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On the equivariant K-groups $K^*_{U_F(n)}(cs^n_F,\ s^n_F)$ *and* $K^*_{SU_F(n)}(cs^n_F,\ s^n_F)$

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§ 1.

1. **Introduction.** Let F denote R, C, or H. Let $U_F(n)$ denote $O(n)$ for $F=R$, $U(n)$ for $F=C$, and $S_P(n)$ for $F=H$. Let S^n_F denote S^{n-1} for $F=R$, S^{2n-1} for $F=C$, and S^{4n-1} for $F=H$. We can regard S^n_F as a $U_F(n)$-space (respectively $SU_F(n)$-space) with a natural $U_F(n)$ action (respectively $SU_F(n)$ action) on F^n. We can consider $F^n \subset F^{n+1}$' where a vector in F^n has zero in its $(n+1)$—cordinate when viewed in F^{n+1}. Consequently, there exists natural inclusions $U_F(n-1) \subset U_F(n)$, $SU_F(n-1) \subset SU_F(n)$. Let CS^n_F be a cone over S^n_F. Then CS^n_F is a $U_F(n)$-space (respectively $SU_F(n)$-space) with group action $g(y,\ t)=(gy,\ t)$.

Now, in this paper we shall prove the following

Theorem. *For* $F=C$, *or,* H, $K^*_{U_F(n)}(CS^n_F,\ S^n_F)$ *is an* $R(U_F(n))$*-module with one generator.* $K^*_{SU(n)}(CS^{2n-1},\ S^{2n-1})$ *is an* $R(SU(n))$*-module with one generator.* $K^*_{SO(2r+1)}(CS^{2r},\ S^{2r})$ *is an* $R(SO(2r+1)$*-module with one generator.* $K^*_{SO(2r)}(CS^{2r-1},\ S^{2r-1})$ *is an* $R(SO(2r))$*-module with two generators.*

Let G be a compact Lie group and H a closed subgroup of G, then from 4.1 and 4.11 of [1], we have the following isomorphism between Z_2-graded $R(G)$-modules :

$$K^*{}_{(G,H)}(\text{point}) \cong K^*_G(C(G/H),\ G/H). \tag{1.1}$$

From 3.5 Theorem of [1], the sequence

$$0 \longrightarrow K^*{}_{(G,H)}(\text{point}) \xrightarrow{j^*} K^*_G(\text{point}) \xrightarrow{i^*} K^*_H(\text{point}) \xrightarrow{\partial} K^*{}_{(G,H)}(\text{point}) \longrightarrow 0 \tag{1.2}$$

is exact, and each of the maps is a K^*_G(point)-module map. Therefore, we have the following isomorphisms between $R(G)$-modules : (Paticulaly, if $(G,\ H)=(U_F(n),\ U_F(n-1))$ or $(SU_F(n),\ SU_F(n-1))$, $G/H=S^n_F$.

(1. 3) $$K_G(C(G/H), G/H) \cong \text{Kernel } i^*$$
$$K_G{}^1(C(G/H), G/H) \cong \text{Cokernel } i^* .$$

§ 2

The following results on the representation theory will be required from [2]. Throughout this paper we use the same notaions as in [2].

2. 1 Theorem. *The ring $R(U(n))$ equals the polynomial ring*

$$Z[\lambda_1(n), \cdots\cdots, \lambda_n(n), \lambda_n(n)^{-1}]$$

where as a subring of $R(T(n))=Z[\alpha_1, \alpha_1^{-1}, \cdots\cdots, \alpha_n, \alpha_n^{-1}]$ the relation

$$\lambda_k(n) = \sum_{i(1)< \cdots\cdots <i(k)} \alpha_{i(1)} \cdots\cdots \alpha_{i(k)}$$

holds. The ring $R(SU(n))$ equals the polynomial ring

$$Z[\lambda_1(n), \cdots\cdots, \lambda_{n-1}(n)].$$

2. 2 Theorem. *The ring $R(Sp(n))$ equals the polynomial ring*

$$Z[\lambda_1(n), \cdots\cdots, \lambda_n(n)],$$

where, as a subring of $Z[\alpha_1, \alpha_1^{-1}, \cdots\cdots, \alpha_n, \alpha_n^{-1}]$ the element $\lambda_k(n)$ is the k-th symmetric function in the $2n$ variables $\alpha_1, \alpha_1^{-1}, \cdots\cdots, \alpha_n, \alpha_n^{-1}$.

