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Integration of Alexander-Spanier Cochains

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Introduction.

From the point of view of the geometry of microbundles, the corresponding notion of the differential forms in the geometry of vector bundles, is the Alexander-Spanier cochains (cf. [3], [4]). For a differential form φ, one of the most important fact is to able consider the integration of φ on an (arbitrary) differentiable singular chain. Therefore, it is natural to ask whether we may consider the integration of Alexander-Spanier cochains on singular chains or not.

The purpose of this paper is to give a definition of the integral of Alexander-Spanier cochains on singular chains and prove some of its properties such as Stokes' theorem.

In fact, if $f(x, y)$ is a function on $I^2=\{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$ such that

$$f(x, x) = 0, \ f(x, y) \text{ is smooth in } y,$$

then we may set

$$f(x, y) = f_y(x, x)(y - x) + O(|y - x|).$$

Hence we obtain

$$\int_0^1 f_y(x, x)dy = \lim_{|a_{i+1}-a_i| \to 0} \sum_{i=0}^{n} f(a_i, a_{i+1}),$$
$$0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1.$$

This shows that for an Alexander-Spanier *1*-cochain $f(x_0, x_1)$ on X, a topological space, and a singular *1*-simplex φ, $\varphi : I \to X$, it is natural to define $\int_{\varphi(I)} f(x_0, x_1)$ by

$$\int_{\varphi(I)} f(x_0, x_1) = \lim_{|a_{i+1}-a_i| \to 0} \sum_{i=0}^{n} f(\varphi(a_i), \varphi(a_{i+1})).$$

Similarly, for an Alexander-Spanier s-cochain $f^s = f(x_0, x_1, \cdots, x_s)$ on X and a (qubical) singular s-simplex φ, $\varphi : \boldsymbol{I}^s \to X$, we define $\int_{\varphi(\boldsymbol{I}^s)} f^s$ by

$$\int_{\varphi(\boldsymbol{I}^s)} f^s$$
$$= \lim_{|a_{i,j_i+1}-a_{i,j_i}|\to 0} \sum_{j_1=0,\cdots,j_s=0}^{j_1=n_1,\cdots,j_s=n_s} f(\varphi(a_{1,j_1}, \cdots, a_{s,j_s}),$$
$$\varphi(a_{1,j_1+1}, a_{2,j_2}, \cdots, a_{s,j_s}, \cdots, \varphi(a_{1,j_1}), \cdots, a_{s-1,j_{s-1}}, a_{s,j_s+1})),$$
$$0 = a_{i,0} < a_{i,1} < \cdots < a_{i,n_i} < a_{i,n_i+1} = 1.$$

The integration $\int_\gamma f^s$ of f^s on a singular s-chain γ is defined as usual. Of course, $\int_\gamma f^s$ may not be exist in general. For example, on $\boldsymbol{I}^1$, give f^1 by $f(x, y) = \sqrt{|x-y|}$ and take φ to be the identity map, then we have

$$\int_{\varphi(\boldsymbol{I}^1)} f(x, y) = \lim. \sum \sqrt{|a_{i+1} - a_i|} = \infty.$$

On the other hand, on $\boldsymbol{I}^2$, give f^1 by $f(x, y) = |x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ and $\varphi(x) = (x, x\sin(1/x))$, $x \neq 0$, $\varphi(0) = (0, 0)$, we get

$$\int_{\varphi(\boldsymbol{I}^1)} f(x, y) = \textit{the length of the curve } \varphi(\boldsymbol{I}^1) = \infty.$$

But there are examples that show the existence of $\int_\gamma f^s$ for non-smooth f^s and γ.

The outline of this paper is as follows: In § 0, we review the definition and properties of Alexander-Spanier cochains (cf. [1], [2], [15], [16]). The definition of the integral is given in § 1. Some elementary properties of the integral are also proved in § 1. § 2 is devoted examples and show some classical integrals such as Stieltjes integral (cf. [7]) and Helinger integral are written as $\int_{\varphi(\boldsymbol{I}^1)} f(x, y)$ with suitable choice of $f(x, y)$. Then we prove the Stokes' theorem

$$\int_{\partial\gamma} f^s = \int_\gamma (\delta f)^{s+1}$$

for this integral in § 3 under suitable assumptions about f and γ. Here δ it the coboundary homomorphism in the Alexander-Spanier cohomology (cf. [1], [2], [15], [16]). We note that, if $s=1$, then this is trivial, because $\delta f(x, y) = f(y) - f(x)$ and

$$\int_{\varphi(\boldsymbol{I}^1)} \delta f = \lim. \sum (f(\varphi(a_{i+1})) - f(\varphi(a_i)) = f(\varphi(1)) - f(\varphi(o)),$$

in this case. In § 4, we treat the volume element $v = v(x_0, x_1, \cdots, x_n)$ with respect to a metric r on X. Here, v is given by

$$v(x_0, x_1, \cdots, x_n) = r(x_0, x_1)r(x_0, x_2)\cdots r(x_0, x_n),$$

and X is assumed to be a CW-complex. The existence of the volume element shows that if X is a topological manifold and dim. $X \geq 6$, then the structure group of the tangent microbundle of X is reduced to the group of germs of Lebesgue measure preserving homeomorphisms of R^n as an $H_*(n)$ -bundle if X has a metric which is invariant under the operation of the connection of X (Theorem 5). Another application of the volume element is the definition of (singular) integral operators on a (compact) CW-complex$X$. For example, if $k(x, y)$ is a continuous function on $X \times X - \Delta(X)$, $\Delta(X)$ is the diagonal of $X \times X$, such that

$$|k(x, y)| = o(r(x, y)^{1-n}), \quad n = \text{dim. } X,$$

then $I(k)[f] = \int_X k(x, y)v(x, x_1, \cdots, x_n)$ is defined for any continuous function f on X and $I(k)[f]$ is continuous on X. Since $I(k)$ is a compact operator on $C(X)$, the Banach space of continuous functions on X with uniform convergence topology, if k is continuous on $X \times X$, we may define the symbol $\sigma(I(k))$ of $I(k)$ by

$$\sigma(I(k)) \textit{ is the class of } k \textit{ mod. } C(X \times X).$$

Similarly, we can define integral operator $I(k)$ as the map

$$I(k): \Gamma(\mathscr{E}) \rightarrow \Gamma(\mathscr{F}),$$

where $\mathscr{E}$ and $\mathscr{F}$ are the vector bundles over X, $\Gamma(\mathscr{E})$ and $\Gamma(\mathscr{F})$ are the spaces of their (continuous) cross -sections. These shows us the possibility of the extension of the theory of elliptic complexes for CW-complexes. In fact, if X is a topological manifold, then we can define the symbol $\sigma(I(k))$ as the bundle map

$$\sigma(I(k)) : p^*(\mathscr{E}) \rightarrow p^*(\mathscr{E}),$$

where $p^*(\mathscr{E})$ and $p^*(\mathscr{F})$ are the induced bundles on the tatal space of the tangent microbundle of X of $\mathscr{E}$ and $\mathscr{F}$. These are stated in § 5.

I would like to thank Prof. Uchiyama who teach me examples of the classical integrals which reduces to the form of $\int_{\varphi(I^1)} f(x, y)$.

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§ 0. Alexander-Spanier cochains.

1. For a topological space X, we set

(1) $$\Delta_s(X) = (x, x, \cdots, x) | x \in X \subset \overbrace{X \times \cdots \times X}^{s+1}.$$

We often denote $\Delta(X)$ instead of $\Delta_1(X)$. We denote by $\mathfrak{R}$ a topological vector space (over $\boldsymbol{R}$ or $\boldsymbol{C}$).

Definition. *Two $\mathfrak{R}$-valued functions f and g on $U(\Delta_s(X))$, a neighborhood of $\Delta_s(X)$, are called equivalent if*

$$f \mid V(\Delta_s(X)) = g \mid V(\Delta_s(X)),$$

for some neighborhood $V(\Delta_s(X))$ of $\Delta_s(X)$ and the equivalence class of f by this relation is called the germ of f (at $\Delta_s(X)$) and denoted by $\bar{f}$ or simply, f.

