x_1, \cdots\cdots, x_{i-1}, x_{i+1}, \cdots\cdots, x_n)$ to be parameters. Hence we also obtain

Theorem 6'. *We may consider the metric ρ given by theorem 6 also sats satisfies the following. To set*

$$v(x_0, \cdots\cdots, x_n) = \rho(x_0, x_1)\rho(x_0, x_2) \cdots\cdots \rho(x_0, x_n),$$

$v(x_0, \cdots\cdots, x_n)$ is a positive Alexander-Spanier n-cochain of X and therefore it defines a measure $m(\rho)$ (cf. [4]). This $m(\rho)$ is a non-trivial positive Radon measure on X and

$$m(\rho)(E) \neq 0,$$

if E is $(m(\rho))$ measurable and contains some non-empty open set of X.

§ 4. Theorems of Hopf-Rinow and Nomizu-Ozeki.

10. ***Definition.*** *On a metric space X with metric ρ, we call a curve γ to be a geodesic with respect to ρ if γ satisfies*

$$\int_{\gamma} k_a \delta\rho = 0,$$

where γ is given by $f: I \to X$ and $a = f(0)$.

Note. For a non closed curve γ given, for example, $f: [0, 1) \to X$, we call γ to be a geodesic with respect to ρ if we have

$$\lim_{t\to 1} \int_{\gamma_t} k_a \delta\rho = 0,$$

where $a = f(0)$ and γ_t is given by $f_t: I \to X$, $f_t(s) = f(ts)$, $0 \leqq t < 1$.

We note that, if γ is a geodesic with respect to ρ, then for any b, c, $0 < b < c < 1$, $\gamma_{b,c}$ is also a geodesic with respect to ρ, because $k_a\delta\rho$ is a positive cochain and therefore we get

$$\int_{\gamma_{b,c}} k_a\delta\rho \leqq \int_{\gamma} k_a\delta\rho.$$

Here $\gamma_{b,c}$ is given by $f_{b,c}: I \to X$, $f_{b,c}(t) = f(b + (c-b)t)$.

As was remarked in § 1, a curve γ satisfies

$$\int_{\gamma} \rho = \rho(f(0), f(1)),$$

if and only if it is a geodesic with respect to ρ. Here γ is given by $f: I \to X$.

On X, we consider the following property (***).

(***). *For any $x \in X$, there exists $\varepsilon = \varepsilon(x) > 0$ such that $\{y \mid \rho(x, y) = \varepsilon\}$ is compact and if z satisfies $\rho(x, z) \leqq \varepsilon$, then there exists aunique geodesic γ with respect to ρ which joins x and z.*

It is known that (***) is important in the proof of the theorem of Hopf-Rinow (cf. [5]), [11]. In fact, the followings are proved only by using (***) and usual properties of metric and curves with minimal length.

(a). *If X is complete by the metric ρ, then a geodesic with respect to is extended to a geodesic with respect to ρ which has infinite length, i. e. if $f: I \to X$ is a geodesic, then there exists $g: [0, 1) \to X$ such that*

$$f = g_a, \ \textit{for some } 0 < a < 1, \ g_a(s) = g(as),$$

$$\lim_{t\to 1} \int_{g_t(I)} k_{g(0)}\delta\rho = 0, \lim_{t\to 1} \int_{g_t(I)} \rho = \infty.$$

(b). *If X is complete by the metric ρ, then for any x, $y \in X$, there exists a geodesic which joins x and y.*

Note. In (a), g may not be a 1 to 1 map.

Since the metric ρ given by theorem 6 satisfies (***), we get

Theorem 7. *If X is complete as a metric space by the metric ρ given by theorem 6, then for any x, $y \in X$, there exists a curve γ which joins x and y and*

(27) $$\int_{\gamma}\rho = \rho(x,\ y).$$

Especially, if X is compact, then for any $x,\ y \in X$, there exists a geodesic γ with respect to ρ which joins x and y and satisfies (27) for any metric ρ given by theorem 6.

11. ***Theorem** 8. If X is complete by the metric ρ given by theorem 6, then a bounded set of X by the metric ρ is relative compact in X.*

Proof. We denote $B(x,\ r) = \{y \mid \rho(x,\ y) \leqq r\}$ and set

(28) $$r(x) = \sup_{r}.\{\ B(x,\ r)\ is\ compact\}.$$

Since X is locally compact, $r(x) > 0$. If $r(x) = \infty$, then the theorem is true. Hence we assume $r(x) \neq \infty$.

We take infinite points $\{y_\alpha\}$, $y_\alpha \in B(x,\ r(x))$. If $B(x,\ r(x)-\varepsilon) \cap \{y_\alpha\}$ is an infinite set for some $\varepsilon > 0$, then $\{y_\alpha\}$ contains a series which converges in X because $B(x,\ r(x)-\varepsilon)$ is compact. Hence we may assume for any $\varepsilon > 0$, to set $\{y_\beta\}$ to be the subset of $\{y_\alpha\}$ such that y_β satisfies

(29) $$\rho(x,\ y_\beta) > r(x) - \varepsilon,$$

$\{y_\beta\}$ is an infinite set.