2. 3 Theorem. In the case $n=2r+1$, *$R(Spin(n))$ equals the polynomial ring*

$$Z[\lambda^1(\rho_n), \cdots\cdots, \lambda^r(\rho_n), \Delta_n],$$

and $R(SO(n))$ equals the polynomial ring

$$Z[\lambda^1(\rho_n), \cdots\cdots, \lambda^r(\rho_n)].$$

The following relation holds :

$$\Delta_n^2 = \lambda^r(\rho_n) + \cdots\cdots + \lambda^1(\rho_n) + 1.$$

In the case $n=2r$, $R(Spin(n))$ equals the polynomial ring

$$Z[\lambda^1(\rho_n), \cdots\cdots, \lambda^r(\rho_n), \Delta_n^+, \Delta_n^-]$$

and $R(SO(n))$ equals the polynomial ring

$$Z[\lambda^1(\rho_n), \cdots\cdots, \lambda^{r-1}(\rho_n), \lambda^r_+(\rho_n), \lambda^r_-(\rho_n)],$$

with one relation

$$(\lambda^r_+ + \lambda^{r-2} + \lambda^{r-4} + \cdots\cdots)(\lambda^r_- + \lambda^{r-2} + \lambda^{r-4} + \cdots\cdots)$$
$$= (\lambda^{r-1} + \lambda^{r-3} + \lambda^{r-5} + \cdots\cdots)^2.$$

In $R(Spin(2r))$ the following relation holds :

$$(\Delta^+_{2r})^2 = \lambda^r_+(\rho_{2r}) + \lambda^{r-2}(\rho_{2r}) + \lambda^{r-4}(\rho_{2r}) + \cdots\cdots,$$
$$(\Delta^+_{2r})(\Delta^-_{2r}) = \lambda^{r-1}(\rho_{2r}) + \lambda^{r-3}(\rho_{2r}) + \cdots\cdots,$$
$$(\Delta^-_{2r})^2 = \lambda^r_-(\rho_{2r}) + \lambda^{r-2}(\rho_{2r}) + \lambda^{r-4}(\rho_{2r}) + \cdots\cdots.$$

From the selection of α_i in [2], throughout this paper we shall write $R(T(n)) = Z[\alpha_1, \alpha_1^{-1}, \cdots\cdots \alpha_n, \alpha_n^{-1}]$ and $R(T(n-1)) = Z[\alpha_1, \alpha_1^{-1}, \cdots\cdots, \alpha_{n-1}, \alpha_{n-1}^{-1}]$ with the same notations α_i.

§ 3

3. 1 Lemma. Let i^*_n : $R(U(n)) \longrightarrow R(U(n-1))$ *be a natural homomorphism (by the inclusion $U(n-1) \subset U(n)$), then we have*

$$i^*_n(\lambda_l(n)) = \begin{cases} \lambda_l(n-1) + \lambda_{l-1}(n-1) & (\text{for } l < n-1) \\ \lambda_{n-1}(n-1) & (\text{for } l = n-1). \end{cases} \tag{3. 1. 1}$$

Proof. The diagram

$$\begin{array}{ccc} R(U(n)) & \longrightarrow & R(T(n)) \\ \downarrow i^*_n & & \downarrow i^*_n \\ R(U(n-1)) & \longrightarrow & R(T(n-1)) \end{array}$$

is commutative, and we have $i^*_n(\alpha_i) = \alpha_i$ for $i \leqq n-1$ and $i^*_n(\alpha_n) = 1$. since $\lambda_l(n)$ is a l-th symmetric function in the n variables $\alpha_1, \cdots\cdots, \alpha_n$, we have easily (3. 1. 1).

3. 2 Lemma. In $R(U(n))$, *we set*

$$\overline{\lambda_l}(n) = \sum_{i=0}^{l}(-1)^i \lambda_{l-i}(n),$$

then $R(U(n))$ equals the polynomial ring

$$Z[\overline{\lambda_1}(n), \cdots\cdots, \overline{\lambda}_n(n), (\overline{\lambda}_n(n) + \overline{\lambda}_{n-1}(n))^{-1}].$$

Proof. Since $\overline{\lambda_1}(n), \cdots\cdots, \overline{\lambda_n}(n)$ are algebraically independent, and $\overline{\lambda}_n(n) + \overline{\lambda}_{n-1}(n) = \lambda_n$, it is trivial.