Definition. *A germ of f at $\Delta_s(X)$ is called an ($\mathfrak{R}$-valued) Alexander-Spanier s-cochain.*

By definition, the set of all Alexander-Spanier s-cochains of X forms an $\boldsymbol{R}$ (or $\boldsymbol{C}$) vector space. If $\mathfrak{R}$ is a ring, then it is also an $\mathfrak{R}$-modul. It is denoted by $C^s(X)$ or $C^s(X, \mathfrak{R})$.

We call an Alexander-Spanier s-cochain $\bar{f}$ is continuous, regular or alternative if a representation f of $\bar{f}$ is continuous or satisfies

$$f(x_0, x_1, \cdots, x_s) = 0, \text{ if } x_i = x_j \text{ for some } i, j \ (i \neq j),$$

or

$$f(x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(s)}) = \operatorname{sgn}(\sigma) f(x_0, x_1, \cdots, x_s),$$
$$\sigma \in \mathfrak{S}^{s+1}.$$

Similarly, if X is a Lipschitz manifold (cf. [13]), smooth manifold or an analytic space (may have singularities), we can define a Lipschitz continuous, smooth or analytic Alexander-Spanier cochain as above.

It is known, that to define the coboundary homomorphism $\delta : C^s(X) \to C^{s+1}(X)$ by

$$\delta f(x_0, x_1, \cdots, x_{s+1}) = \sum_{i=0}^{s+1}(-1)^i f(x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{s+1}),$$

we obtain

$$H^s(X, \mathfrak{R}) \simeq B_s(X, \mathfrak{R}) / Z_s(X, \mathfrak{R}), \tag{2}$$
$$B^s(X, \mathfrak{R}) = ker. [\delta : C^s(X, \mathfrak{R}) \to C^{s+1}(X, \mathfrak{R})], \quad Z^s(X, \mathfrak{R}) = \delta C^{s-1}(X, \mathfrak{R}),$$

if X is a normal paracompact space ([2], [15], [16]). Here $H^s(X, \mathfrak{R})$ is the Cech cohomology group. We note that (2) is true although we restrict $C^s(X, \mathfrak{R})$ to the group of continuous, regular (regular and continuous), alternative (alternative, regular and continuous) s-cochains on X. Similarly, (2) is true for the group of

Lipschitz continuous or smooth s-cochains on X if X is a Lipschitz manifold or a smooth manifold.

2. If $\mathfrak{R}$ is a ring, then we can define the product for $f^r \in C^r(X, \mathfrak{R})$ and $g^s \in C^s(X, \mathfrak{R})$ by

$$\begin{aligned}&(f^s \cdot g^s)(x_0, x_1, \cdots, x_{r+s})\\&= f(x_0, x_1, \cdots, x_r)g(x_r, x_{r+1}, \cdots, x_{r+s}).\end{aligned}$$

If f^r and g^s are both alternative, then we use the alternative product $f^r \wedge g^s$ given by

$$\begin{aligned}&(f^r \wedge g^s)(x_0, x_1, \cdots, x_{r+s})\\&= A((f^r \cdot g^s)(x_0, x_1, \cdots, x_{r+s})).\end{aligned}$$

Here $A(h^t)$ is the alternation of $h^t \in C^t(X, \mathfrak{R})$ defined by

$$\begin{aligned}&A(h^t)(x_0, x_1, \cdots, x^t)\\&= \frac{1}{(t+1)!} \sum_{\sigma \in \mathfrak{S}\, t+1} \operatorname{sgn}(\sigma) h(x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(t)}).\end{aligned}$$

By definition, if h is alternative, then $Ah = h$. Hence we get

$$C^t(X, \mathfrak{R}) = AC^t(X, \mathfrak{R}) + ker.\ A,$$

and $AC^t(X, \mathfrak{R})$ is the space of all alternative t-cochains of X. We note that, if h is symmetric, that is, h satisfies

$$h(x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(t)}) = h(x_0, x_1, \cdots, x_t), \quad \sigma \in \mathfrak{S}^{t+1},$$

then h belongs in $ker.\,A$.

Definition. *If $\mathfrak{R}$ is a (semi) normed vector space, then we denote by $||\bar{f}^s||$ the ($\boldsymbol{R}$-valued) s-cochain with representation $||f^s||$, $f^s \in f^s$, where $||\bar{f}^s||$ is given by*

$$||f^s||(x_0, x_1, \cdots, x_s) = ||f^s(x_0, x_1, \cdots, x_s)||,$$

and $||a||$ is the norm of a in $\mathfrak{R}$.

By definition, if $\bar{f}^s$ is alternative, then $||\bar{f}^s||$ is a symmetric ($\boldsymbol{R}$-valued) s-cochain of X.

Note. If $\mathfrak{R}=\boldsymbol{R}$ or $\boldsymbol{C}$, then we denote $|\bar{f}^s|$ the s-cochain with representation $|f^s|$, $f^s \in \bar{f}^s$.

Definition. *We call an $\boldsymbol{R}$-valued s-cochain $\bar{f}^s$ to be positive if some $f^s \in \bar{f}^s$ satisfies*

$$f(x_0, x_1, \cdots, x_s) \geqq 0, \ (x_0, x_1, \cdots, x_s) \in U(\Delta_s(X)),$$

for some $U(\Delta_s(X))$.

By definition, $||\bar{f}^s||$ (or $|\bar{f}^s|$) is a positive s-cochain.

§ 1. Definition of the integral.

3. We use the following notations.

$$I^s = (t_1, , t_s) | 0 \leqq t_1 \leqq 1, \cdots, 0 \leqq t_s \leqq 1,$$
$J = (j_1, \cdots, j_s)$, $j_1, \cdots, j_s$ *are 0 or natural numbers,*
$$J + 1_k = (j_1, \cdots, j_{k-1}, j_k+1, j_{k+}{}^1, \cdots, j_s),$$
$$a_J = (a_{1,j_1}, \cdots, a_{s,j_s}), \ 0 \leqq a_{1,j_1} \leqq 1, \cdots, 0 \leqq a_{s,j_s} \leqq 1.$$

Definition. *If* $f^s = f(x_0, x_1, \cdots, x_s)$ *is defined on some neighborhood* $U(\Delta_s(X))$ *of* $\Delta_s(X)$ *and* $\varphi : I^s \to X$ *is a (qubical) singular s-simplex of X* (cf. [12]). *then we set*

$$(3) \qquad \int_{\varphi(I^s)} f^s = \lim_{|a_{J+1_k} - a_J| \to 0} \sum_{J=(0,\cdots,0)}^{J=(n_1,\cdots,n_s)} f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})),$$

if the limit exists. Here $\{a_{i,j_i}\}$ *is a series of real numbers such that*

$$0 = a_{0,i} < a_{i,1} < \cdots < a_{i,n_i} < a_{i,n_i+1} = 1.$$

Lemma 1. *The existence and non-existence and the vlaue of* $\int_{\varphi(I^s)} f^s$ *(if it exists) are depends on the germ of* f^s.

Proof. If f^s are equivalent, then $f^s | V(\Delta_s(X)) = g^s | V(\Delta_s(X))$ for some $V(\Delta_s(X))$. Then taking $|a_{i,j_i+1} - a_{i,j_i}|$ sufficiently small for each i and j_i, $(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s}))$ belongs in $V(\Delta_s(X))$ and we get

$$f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_1}))$$
$$= g(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})).$$

This shows the lemma.

Definition. *We define the integral* $\int_{\varphi(I^s)} \bar{f}^s$ *of an Alexander-Spanier s-cochain* $\bar{f}^s$ *on a (cubical) singular simplex* $\varphi : I^s \to X$ *by*

$$(4) \qquad \int_{\varphi(I^s)} \bar{f}^s = \int_{\varphi(I^s)} f^s,$$

where f^s *is a representation of* $\bar{f}^s$.

Definition. *If* $\int_{\varphi(I^s)} \bar{f}^s$ *exists, then we call* $\bar{f}^s$ *is integrable on* $\varphi(I^s)$.