By theorem 7, we can join each y_β which satisfies (29) and x by a geodesic γ_β with respect to ρ which satisfies (27). Then by (29), we can take unique point $y_{\beta,\varepsilon}$ on γ_β such that

$$\rho(x,\ y_{\beta,\varepsilon}) = r(x) - \varepsilon.$$

Then since $B(x,\ r(x)-\varepsilon)$ is compact, the derived set of $\{y_{\beta,\varepsilon}\}$ is not an empty set.

If for any $\{y_{\beta_i,\varepsilon}\}$ which converges in X, satisfies

$$\overline{\lim_{i\to\infty}}.\ \rho(x,\ y_{\beta_i}) \leqq r(x) - \mu,\ \ \mu > 0,$$

then $\{y_\alpha \mid \rho(x,\ y_\alpha) > r(x) - \mu\}$ should be a finite set, and therefore $\{y_\alpha\}$ contains a sequence which converges in X. Hence we may assume that there exists a sequence $\{y_{i,\varepsilon}\}$ contained in $\{y_{\beta,\varepsilon}\}$ such that

$$\lim_{i\to\infty}.\ y_{i,\varepsilon} = y_\varepsilon,\ \ \lim_{i\to\infty}.\ \rho(x,\ y_i) = r(x).$$

We join y_ε and x by a curve γ_ε whose length is $r(x) - \varepsilon$. Then by (a) of n°10, γ_q can be extended to satisfy (2) with arbitrary length. We take one of such extension $\bar{\gamma}_\varepsilon$ of γ_ε and take the point y on $\bar{\gamma}^\varepsilon$ snch that

$$\rho(x, y) = r(x), \quad \rho(y_\varepsilon, y) = \varepsilon.$$

Then, by the definition of ρ, if $\varepsilon - \varepsilon'$ is sufficiently small, we have

$$\lim_{i\to\infty} y_{i,\varepsilon}' = y_\varepsilon', \tag{30}$$

where $y_{i,\varepsilon}'$ and y_ε' are the unique points on γ_i and $\bar{\gamma}_\varepsilon$ which satisfy $\rho(x, y_{i,\varepsilon'}) = \rho(x, y_{\varepsilon'}) = r(x) - \varepsilon'$. moreover, by the definition of ρ, there exists $\delta > 0$ such that if (30) is hold and $|\varepsilon' - \varepsilon''| < \delta$, then

$$\lim_{i\to\infty} y_{i,\varepsilon''} = y_{\varepsilon''} \tag{30$'$}$$

is hold and δ does not depend on ε'. Hence we have

$$\lim_{i\to\infty} y_i = y.$$

Therefore $B(x, r(x))$ is compact.

Then, since $B(x, r(x))$ is compact, $\{y \mid \rho(x, y) = r(x)\}$ is also compact. Hence we can take $\varepsilon > 0$ such that $B(x, r(x) + \varepsilon)$ to be compact, because X is locally compact. But this contradicts to the definition of $r(x)$. Therefore $r(x)$ is equal to ∞ and we have the theorem.

Note. If a metric ρ of X satisfies this theorem, then X is complete with respect to ρ. Because if $\{x_n\}$ is a Cauchy sequence, then for some m and $r < \infty$, we have $x_n \in B(x_m, r)$ for all n. Then, since $B(x_m, r)$ is compact, a subsequence of $\{x_n\}$ converges in X. But since $\{x_n\}$ is a Cauchy sequence, $\{x_n\}$ itself converges in X.

12. If X is not complete by the metric ρ given by theorem 6, then $r(x)$, defined by (28), is a positive continuous function on X (cf. [10]). Since $r(x) > 0$, we can take a continuous function $\omega(x)$ on X such that $\omega(x) > 1/r(x)$.

We set

$$d(x, y) = \sqrt{\omega(x)\omega(y)}\,\rho(x, y), \tag{31}$$

then $d(x, y)$ satisfies

(i). $d(x, y) \geqq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

(ii). $d(x, y) = d(y, x)$.

(iii). *If X is arcwise connected, then for any $x, y \in X$, there exists a curve γ which joins x and y and*

$$\int_\gamma d < \infty.$$

Hence to set

$$\rho'(x, y) = \inf_{\gamma,\ \gamma \text{ joins } x \text{ and } y} \int_\gamma d,$$

$\rho'(x, y)$ is a metric on X (cf. n°2). Moreover, by the definition of ρ', we obtain (cf. [10]),

***Lemma* 11.** *Setting*

$$B'(x, a) = \{y \mid \rho'(x, y) \leq a\},$$

we have

$$B'(x, \frac{1}{3}) \subset B(x, \frac{r(x)}{2}). \tag{32}$$

We note that, by the definitions of d and ρ, if y is contained in a coordinate neighborhood U of x and γ is contained in U, then

$$\int_\gamma d$$

$$= \lim_{|t_{i+1}-t_i|\to 0} \sum \sqrt{\phi(f(t_i))\phi(f(t_{i+1}))}\, \| f(t_i) - f(t_{i+1}) \|.$$

Here f is a continuous map from $\boldsymbol{I}$ into $\boldsymbol{R}^n$ such that $f(0) = h_U(x)$, $f(1) = h_U(y)$ and $\phi(x)$ is a positive continuous function on $\boldsymbol{R}^n$ (h_U means the homeomorphism from U onto $\boldsymbol{R}^n$ by which the manifold structure of X is given).