3. 3 Theorem. *We have the following isomorphism between Z_2-graded $R(U(n))$-*

modules and $R(SU(n))$-modules :

$$K^*_{U(n)}(CS^{2n-1},\ S^{2n-1}) \cong \{\overline{\lambda}_n(n)\}$$
$$K^*_{SU(n)}(CS^{2n-1},\ S^{2n-1}) \cong \{(\overline{\lambda}_{n-1}(n)' - 1)\}$$

where $\{\overline{\lambda}_n(n)\}$ is an $R(U(n))$-module generated by $\overline{\lambda}_n(n)$ and $\{(\overline{\lambda}_{n-1}(n)-1)\}$ is an $R(SU(n))$-module generated by $\overline{(\lambda}_{n-1}(n) - 1)$.

Proof. From the definitions of $\overline{\lambda}_l(n)$, we have

$$(3.3.1)\qquad i^*_n(\overline{\lambda}_l(n)) = \begin{cases} \lambda_{l-1}(n-1) & (\text{for } l \leqq n-1) \\ 0 & (\text{for } l = n). \end{cases}$$

Therefore, from the last statement in **§1**, it is sufficient to determine Kernel i^*_n.

From (3.3.1), Kernel i^*_n is an ideal generated by $\overline{\lambda}_n(n)$ in $R(U(n))$, so kernel i^*_n is an $R(U(n))$-module generated by $\overline{\lambda}_n(n)$. For $R(SU(n))$, observe that the ring $R(ST(n))$ is an ideal genelated by $(\lambda_n - 1)$. Therefore Kernel i^*_n ($i^*_n : R(SU(n)) \longrightarrow R(SU(n-1))$) is an $R(SU(n))$-module genelated by $\overline{(\lambda}_{n-1}(n)-1)$.

Remark. For Λ^e, Λ^o we take the representations of $U(n)$ on the even and odd parts of the exterior algebra $\Lambda^*(C^n)$, and we identify these two parts by exterior multiplication with the n-th basic vector e_n of C^n. Since $U(n\text{-}1)$ keeps e_n fixed, this identification is compatible with the action of $U(n-1)$. So we obtain an element $[\gamma]$ of $K_{(U(n),\,U(n-1))}$ (point) such that $j^*([\gamma]) = \overline{\lambda}_n(n)$. Therefore $K^*_{(U(n),\,U(n-1))}$ (point) equals the $R(U(n))$-module generated by $[\gamma]$.

§4

4.1 Let $\lambda_l(n)$ be a l-th symmetric function in the $2n$ variables $\alpha_1,\ \alpha_1^{-1},\ \cdots\cdots,\ \alpha_n,\ \alpha_n^{-1}$, and $\lambda'_l(n)$ be a l-th symmetric function in the $(2n+1)$ variables $\alpha_1,\ \alpha_1^{-1},\ \cdots\cdots,\ \alpha_n,\ \alpha_n^{-1}$, 1 and $i^*(\lambda_l(n))$ be a l-th symmetric function in the $2n$ variables $\alpha_1,\ \alpha_1^{-1},\ \cdots\cdots,\ \alpha_{n-1},\ \alpha_{n-1}^{-1}$, 1, 1. Then

$$(4.1.1)\qquad \prod_{i=0}^{n}(x-\alpha_i)(x-\alpha_i^{-1}) = \sum_{i=0}^{n}(-1)^i\lambda_i(n)x^{2n-i} + \sum_{i=0}^{n-1}(-1)^i\lambda_i(n)x^i$$

implies

$$(4.1.2)\qquad \lambda_l(n) = \lambda_l(n-1) + (\alpha_n + \alpha_n^{-1})\lambda_{l-1}(n-1) + \lambda_{l-2}(n-1)$$

$$(4.1.3)\qquad \lambda'_l = \lambda_l(n) + \lambda_{l-1}(n)$$

$$(4.1.4)\qquad \begin{aligned} i^*(\lambda_l(n)) &= \lambda_l(n-1) + 2\lambda_{l-1}(n-1) + \lambda_{l-2}(n-1) \\ &= \lambda'_l(n-1) + \lambda'_{l-1}(n-1), \end{aligned}$$

where $\lambda_0(n)=1$ and $\lambda_{-1}(n)=0$.