Similarly, we call f^s is integrable on $\varphi(I^s)$ if $\int_{\varphi(I^s)} f^s$ exists, where f^s is a function on $U(\Delta_s(X))$.

Definition. *If* $\mathfrak{N}$ *is a (semi) normed vector space and* $\bar{f}^s$ *and* $||f^s||$ *are both in-*

tegrable on $\varphi(\boldsymbol{I}^s)$, *then we call* $\bar{f}^s$ *is absolutely integrable on* $\varphi(\boldsymbol{I}^s)$.

Similarly, we define the absolute integrability of f^s, a function on $U(\Delta_s(X))$.

Note. If $\mathfrak{R}=\boldsymbol{R}$ or $\boldsymbol{C}$, then the absolute integrability of f^s follows from the integrability of $|f^s|$.

By definition, we get

$$\int_{\varphi(\boldsymbol{I}^s)}(\alpha f^s+\beta g^s)=\alpha\int_{\varphi(\boldsymbol{I}^s)}f^s+\beta\int_{\varphi(\boldsymbol{I}^s)}g^s, \tag{5}$$

and if f^s is absolutely integrable on $\varphi(\boldsymbol{I}^s)$ and $\psi(\boldsymbol{I}^s)$, then

$$\int_{\varphi(\boldsymbol{I}^s)+\psi(\boldsymbol{I}^s)}f^s=\int_{\varphi(\boldsymbol{I}^s)}f^s+\int_{\psi(\boldsymbol{I}^s)}f^s. \tag{6}$$

By (6), we define

Definition. *If* $\gamma=\sum\alpha_i\varphi_i(\boldsymbol{I}^s)$ *is a (qubical) singular s-chain on X and* f^s *is absolutely integrable on each* φ_i $(\boldsymbol{I}^s)$, *then we define the integral* $\int_\gamma f^s$ *of an Alexander-Spanier s-cochain* f^s *on* γ *by*

$$\int_\gamma f^s=\sum\alpha_i\int_{\varphi_i(\boldsymbol{I}^s)}f^s. \tag{7}$$

For the integration on γ, we define the integrability and the absolute integrability of f^s on γ as above. Then we get

$$\int_\gamma(\alpha f^s+\beta g^s)=\alpha\int_\gamma f^s+\beta\int_\gamma g^s, \tag{5$'$}$$

and if f^s is absolutely integrable on γ_1 and γ_2, then

$$\int_{\gamma_1+\gamma_2}f^s=\int_{\gamma_1}f^s+\int_{\gamma_2}f^s. \tag{6$'$}$$

4. ***Theorem 1.*** *If* φ *does not depend on* t_i *and* f^s *satisfies*

$$f(x_0,\ x_1,\ \cdots,\ x_s)=0 \text{ if } x_0=x_i,$$

then $\int_{\varphi(\boldsymbol{I}^s)}f^s$ *is equal to 0.*

Proof. By the assumption about φ, we have $\varphi(a_J)=\varphi(a_{J+1_i})$ for all $\boldsymbol{J}$. Then by the assumption about f^s, we have

$$f(\varphi(a_J),\ \varphi(a_{J+1_1}),\ \cdots,\ \varphi(a_{J+1_s}))=0,$$

for all $\boldsymbol{J}$. Hence we have the theorem.

Corollary. *If* f^s *is regular and* $\varphi(\boldsymbol{I}^s)$ *is degenerated* (*cf.* [12]), *then*

(8) $$\int_{\varphi(I^s)} f^s = 0.$$

Note. To get this corollary, it issufficient f^s to satisfy

$$f(x_0, \cdots, x_s) = 0, \text{ if } x_0 = x_i \text{ for some } i.$$

Theorem 2. *If f^s is alternative and absolutely integrable on $\varphi(I^s)$, then*

(9) $$\int_{\varphi(\sigma(I^s))} f^s = \operatorname{sgn}(\sigma)\int_{\varphi(I^s)} f^s, \quad \sigma \in \mathfrak{S}^s.$$

Here σ operates on I^s by

$$(t_1, \cdots, t_s) = (t_{\sigma(1)}, \cdots, t_{\sigma(s)}).$$

Proof. To show the theorem, it is sufficient to show

(9)′ $$\int_{\varphi(\tau_{ij}(I^s))} f^s = -\int_{\varphi(I^s)} f^s, \quad i \neq j,$$
$$\tau_{ij}(t_1, \cdots, t_i, \cdots, t_j, \cdots, t_s) = (t_1, \cdots, t_j, \cdots, t_i, \cdots, t_s).$$

But since we have

$$\sum_{J=(0,\cdots,0)}^{(n_1,\cdots,n_s)} f(\varphi(\tau_{ij}(a_J)), \varphi(\tau_{ij}(a_{J+1_1})), \cdots, \varphi(\tau_{ij}(a_{J+1_i})), \cdots, \varphi(\tau_{ij}(a_{J+1_j})), \cdots, \varphi(\tau_{ij}(a_{J+1_s})))$$
$$= \sum_{J=(0,\cdots,0)}^{(n_1,\cdots,n_s)} f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_j}, \cdots, \varphi(a_{J+1_i}), \cdots, \varphi(a_{J+1_s})),$$

we get by the alternativity of f,

$$\sum_J f(\varphi(\tau_{ij}(a_J)), \varphi(\tau_{ij}(a_{J+1_1})), \cdots, \varphi(\tau_{ij}(a_{J+1_s})))$$
$$= -\sum_J f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})).$$

Hence we obtain (9)′.

Note. To get (9), it is sufficient to assume f^s being alternative only in $x_1, \cdots, x_s$. That is, f^s to satisfy only

$$f(x_0, x_{\sigma(1)}, \cdots, x_{\sigma)s)}) = \operatorname{sgn}(\sigma) f(x_0, x_1, \cdots, x_s), \quad \sigma \in \mathfrak{S}^s.$$

Similarly, if f^s is symmetric in $x_1, \cdots, x_s$ and absolutely integrable on $\varphi(I^s)$, then

$$\int_{\varphi(\sigma(I^s))} f^s = \int_{\varphi(I_s)} f^s.$$

§ 2. Examples.

5. As stated in introduction, if $s=1$, f is regular and $f(\varphi(s), \varphi(t))$ is smooth in t, then

$$\int_{\varphi(I^1)} f(x, y) = \int_0^1 \frac{\partial f}{\partial t}(\varphi(s), \varphi(s))ds.$$

In general, we get

Theorem 3. *If f is regular and setting*

$$f(\varphi(t_1, \cdots, t_s), \varphi(t_{11}, \cdots, t_{1s}), \cdots, \varphi(t_{s1}, \cdots, t_{ss}))$$
$$= g(t_1, \cdots, t_s, t_{11}, \cdots, t_{1s}, \cdots, t_{s1}, \cdots, t_{ss}),$$
$$0 \leqq t_{ij} \leqq 1, \ i=1, \cdots, s, \ j=1, \cdots, s,$$

g is smooth in each t_{ii}, then

(10)
$$\int_{\varphi(I^s)} f^s$$
$$= \int_0^1 \cdots \int_0^1 \frac{\partial^s g}{\partial t_{11} \cdots \partial t_{ss}}\Big|_{t_{ij}=t_j} dt_1 \cdots dt_s.$$

Proof. By assumption, we get

$$g(t_1, \cdots, t_s, t_{11}, t_2, \cdots, t_s, \cdots, t_1, \cdots, t_{s-1}, t_{ss})$$
$$= \frac{\partial^s g}{\partial t_{11} \cdots \partial t_{ss}}(t_1, \cdots, t_s, t_1, \cdots, t_s, \cdots, t_1, \cdots, t_s)(t_{11}-t_1)(t_{ss}-t_s)$$
$$+ o(|t_{11}-t_1| \, |t_{ss}-t_s|).$$