***Lemma* 12.** *Under the above assumption, there exists a curve γ which join $h_U(x)$ and $h_U(y)$ and gives the minimal value of $\int_\gamma d$ and such γ is unique up to the change of parameters.*

Proof. If ϕ is smooth, then

$$\int_\gamma d = \int_0^1 \phi(f(t)) \sqrt{\sum \left(\frac{df_i(t)}{dt} \right)^2}\, dt,$$

if γ is a smooth curve given by $f : \boldsymbol{I} \to \boldsymbol{R}^n$, $f(t) = f_1(t), \cdots\cdots, (f_n(t))$ where each f_i is differentiable. Hence we have the lemma by the theory of calculus of variation.

If ϕ is not smooth, then as in n°7, we set $\phi_s = \phi * e_s$, and set

$$\int_\gamma d_s \tag{33}$$

$$= \lim_{|t_{i+1}-t_i| \to 0} \sum_i \sqrt{\phi_s(f(t_i))\phi_s(f(t_{i+1}))}\, \| f(t_i) - f(t_{i+1}) \|.$$

Then, since ϕ_s is smooth, there exists unique curve γ_s which gives the minimal value of $\int_\gamma d_s$, and by the theorem of Ascoli-Arzéla, $\lim._{s\to 0}\gamma_s$ exists. We set the limit by γ_0.

On the other hand, since there exists a curve γ_a which gives the minimal value of $\int_\gamma d$ (cf. the proof of theorem 3), we approximate γ_a by smooth curves $\gamma_{a,u}$ as follows : Assume γ_a is given by $f_a : \boldsymbol{I} \to X$, $f_a(t) = (f_{a,1}(t) \cdots\cdots f_{a,n}(t))$ and set

$$f_{a,u,i}(t)$$
$$= f_{a,i}*e_u(t) + (1-t)(f_{a,i}(0) - f_{a,i}*e_u(0)) + t(f_{a,i}(1) - f_{a,i}*e_u(1)),$$
$$i = 1, \cdots\cdots, n, \ 0 < u \leqq 1,$$

then $\gamma_{a,u}$ is given by $f_{a,u} : I \to X$, $f_{a,u}(t) = (f_{a,u,1}(t), \cdots\cdots, f_{a,u,n}(t))$. By definition, $lim._{u\to 0}\gamma_{a,u} = \gamma_a$.

Then, since

$$\lim_{\substack{s\to 0 \\ u\to 0}} \int_{\gamma_{a,u}} d_s = \int_{\gamma a} d, \tag{34}$$

we have

$$\int_{\gamma 0} d = \int_{\gamma a} d, \tag{35}$$

because $\int_{\gamma 0} d \geqq \int_{\gamma a} d$ by the definition of γ_a and $\int_{\gamma a,u} d_s \geqq \int_{\gamma s} d_s$ and (34) shows $\int_{\gamma a} d \geqq \int_{\gamma 0} d$.

If γ_a is not equal to γ_0, then we have for some t_0 and ε

$$min. \ ||f_{a,u}(t) - f_u(t)|| > \delta, \ |t - t_0| < \varepsilon,$$

for some δ if $u < u_0$, where u_0 is a positive constant. Then, since γ_s converges uniformly to γ, we have

$$\int_{\gamma a,u} d_u > \int_{\gamma u} d_u + \varepsilon, \ u < u_0.$$

But this shows $\lim_{\substack{s\to 0 \\ u\to 0}} \int_{\gamma a,u} d_s > \int_{\gamma 0} d + \varepsilon$. This contradicts to (35). Hence we have the lemma.

By this lemma, ρ' also satisfies (***). Therefore we obtain by (32)

Theorem 9. *If X is a paracompact arcwise connected (topological) manifold, then X has a metric ρ which has following properties.*

(i). *X is complete as a metric space with metric.*

(ii). *For any x, $y \in X$, there exists a curve γ which joins x and y and*

$$\int_\gamma \rho = \rho(x, y).$$

Moreover, such γ is unique up to the change of parameters if $\rho(x, y)$ is sufficiently small.

(iii). *The Alexander-Spanier n-cochain $v(\rho)$ given by*

$$v(\rho)(x_0, x_1, \cdots\cdots, x_n)$$
$$= \rho(x_0, x_1)\rho(x_0, x_2)\cdots\cdots\rho(x_0, x_n), \ n = dim. X,$$

defines a positive Radon measure $m = m(\rho)$ on X and it satisfies

$$m(\rho)(E) \neq 0,$$

if E is $m(\rho)$-measurable and contains some non-empty open set of X (cf. [4]).

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