4.2 Lemma. *Let $i^*_n : \boldsymbol{R}(S_P(n)) \longrightarrow \boldsymbol{R}(S_P(n-1))$ be a natural homomorphism (by the inclusion $S_P(n-1)\subset S_P(n)$), then we have*

$$(4.2.1)\qquad i^*_n(\lambda_l(n)) = \begin{cases} 2(\lambda_{n-1}(n-1)+\lambda_{n-2}(n-1)) & (\textit{for } l=n) \\ \lambda_l(n-1)+2\lambda_{l-1}(n-1)+\lambda_{l-2}(n-1) & (\textit{for } l<n). \end{cases}$$

Proof. From (4.1.4), it is trivial.

4.3 Lemma. *we set $a_{-1}=0$, $a_0=1$, $a_1=-2$, $a_2=3$, and inductively $a_l=-2a_{l-1}-a_{l-2}$ for $l=3,4,\cdots\cdots, n-1$. (In fact, $a_l=(-1)^l(l+1)$). we set*

$$\overline{\lambda_l}(n)=\sum_{i=0}^{l}(-1)^i(i+1)\lambda_{l-i}(n) \qquad (\textit{for } l<n)$$

$$\overline{\lambda_n}(n)=\lambda_n-2(\sum_{i=0}^{n}(-1)^{n+1}\lambda_{n-i-1}(n)),$$

then we have

$$(4.3.1)\qquad i^*_n(\overline{\lambda_l}(n)) = \begin{cases} \overline{\lambda_l}^{(n-1)} & (\textit{for } l<n) \\ 0 & (\textit{for } l=n). \end{cases}$$

Proof, From (4.1.4), it is trivial.

4.4 Theorem. *We have the following isomorphism between $\boldsymbol{Z}_2$-graded $\boldsymbol{R}(S_P(n))$-modules :*

$$K^*_{S_P}(n)(CS^{4n-1}, S^{4n-1}) \cong \{\overline{\lambda_n}(n)\},$$

where $\overline{\lambda_n}(n)$ is as in **4.3**, *and { } is the same notation as in* **3.3**.

Proof. Since $\overline{\lambda_1}(n), \cdots\cdots, \overline{\lambda_n}(n)$ are algebraically independent and $i^*_n : \boldsymbol{R}(S_P(n)) \longrightarrow \boldsymbol{R}(S_P(n-1))$ is an onto homomorphism, the result follows at once.

§ 5

5.1 Lemma. *Let $i^*_r : \boldsymbol{R}(SO(2r)) \longrightarrow \boldsymbol{R}(SO(2r-1))$ be a natural homomorphism (by the inclusion $SO(n-1)\subset SO(n)$), then we have*

$$(5.1.1)\qquad i^*_r(\lambda^l(\rho_{2r})) = \begin{cases} \lambda^l(\rho_{2r-1})+\lambda^{l-1}(\rho_{2r-1}) & (\textit{for } l<r) \\ 2\lambda^{r-1}(\rho_{2r-1}) & (\textit{for } l=r) \end{cases}$$

$$(5.1.2)\qquad i^*_r(\lambda^r_{\pm}(\rho_{2r})) = \lambda^{r-1}(\rho_{2r-1}).$$

Proof. For (5.1.1), from (4.1.4), it is trivial. For (5.1.2), from **2.3** we have

$$\lambda^r_{\pm}(\rho_{2r}) = (\Delta^{\pm}_{2r})^2 - (\lambda^{r-2}(\rho_{2r}) + \lambda^{r-4}(\rho_{2r}) + \cdots\cdots).$$

$$i^*_r((\Delta^{\pm}_{2r})^2) = (\Delta_{2r-1})^2 \quad \text{(c. f, Proposition 9. 4 of [2])}$$

$$(\Delta_{2r-1})^2 = \lambda^{r-1}(\rho_{2r-1}) + \cdots\cdots + \lambda^1(\rho_{2r-1}) + 1.$$

Therefore (5. 1. 1) implies (5. 1. 2).