Hence we have

$$\int_{\varphi(I^s)} f^s$$
$$= \lim_{|a_{J+1_i} - a_J| \to 0} \sum_J f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s}))$$
$$= \lim_{|a_{J+1_i} - a_J| \to 0} \sum_J g(a_J, a_{J+1_1}, \cdots, a_{J+1_s})$$
$$= \lim_{|a_{J+1_i} - a_J| \to 0} \sum_J \frac{\partial^s g}{\partial t_{11} \cdots \partial t_{ss}}(a_{1,j_1}, \cdots, a_{s,j_s}, a_{1,j_1}, \cdots, a_{s,j_s},$$

$$, \cdots, a_{1,j_1}, \cdots, a_{s,j_s})(a_{1,j_1+1}-a_{1,j_1})\cdots(a_{s,j_s+1}-a_{s,j_s})+$$

$$+\sum_J o(|a_{1,j_1+1}-a_{1,j_1}|\cdots|a_{s,j_s+1}-a_{s,j_s}|)$$

$$=\int_0^1\cdots\int_0^1\frac{\partial^s g}{\partial t_{11}\cdots\partial t_{ss}}\Big|_{t_{ij}=t_j}dt_1\cdots dt_s.$$

Corollary. *If X is a smooth manifold, f^s is a regular smooth cochain and φ is a smooth singular chain, then $\int_{\varphi(I^s)} f^s$ exists.*

6. In this n⁰, we assume $X=I^1$ and φ is the identity map and denote $\int_{I^1} f(x_0, x_1)$ instead of $\int_{\varphi(I^1)} f(x_0, x_1)$.

We give some examples of classical integrals which are written in the form $\int_{I^1} f(x_0, x_1)$. I owe these examples to Prof. Uchiyama.

(i). Taking $f(x_0, x_1)=g(x_0)(x_1-x_0)$, we get

$$\int_{I^1} f(x_0, x_1)=\int_0^1 g(x)dx,$$

where the right hand side is the usual Riemannian integral of $g(x)$.

(ii). Taking $f(x_0, x_1)=g(x_0)(h(x_1)-h(x_0))$, we get

$$\int_{I^1} f(x_0, x_1)=\int_0^1 gdh,$$

where the right hand side is the Young-Stieltjes integral (cf. [7]). It is known that setting

$$\omega(t, g)=\sup_{|x-y|\leq t}.|g(x)-g(y)|,\quad \omega(t, h)=\sup_{|x-y|\leq t}.|(x)-h(y)|,$$

if $\int_0^1(\omega(t, g)\omega(t, h)/t^2)dt$ exists, then $\int_0^1 gdh$ exists.

(ii)'. If we use alternative *1*-cochain $f(x_0, x_1)$ given by

$$f(x_0, x_1)=\frac{1}{2}(g(x_0)+g(x_1))(h(x_1)-h(x_0)),$$

instead of $g(x_0)(h(x_1)-h(x_0))$, then

$$\int_{I^1} f(x_0, x_1)=L\int_0^1 gdh.$$

Here $L\int_0^1 gdh$ is the Lane-Stieltjes integral ([9], [14]).

(iii). Setting

$$f(x_0, x_1)=\frac{(g_1(x_1)-g_1(x_0))(g_2(x_1)-g_2(x_0))}{h(x_1)-h(x_0)},$$

where if $h(a)=h(b)$, then $g_i(a)=g_i(b)$, $i=1$ or 2 and set $f(a, b)=0$, we get

$$\int_{I^1} f(x_0, x_1)=\int_0^1 \frac{df_1(x)df_2(x)}{dh(x)},$$

where the right hand side is the Hellinger integral. It is known that to set

$$\Delta f(x)=f(x+\delta)-f(x),\ \delta>0,$$

if $h(x)$ and $k(x)$ are bounded decreasing continuous functions on I and for any $\delta>0$, we get

$$(\Delta g_1)^2 \leqq \Delta h\cdot\Delta k,\ (\Delta g_2)^2 \leqq \Delta g\cdot\Delta k,$$

then $\int_0^1 (df_1(x)df_2(x)/dh(x))$ exists.

Note. For $X=I^s$, taking φ to be the identity map and denote $\int_{I^s} f^s$ instead of $\int_{\varphi(I^s)} f^s$, we obtain

(i)'. Setting $x_0=(t_1, \cdots, t_n)$, $x_i=(t_{i,1}, \cdots, t_{i,n})$, $i=1, \cdots, s$ and

$$f(x_0, x_1, \cdots, x_s)$$
$$=g(x_0)p_1(x_1-x_0)\cdots p_s(x_s-x_0),\ p_i(x_i-x_0)=t_{ii}-t_i,$$

we have

$$\int_{I^s} f^s=\int_0^1\cdots\int_0^1 g(t_1, \cdots, t_s)dt_1\cdots dt_s.$$

Here the right hand side is the Riemannian integral of g.

7. In this n°, we assume that X is a metric space and denote its metric by $r=r(x, y)$.

By definition, $r(x_0, x_1)$ defines a positive 1-cochain of X. We also denote it by r. Then $\int_{\omega(I^1)} r$ is the length of the curve $\varphi(I^1)$ by the metric r. For example, if $X=\mathbf{R}^n$, r is the usual euclid metric, then setting

$$\varphi(t)=(f_1(t), \cdots, f_n(t)),\ 0\leqq t\leqq 1,$$

r is integrable on $\varphi(I^1)$ if and only if each f_i is the function of bounded variation.

Similarly, using r, to define a positive s-cochain $v^s=v(r)^s$ of X by

$$v^s(x_0, x_1, \cdots, x_s)=r(x_0, x_1)\,r(x_0, x_2)\cdots r(x_0, x_s), \tag{11}$$

we may consider v^s to be the s-dimensional volume element of X with respect to the metric r. In fact, if $\varphi(I^s)$ is non-degenerated, then $\int_{\varphi(I^s)} v^s\neq 0$, and $\int_{\varphi(I^s)}$

$v^s = 0$ if (I^s) is degenerated. But $\int_{\varphi(I^s)} v^s$ may be equal to ∞ for non-degenerated φ.

Definition. *If* dim. $X = n$, *then we denote* v^n *by* v *or* $v(r)$ *and call the volume element of* X *with respect to the metric* r.

We note that $Ar = 0$ and $Av^s = 0$ for all s. On the other hand, on $\boldsymbol{R}^n$, we may take

$$V(x_0, x_1, \cdots, x_n) = p_1(x_1 - x_0)p_2(x_2 - x_0) \cdots p_n(x_n - x_0)$$

to be the volume element and for this V, $AV \neq 0$.

§ 3. Stokes' theorem.

8. ***Lemma 2.*** *If* f^s *is absolutely integrable on* $\varphi(\boldsymbol{I}^s)$, *then for any* $\delta > 0$, *there exist* $\varepsilon > 0$ *and* $N = N(\delta) > 0$ *such that*

$$\begin{aligned} \text{(12)} \qquad & || f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) || \\ & \leqq N\, |a_{1,j_1+1} - a_{1,j_1}| \cdots |a_{s,j_s+1} - a_{s,j_s}|, \\ & \textit{if } a_J \in \boldsymbol{I}_\delta^1 \times \cdots \times \boldsymbol{I}_\delta^s \textit{ and } |a_{i,j_i} - a_{i,j_i}| < \varepsilon, \ i = 1, \cdots, s, \end{aligned}$$

where $\boldsymbol{I}_\delta^k$ is given by

$$\text{(12)}' \qquad \boldsymbol{I}_\delta^k = \bigcup_{i=0}^{mk} [b^k_{2i}, b^k_{2i+1}],$$

$$0 \leqq b^k_0 < b^k_1 < \cdots < b^k_{2m_k} < b^k_{2m_k+1} \leqq 1, \quad \sum (b^k_{2i+1} - b^k_{2i}) > 1 - \delta.$$

Proof. If f^s does not satisfy (12), then we can take c^k_i, $k = 1, \cdots, s$ and $i = 1, 2$ such that $0 \leqq c^k_1 < c^k_2 \leqq 1$ and

$$\begin{aligned} & || f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) || \\ & > N |a_{1,j_1+1} - a_{1,j_1}| \cdots |a_{s,j_s+1} - a_{s,j_s}|, \\ & \textit{for some } a_J \in K, \ K = (c^1_1, c^1_2) \times \cdots \times (c^s_1, c^s_2), \end{aligned}$$

and the set of such a_J is dense in K for any N. Hence we have

$$\int_{\varphi(K)} ||f^s|| > N |c^1_2 - c^1_1 - \alpha| \cdots |c^s_2 - c^s_1 - \alpha|,$$

for any N and α. Then since $\int_{\varphi(\boldsymbol{I}^s)} ||f^s|| \geqq \int_{\varphi(K)} ||f^s||$, $\int_{\varphi(\boldsymbol{I}^s)} ||f^s||$ should be equal to ∞. This contradicts to the assumption.

Definition. *We call* f^s *is uniformly integrable on* $\varphi(\boldsymbol{I}^s)$ *if for any* $\boldsymbol{I}_\delta^1 \times \cdots \times \boldsymbol{I}_\delta^s$ *we have*

(13) $$\int_{\varphi(I^s)} f^s = \lim_{\delta\to 0} \int_{\varphi(I_\delta^1 \times \cdots \times I_\delta^s)} f^s.$$

For example, if f^s is regular and $f(\varphi(t_1, \cdots, t_s), \varphi(t_{11}, \cdots, t_{1s}), \cdots, \varphi(t_{s1}, \cdots, t_{ss}))$ is Lipschitz continuous in t_{ii}, $i = 1, \cdots, s$, then f^s is uniformly integrable on $\varphi(I^s)$ if $\int_{\varphi(I^s)} f^s$ exists.

By lemma 2, we have

Lemma 2'. *If f^s is absolutely and uniformly integrable on $\varphi(I^s)$, then for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, $N = N(\varepsilon) > 0$ and $\alpha = \alpha(\varepsilon) > 0$ such that*

(14) $$\left|\left|\int_{\varphi(I^s)} f^s - \sum_{a_J \in I_\delta^1 \times \cdots \times I_\delta^s} f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s}))\right|\right| < \varepsilon,$$

$$||f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s}))||$$
$$\leqq N|a_{1,j_1+1} - a_{1,j_1}| \cdots |a_{s,j_s+1} - a_{s,j_s}|,$$
$$\textit{if } a_J \in I_\delta^1 \times \cdots \times I_\delta^s \textit{ and } |a_{i,j_i+1} - a_{i,j_i}| < \alpha.$$

9. ***Theorem 4.*** *Let f^{s-1} be a continuous alternative regular $(s-1)$-cochain and $\gamma = \sum \alpha_i \varphi_i(I^s)$ is a singular s-chain such that*

(i). *$(\delta f)^s$ is absolutely and uniformly integrable on each $\varphi_i(I^s)$.*

(ii). *f^{s-1} is absolutely and uniformly integrable on each singular simplex of $\partial\varphi_i(I^s)$.*

Then we have

(15) $$\int_\gamma (\delta f)^s = \int_{\partial\gamma} f^{s-1}.$$

Proof. By definition, to show (15), it is sufficient to show

(15)' $$\int_{\varphi(I^s)} (\delta f)^s = \int_{\partial\varphi(I^s)} f^{s-1}.$$

To calculate this right hand side, we set

$$\int_{\varphi(I^s)} (\delta f)^s$$
$$= \lim_{|a_{J+1_i} - a_J| \to 0} \sum_{J=(0,\cdots,0)}^{(n_1,\cdots,n_s)} \{f(\varphi(a_{J+1_1}, \cdots, \varphi(a_{J+1_s})) -$$
$$- f(\varphi(a_J), \varphi(a_{J+1_2}), \cdots, \varphi(a_{J+1_s})) + \cdots +$$
$$+ (-1)^k f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_{k-1}}), \varphi(a_{J+1_{k+1}}),$$
$$, \cdots, \varphi(a_{J+1_s})) + \cdots + (-1)^s f(\varphi(a_J), \varphi(a_{J+1_1}), \cdots,$$
$$\varphi(a_{J+1_{s-1}}))\}.$$

Then, since the limit exists for any partition of I^s, we may set $n_1 = \cdots = n_s = n$

in this right hand side. Then we get

$$\sum_{J=(0,\cdots,0)}^{(n,\cdots,n)}\{f(\varphi(a_{J+1_1}),\cdots,\varphi(a_{J+1_s}))-f(\varphi(a_J),\varphi(a_{J+1_2}),\cdots,$$
$$\varphi(a_{J+1}))+\cdots+(-1)^s f(\varphi(a_J),\varphi(a_{J+1_1}),\cdots,\varphi(a_{J+1_{s-1}}))\}$$
$$=\sum_{k=1}^{s}\{\sum_{J(k)=(0,\cdots,0)}^{(n-1,\cdots,\overset{(k-1)}{n-1},\overset{(k+1)}{n},\cdots,n)} f(\varphi(a_{J+1_1}|_{k=n}),\cdots,\varphi(a_{J+1_s}|_{k=n}))\}+$$
$$+\sum_{k=1}^{s}\{\sum_{J(k)=(0,\cdots,0)}^{(n,\cdots,n)}(-1)^k f(\varphi(a_J|_{k=0}),\varphi(a_{J+1_1}|_{k=0}),\cdots,\varphi(a_{J+1_{k-1}}|_{k=0})$$
$$\varphi(a_{J+1_{k+1}}|_{k=0}),\cdots,\varphi(a_{J+1_s}|_{k=0}))\}+$$
$$+\sum_{J=(0,\cdots,0)}^{(n-1,\cdots,n-1)} f(\varphi(a_{J+1_1}),\cdots,\varphi(a_{J+1_s}))+$$
$$+\sum_{k=1}^{s}\{\sum_{J=(0,\cdots,\underset{(k)}{0,1},0,\cdots,0)}^{(n,\cdots,n)}(-1)^k f(\varphi(a_J),\varphi(a_{J+1_1}),\cdots,\varphi(a_{J+1_{k-1}}),$$
$$\varphi(a_{J+1_{k+1}}),\cdots,\varphi(a_{J+1_s}))\}.$$

Here J_k and $a_J|_{k=m}$ mean

$$J_k=(j_1,\cdots,j_{k-1},j_{k+1},\cdots,j_s),$$
$$a_J|_{k=m}=(a_{1,j_1},\cdots,a_{k-1,j_{k-1}},a_{k,m},a_{k+1,j_{k+1}},\cdots,a_{s,j_s}).$$

We note that in this notation, $a_{J+1_k}|_{k=m}$ means $(a_{1,j_1},\cdots,a_{k-1,j_{k-1}},a_{k,m+1},a_{k+1,j_{k+1}},\cdots,a_{s,j_s})$. Hence we have

$$a_{J+1_k}|_{k=n}=a_J|_{k=n+1},$$

and therefore we get by the alternativity

$$f(\varphi(a_{J+1_1}|_{k=n}),\cdots,\varphi(a_{J+1_k}|_{k=n}),\cdots,\varphi(a_{J+1_s}|_{k=n}))$$
$$=(-1)^{k-1}f(\varphi(a_J|_{k=n+1}),\varphi(a_{J+1_1}|_{k=n}),\cdots,\varphi(a_{J+1_{k-1}}|_{k=n}),$$
$$\varphi(a_{J+1_{k+1}}|_{k=n}),\cdots,\varphi(a_{J+1_s}|_{k=n})).$$

Hence we have

$$\lim_{|a_{J+1_i}-a_J|\to 0}[\sum_{k=1}^{s}\{\sum_{J_k=(0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)} f(\varphi(a_{J+1_1}|_{k=n}),\cdots,\varphi(a_{J+1_s}|_{k=n}))\}+ \tag{16}$$
$$+\sum_{k=1}^{s}\{\sum_{J_k=(0,\cdots,0)}^{(n,\cdots,n)}(-1)^k f(\varphi(a_J|_{k=0}),\varphi(a_{J+1_1}|_{k=0}),\cdots,\varphi(a_{J+1_{k-1}}|_{k=0}),$$

$$\varphi(a_{J+1_{k+1}}|_{k=0}),\ \cdots,\ \varphi(a_{J+1_s}|_{k=0}))$$
$$=\int_{\partial\varphi(I^s)} f^{s-1}.$$