5. 2 Lemma. *We set*

$$\overline{\lambda^l}(\rho_{2r}) = \sum_{i=0}^{l}(-1)^i\lambda^{l-i}(\rho_{2r}) \qquad (\text{for } l=1, 2, 3, \cdots\cdots, r-1),$$

then we have

$$\begin{aligned} &\lambda^l(\rho_{2r}) = \overline{\lambda^l}(\rho_{2r}) + \overline{\lambda^{l-1}}(\rho_{2r}) \\ (5.\ 2.\ 1) \qquad &i^*_r(\overline{\lambda^l}(\rho_{2r})) = \lambda^l(\rho_{2r-1}) \\ &i^*_r(\lambda^r_{\pm}(\rho_{2r})) = \lambda^{r-1}(\rho_{2r-1}). \end{aligned}$$

Proof. From **5. 1**, it is trivial.

5. 3 Theorem. *We have the following isomorphism between* Z_2*-graded* $R(SO(2r))$*-modules :*

$$K^*_{SO(2r)}(CS^{2r-1},\ S^{2r-1}) \cong \{(\overline{\lambda^{r-1}}(\rho_{2r}) - \lambda^r_+(\rho_2{}^r)),\ (\overline{\lambda^{r-1}}(\rho_{2r}) - \lambda^r_-(\rho_{2r}))\}$$

, *where* { } *is the same notation as in* **3. 3.**

Proof. From 5. 2, i^*_r : $R(SO(2r)) \longrightarrow R(SO(2r-1))$ is an onto homomorphism, so it is sufficient to determine Kernel i^*_r. Now, from the definition of $\overline{\lambda^l}(\rho_{2r})$, $\overline{\lambda^1}(\rho_{2r})$, $\cdots\cdots$, $\overline{\lambda^{r-1}}(\rho_{2r})$ are algebraically independent. Therefore (5. 2. 2) and (5. 2. 3) implies the result.

§ 6

6. 1 Lemma. Let i^*_r : $R(SO(2r+1)) \longrightarrow R(SO)2r))$ *be a natural homomrphism (by the inclusion* $SO(2r) \subset SO(2r+1)$*) then we have*

$$i^*_r(\lambda^l(\rho_{2r+1})) = \lambda^l(\rho_{2r}) + \lambda^{l-1}(\rho_{2r}).$$

Proof. From (4. 1. 4), it is trivial.

6. 2 Theorem. *We have the following isomorphism between* Z_2*-graded* $R(SO(2r+1))$*-modules :*

$$K^*_{SO(2r+1)}(CS^{2r},\ S^{2r}) \cong \{\lambda^r_+(\rho_{2r})\}$$

Proof. From **6. 1**, i^*_r : $R(SO(2r+1)) \longrightarrow R(SO(2r))$ is a monomorphism, and

Image i_r^* equals the subring

$$\boldsymbol{Z}[\lambda^1(\rho_{2r}), \cdots\cdots, \lambda^{r-1}(\rho_{2r}), (\lambda^r_+(\rho_{2r})+\lambda^r_-(\rho_{2r}))]$$

in $\boldsymbol{R}(SO(2r))$. So it is sufficient to determine Cokernel i_r^*. From the relation in $\boldsymbol{R}(SO(2r))$, $(\lambda^r_-(\rho_{2r}))(\lambda^r_+(\rho_{2r})$ is an element of Image i_r^*. We set $x=\lambda^r_+(\rho_{2r})$, $y=\lambda^r_-(\rho_{2r})$ and $B=$ Image i_r^*. Since

$$x^2=(x+y)x-yx$$
$$x^k=(x+y)x^{k-1}-(yx)x^{k-2}, \qquad (for\ k\geqq 3),$$

we can write

(6. 1. 1) $\qquad x^k=\sum a_i x \quad (\text{mod } B)$

where a_i is an element of B. Since

$$y^k=(x+y-x)^k=\sum_{i=0}^{k}\binom{k}{i}(x+y)^i(-1)^{k-i}x^{k-i},$$

from (6. 1. 1), we can write

(6. 1. 2) $\qquad y^k=\sum b_i x \quad (\text{mod } B),$

where b_i is an element of B. Therefore Cokernel i_r^* is an $\boldsymbol{R}(SO(2r+1))$-module generated by $\lambda^r_+(\rho_{2r})$.

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