On the other hand, we get

$$\sum_{J=(0,\cdots,0)}^{(n-1,\cdots,n-1)} f(\varphi(a_{J+1_1}),\ \cdots,\ \varphi(a_{J+1_s}))+$$
$$+\sum_{k=1}^{s}\{\sum_{J=(0,\cdots,0,1,0,\cdots,0)}^{(n,\cdots,n)} (-1)^k f(\varphi(a_J),\ \varphi(a_{J+1_1}),\ \cdots,\ \varphi(a_{J+1_{k-1}}),$$
$$\varphi(a_{J+1_{k+1}}),\ \cdots,\ \varphi(a_{J+1_s}))\}$$
$$=\sum_{J=(0,\cdots,0)}^{(n-1,\cdots,n-1)}\{f(\varphi(a_{J+1_1}),\ \cdots,\ \varphi(a_{J+1_s}))-f(\varphi(a_{J+1_1}),\ \varphi(a_{J+1_1+1_2}),$$
$$,\ \cdots,\ \varphi(a_{J+1_1+1_s}))+\cdots+(-1)^k f(\varphi(a_{J+1_k}),$$
$$\varphi(a_{J+1_k+1_1}),\ \cdots,\ \varphi(a_{J+1_k+1_{k-1}}),\varphi(a_{J+1_k+1_{k+1}}),$$
$$,\ \cdots,\ \varphi(a_{J+1_k+1_s}))+\cdots+(-1)^s f(\varphi(a_{J+1_s}),$$
$$\varphi(a_{J+1_s+1_1}),\ \cdots,\ \varphi(a_{J+1_s+1_{s-1}}))\}+$$
$$+\sum_{k=1}^{s}[\sum_{l<k}\{\sum_{J(l)=(0,\cdots,0,\underset{(k)}{1},0,\cdots,0)}^{(n-1,\cdots,\overset{(l-1)(l+1)}{n-1,n},\cdots,n)} (-1)^k f(\varphi(a_{J+1_k}|_{k=n}),$$
$$\varphi(a_{J+1_k+1_1}|_{l=n}),\ \cdots,\ \varphi(a_{J+1_k+1_{k-1}}|_{l=n}),$$
$$\varphi(a_{J+1_k+1_{k+1}}|_{l=n}),\cdots,\varphi(a_{J+1_k+1_s}|_{l=n}))\}+$$
$$+\sum_{k<l}\{\sum_{J(l)=0,\cdots,0,\underset{(k)}{1},0,\cdots,0)}^{(n-1,\cdots,\overset{(k)(l-1)(l+1)}{n-1,n,n-1},\cdots,n-1,n,\cdots,n)} (-1)^k f(\varphi(a_{J+1_k}|_{l=n}),$$
$$\varphi(a_{J+1_k+1_1}|_{l=n}),\ \cdots,\ \varphi(a_{J+1_k+1_s}|_{l=n}))\}].$$

Here $a_{J+1_k+1_m}$ means (assuming k < m)

$$a_{J+1_k+1_m}$$
$$=(a^{1,j_1,\cdots,}a_{k-1,j_{k-1}},\ a_{k,j_k+1},\ a_{k+1,j_{k+1}},\ \cdots,\ a_{m-1,j_{m-1}},\ a_{m,j_m+1},$$
$$a_{m+1,j_{m-1}},\cdots,a_{s,j_s}).$$

Then we set

(17)
$$\sum_{k=1}^{s}[\sum_{l<k}\{\sum_{J(l)=(0,\cdots,0,1,0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)} (-1)^k f(\varphi(a_{J+1_k}|_{l=n}),$$
$$\varphi(a_{J+1_k+1_1}|_{l=n}),\ \cdots,\ \varphi(a_{J+1_k+1_s}|_{l=n}))\}+$$
$$+\sum_{k<l}\{\sum_{J(l)=(0,\cdots,0,1,0,\cdots,0)}^{(n-1,\cdots,n-1,n,n-1,\cdots,n-1,n,\cdots,n)} (-1)^k f(\varphi(a_{J+1_k}|_{l=n}),$$

$$\varphi(a_{J+1_k+1_1}|_{l=n}), \cdots, \varphi(a_{J+1_k+1_s}|_{l=n}))\}]$$

$$=\sum_{k=1}^{s}\sum_{j_k=1}^{n}[\sum_{l<k}\sum_{J(l,k)=0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}+$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}].$$

Here $a_J|_{l=p,k=q}$ and $J_{(l,k)}$ mean

$$a_J|_{l=p,k=q} =(a_{1,j_1}, \cdots, a_{l-1,j_{l-1}}, a_{l,p}, a_{l+1,j_{l+1}}, \cdots, a_{k-1,j_{k-1}}, a_{k,q}, a_{k+1,j_{k-1}}, \cdots, a_{s,j_s}),$$

$$J_{(l,k)} = (j_1, \cdots, j_{l-1}, j_{l+1}, \cdots, j_{k-1}, j_{k+1}, \cdots, j_s).$$

On the other hand, since f^{s-1} is continuous and integrable, we have

(18)
$$\lim_{|a_{J+1_i}-a_J|\to 0}\sum_{k=1}^{s}\sum_{j_k=1}^{n}[\sum_{l<k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))] +$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-_,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}]$$

$$= \lim_{|a_{J+1_i}-a_J|\to 0}\sum_{k=1}^{s}\sum_{j_k=1}^{s}[\sum_{l>k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}+$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}].$$

Then we obtain by the alternativity

$$\sum_{k=1}^{s}\sum_{j_k=1}^{n}[\sum_{l<k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\} +$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=n,k=j_k+1}),$$

$$, \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}]$$

$$=\sum_{k=1}^{s}\sum_{j_k=1}^{n}[\sum_{l<k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}),$$

$$\varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_l}|_{l=n,k=j_k+1}), \cdots,$$

$$\varphi(a_{J+1_s}|_{l=n,k=j_k+1})) - f(\varphi(a_J|_{l=n,k=j_k+1}),$$

$$\varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_{k-1}}|_{l=n,k=j_k+1}),$$

$$\varphi(a_{J+1_l}|_{l=n,k=j_k+1}), \varphi(a_{J+1_{k+1}}|_{l=n,k=j_k+1}), \cdots,$$

$$\varphi(a_{J+1_{l-1}}|_{l=n,k=j_k+1}), \varphi(a_{J+1_{l+1}}|_{l=n,k=j_k+1}), \cdots,$$

$$\varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}].$$

Then by (14) and the continuity of f, we have

$$||\sum_{k=1}^{s}\sum_{j_k=1}^{n}[\sum_{l<k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}),$$

$$\varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\} +$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-1)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \varphi(a_{J+1_1}|_{l=,k=j_k+1}),$$

$$, \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}]||$$

$$\leqq \varepsilon + 2N\alpha,$$

where N depends on ε but α is independent to N. Hence we get

$$\lim_{|a_{J+1_i}-a_J|\to 0}\sum_{k=1}^{s}\sum_{j_k=1}^{m}[\sum_{l<k}\{\sum_{J(l,k)=(0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}), \tag{19}$$

$$\varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\} +$$

$$+\sum_{k<l}\{\sum_{J(k,l)=(0,\cdots,0)}^{(n-1,\cdots,n-1,n,\cdots,n)}(-1)^k f(\varphi(a_J|_{l=n,k=j_k+1}),$$

$$\varphi(a_{J+1_1}|_{l=n,k=j_k+1}), \cdots, \varphi(a_{J+1_s}|_{l=n,k=j_k+1}))\}]$$

$$= 0.$$

On the other hand, since

$$\sum_{J=(0,\cdots,0)}^{(n-1,\cdots,n-1)}\{f(\varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) - f(\varphi(a_{J+1_1}), \varphi(a_{J+1_1+1_2}),$$

$$, \cdots, \varphi(a_{J+1_1+1_s})) + \cdots + (-1)^s f(\varphi(a_{J+1_s}),$$

$$\varphi(a_{J+1_s+1_1}), \cdots, \varphi(a_{J+1_s+1_{s-1}}))\}$$

$$= \sum_{J=(0,\cdots,0)}^{(n-2,\cdots,n-2)}\{f(\varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) - f(\varphi(a_{J+1_1}), \varphi(a_{J+1_1+1_2}),$$

$$, \cdots, \varphi(a_{J+1_1+1_s})) + \cdots + (-1)^s f(\varphi(a_{J+1_s}),$$

$$\varphi(a_{J+1_s+1_1}), \cdots, \varphi(a_{J+1_s+1_{s-1}}))\} +$$

$$+\sum_{k=1}^{s}[\sum_{J(k)=0,\cdots,0}^{(n-2,\cdots,\overset{(k-1)}{n-2},\overset{(k+1)}{n-1},\cdots,n-1)}\{f(\varphi(a_{J+1_1}|_{k=n-1}), \cdots,$$

$$\varphi(a_{J+1_s}|_{k=n-1})) - f(\varphi(a_{J+1_1}|_{k=n-1}), \varphi(a_{J+1_1+1_2}|_{k=n-1}), \cdots,$$

$$\varphi(a_{J+1_1+1_s}|_{k=n-1})) + \cdots + (-1)^s f(\varphi(a_{J+1_s}|_{k=n-1}),$$

$$\varphi(a_{J+1_s+1_1}|_{k=n-1}), \cdots, \varphi(a_{J+1_s+1_{s-1}}|_{k=n-1}))\}],$$

we get

$$\sum_{J=(0,\cdots 0)}^{(n-1,\cdots,n-1)}\{f(\varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_s})) - f(\varphi(a_{J+1_1}), \varphi(a_{J+1_1+1_2}), \tag{20}$$

$$, \cdots, \varphi(a_{J+1_1+1_s})) + \cdots + (-1)^s f(\varphi(a_{J+1_s}),$$

$$\varphi(a_{J+1_s+1_1}), \cdots, \varphi(a_{J+1_s+1_{s-1}}))\}$$

$$=\sum_{l=1}^{n-1}[\sum_{k=1}^{s}[\sum_{J(k)=(0,\cdots,0)}^{(n-l-1,\cdots,\overset{(k-1)}{n-l-1},\overset{(k+1)}{n-l},\cdots,n-l)}\{f(\varphi(a_{J+1_1}|_{k=n-l}), \cdots,$$

$$\varphi(a_{J+1_s}|_{k=n-l})) - f(\varphi(a_{J+1_1}|_{k=n-l}), \varphi(a_{J+1_1+1_2}|_{k=n-l}),$$

$$, \cdots, \varphi(a_{J+1_1+1_s}|_{k=n-l})) + \cdots + (-1)^s f(\varphi(a_{J+1_s}|_{k=n-l}),$$

$$\varphi(a_{J+1_s+1_1}|_{k=n-l}), \cdots, \varphi(a_{J+1_s+1_{s-1}}|_{k=n-l}))\}]].$$

Then, since f^{s-1} is continuous and integrable, we have

$$\lim_{|a_{J+1_i}-a_J|\to 0}\sum_{l=1}^{n-1}[\sum_{k=1}^{s}[\sum_{J(k)=(0,\cdots,0)}^{(n-l-1,\cdots,n-l-1,n-l,\cdots,n-l)}\{f(\varphi(a_{J+1_1}|_{k=n-l}), \tag{18$'$}$$

$$, \cdots, \varphi(a_{J+1_s}|_{k=n-l})) - f(\varphi(a_{J+1_1}|_{k=n-l}),$$

$$\varphi(a_{J+1_1+1_2}|_{k=n-l}), \cdots, \varphi(a_{J+1_s+1_{s-1}}|_{k=n-l})) +$$

$$+ \cdots + (-1)^s f(\varphi(a_{J+1_s}|_{k=n-l}), \varphi(a_{J+1_s+1_1}|_{k=n-l}),$$

$$, \cdots, \varphi(a_{J+1_s+1_{s-1}}|_{k=n-l}))\}]]$$

$$= \lim_{|a_{J+1_i}-a_J|\to 0}\sum_{l=1}^{n-1}[\sum_{k=1}^{s}[\sum_{J(k)=(0,\cdots,0)}^{(n-l-1,\cdots,n-l-1)}\{f(\varphi(a_{J+1_1}|_{k=n-l}), \cdots,$$

$$\varphi(a_{J+1_s}|_{k=n-l})) - f(\varphi(a_{J+1_1}|_{k=n-l}),$$

$$\varphi(a_{J+1_1+1_2}|_{k=n-l}), \cdots, \varphi(a_{J+1_1+1_s}|_{kk=n-l})) +$$

$$+ \cdots +(-1)^s f(\varphi(a_{J+1_s}|_{k=n-l}), \varphi(a_{J+1_s+1_1}|_{k=n-l}),$$

$$, \cdots, \varphi(a_{J+1_s+1_{s-1}}|_{k=n-l}))\}]].$$

Then, since f^{s-1} is alternative, we obtain

$$\sum_{l=1}^{n-1}\Big[\sum_{k=1}^{s}\Big[\sum_{J(k)=(0,\cdots,0)}^{(n-l-1,\cdots,n-l-1)}\{f(\varphi(a_{J+1_1}|k=n-l),\ \cdots,\varphi(a_{J+1_s}|k=n-l))-$$
$$-f(\varphi(a_{J+1_1}|k=n-l),\ \varphi(a_{J+1_1+1_2}|k=n-l),\ \cdots,\ \varphi(a_{J+1_1+1_s}|k=n-l))+$$
$$+\cdots+(-1)^s f(\varphi(a_{J+1_s}|k=n-l),\ \varphi(a_{J+1_s+1_1}|k=n-l),\ \cdots,$$
$$\varphi(a_{J+1_s+1_{s-1}}|k=n-l))\}\Big]\Big]$$
$$=\sum_{l=1}^{n-1}\Big[\sum_{J(1)=(0,\cdots,0)}^{(n-l-1,\cdots,n-l-1)}\Big[\{f(\varphi(a_{J+1_1}|1=n-l),\ \cdots,\ \varphi(a_{J+1_s}|1=n-l))-$$
$$-f(\varphi(a_{J+1_1}|1=n-l),\ \varphi(a_{J+1_1+1_2}|1=n-l),\ \cdots,\ \varphi(a_{J+1_1+1_s}|1=n-l))+$$
$$+\cdots+(-1)^s f(\varphi(a_{J+1_s}|1=n-l),\ \varphi(a_{J+1_s+1_1}|1=n-l),\ \cdots,$$
$$\varphi(a_{J+1_s+1_{s-1}}|1=n-l)-f(\varphi(a_{J+1_2}|1=n-l),\ \varphi(a_{J+1_1}|1=n-l),\ \cdots,$$
$$\varphi(a_{J+1_s}|1=n-l))-f(\varphi(a_{J+1_2}|1=n-l),\ \varphi(a_{J+1_2+1_1}|1=n-l),\ \cdots,$$
$$\varphi(a_{J+1_2+1_s}|1=n-l))+\cdots+(-1)^s f(\varphi(a_{J+1_s}|1=n-l),$$
$$\varphi(a_{J+1_s+1_1}|1=n-l),\ \cdots,\ \varphi(a_{J+1_s+1_{s-1}}|1=n-l))\}+\cdots+$$
$$+(-1)^k\{f(\varphi(a_{J+1_k}|1=n-l),\ \varphi(a_{J+1_2}|1=n-l),\ \cdots,\ \varphi(a_{J+1_{k-1}}|1=n-l),$$
$$\varphi(a_{J+1_1}|1=n-l),\ \varphi(a_{J+1_{k+1}}|1=n-l),\ \cdots,\ \varphi(a_{J+1_s}|1=n-l))+\cdots+$$
$$+(-1)^s f(\varphi(a_{J+1_s}|1=n-l),\ \varphi(a_{J+1_s+1_k}|1=n-l),\ \cdots,$$
$$\varphi(a_{J+1_s+1_{s-1}}|1=n-l))\}+\cdots+(-1)^s\{f(\varphi(a_{J+1_s}|1=n-l),$$
$$\varphi(a_{J+1_2}|1=n-l),\ \cdots,\ \varphi(a_{J+1_{s-1}}|1=n-l),\ \varphi(a_{J+1_1}|1=n\varphi))+\cdots+$$
$$+(-1)^s f(\varphi(a_{J+1_1}|1=n-l),\ \cdots,\ \varphi(a_{J+1_1+1_{s-1}}|1=n-l))\}\Big]\Big].$$

Hence we get by (14), the uniform integrability and the continuity of f

$$\Big|\Big|\sum_{l=1}^{n-1}\Big[\sum_{k=1}^{s}\Big[\sum_{J(k)=(0,\ \cdots,\ 0)}^{(n-l-1,\cdots,n-l-1)}\{f(\varphi(a_{J+1_1}|k=n-l),\ \cdots,\ \varphi(a_{J+1_s}|k=n-l))-$$
$$-f(\varphi(a_{J+1_1}|k=n-l),\ \varphi(a_{J+1_1+1_2}|k=n-l),\ \cdots,$$
$$\varphi(a_{J+1_1+1_s}|k=n-l))+\cdots+(-1)^s f(\varphi(a_{J+1_s}|k=n-l),$$
$$\varphi(a_{J+1_s+1_1}|k=n-l),\ \cdots,\ \varphi(a_{J+1_s+1_{s-1}}|k=n-l))\Big]\Big]\Big|\Big|$$
$$\leqq \varepsilon+s(s+1)N\alpha+\beta,$$

where N depends on ε but α and β are independent to N. Hence we get by (18)′

$$(19)'\qquad \lim_{|a_{J+1_i}-a_J|\to 0}\sum_{l=1}^{n-1}\Big[\sum_{k=1}^{s}\Big[\sum_{J(k)=(0,\cdots,0)}^{(n-l-1,\cdots,n-l-1,n-l,\cdots,n-l)}\{f(\varphi(a_{J+1_1}|k=n-l),$$
$$,\ \cdots,\ \varphi(a_{J+1_s}|k=n-l))-f(\varphi(a_{J+1_1}|k=n-l),$$
$$\varphi(a_{J+1_1+1_2}|k=n-l),\ \cdots,\ \varphi(a_{J+1_1+1_s}|k=n-l))+$$
$$+\cdots+(-1)^s f(\varphi(a_{J+1_s}|k=n-l),\ \varphi(a_{J+1_s+1_1}|k=n-l),$$

$$, \cdots, \varphi(a_{J+1_s+1_{s-1}} | _{k=n-l}))\}]]$$

$$= 0.$$

By (16), (19) and (19)′, weobtain (15)′. There fore we have the theorem.

Note. In this proof, we use the alternativity of f^{s-1} only in $x_1, \cdots, x_{s-1}$. Hence we have the Stokes' theorem for those f^{s-1} that are alternative in $x_1, \cdots, x_{s-1}$.

10. If $s=1$, then we have for any f

$$\int_{\varphi(I^1)} \delta f = \lim_{|a_{i+1}-a_i| \to 0} \sum f(\varphi(a_{i+1})) - f(\varphi(a_i))$$
$$= f(\varphi(1)) - f(\varphi(0)).$$

Hence if $s=1$, we have

$$\int_r (\delta f)^1 = \int_{\partial r} f^0, \tag{21}$$

for any f.

Example. If $X = I^1$ and φ is the identity, then for a (Riemannian integrable) function f, taking $F(x)$ such that

$$\frac{dF(x)}{dx} = f(x),$$

we have

$$\int_{I^1} (\delta F)(x, \ y) = F(1) - F(0).$$

On the other hand, since we obtain

$$(\delta F)(x, \ y) = F(y) - F(x) = f(x)(y-x) + o(|y-x|),$$

we have

$$\int_{I^1} (\delta F)(x, \ y) = \int_0^1 f(x) dx,$$

where the right hand side is the Riemannian integral of $f(x)$.

§ 4. Volume element with respect to a metric.

11. If X is an n-dimensional CW-complex, then to fix its CW-complex structure, we may consider $\int_X f^n$ for any n-cochain f^n of X. Especially, if the topology of X is given by a metric r, then we may consider $\int_X v^n$, where v^n is the volume element of X with respect to the metric r given by (11).

Note. To define v, it is sufficient that r is defined only on some neighborhood $U(\Delta(X))$ of $\Delta(X)$ in $X\times X$. We call such r to be a local metric of X.

Lemma 3. *If X is a paracompact topological manifold with manifold structure $\{(U, h_U)\}$, h_U is the homeomorphism from U onto $\boldsymbol{R}^n$ by which the manifold structure of X is given, then X has a local metric r such that*

$$\int_{h_{U,\lambda}(\boldsymbol{I}^n)} v^n \neq \infty \tag{22}$$

for all U and $\lambda>0$. Here v^n is the volume element with respect to r, and $h_{U,\lambda}$ is given by

$$h_{U,\lambda}(\xi)=h_U(\lambda^{-1}\xi),\quad \xi\in \boldsymbol{I}^n.$$

Proof. We may assume $\{U\}$ is a locally finite covering of X and take a partition of unity $\{e_U(x)\}$ corresponding to $\{U\}$. Then setting

$$r(x, y)=\sum_U e_U(x)e_U(y)\,||h_U(x)-h_U(y)||,\quad x,\ y\in U, \tag{23}$$

r is the local metric which satisfies (22). Here $||\xi||$ is the (euclidean) norm of $\xi\in\boldsymbol{R}^n$.

Note. Similarly, setting

$$\begin{aligned} &V^n(x_0, x_1, \cdots, x_n) \\ &=\sum_U e_U(x_0)e_U(x_1)\cdots e_U(x_n)P_1(h_U(x_1)-h_U(x_0))\cdots P_n(h_U(x_n)-h_U(x_0)), \\ &P_i(\xi)=\xi_i,\quad \xi=(\xi_1, \cdots, \xi_n), \end{aligned} \tag{24}$$

we obtain an alternative (non-trivial) volume element on X, a paracompact manifold.

12. ***Lemma 4.*** *If $X=\{U, h_U\}$ is a paracompact oriented manifold, then there exstis a measure m on X such that*

(25) *$h_U{}^*(m)$ and the Lebesgue measure of $\boldsymbol{R}^n$ are biabsolutely continuous each other for any U.*

Proof. We denote the Lebesgue measure of $\boldsymbol{R}^n$ by μ. Then we define a measure m_U on X by

$$m_U(M)=\mu(h_U(U\cap M)),$$

and to define m by

$$m(M)=\sum_U\int_M e_U(x)dm_U,$$

where $\{e_U(x)\}$ is the partition of unity corresponding to $\{U\}$. Then m satisfies (25).

Lemma 5. *If $X=\{(U,\ h_U)\}$ is an n-dimensional oriented paracompact manifold with a connection t (cf. [4]), then X has a measure m such that*

(i) *m is invariant under the operation of t,*

(ii) *m satisfies (25),*

if and only if the connected component of the identity of the structure group of the tangent microbundle of X is reduced to the group of the germs of those homeomorphisms of R^n which preserve the Lebesgue measure of R^n as an $H_*(n)$-bundle.

Proof. If there exists a measure m on X which satisfy (i) and (ii), then setting

$$m_{U,x}=h_{U,x}{}^*(m),$$

we have

$$s_U(x,\ y)^*m_{U,y}=m_{U,x},$$

where $h_{U,x}(y)=h_U(y)-h_U(x)$ and $s_U(x,y)=h_{U,x}t(x,\ y)h_{U,y}{}^{-1}$.

Then denoting the Lebesgue measure of R^n by μ, we get by the theorem of Radon-Nykodim ([8]),

$$(26)'\qquad d_\mu(s_U(x,y))s_U(x,\ y)^*d_\mu m_{U,y}=d_\mu m_{U,x}.$$

By (26)', we obtain by (ii),

$$(26)\qquad d_\mu(s_U(x,\ y))=\frac{d_\mu m_{U,x}}{s_U(x,\ y)^*d_\mu m_{U,y}}.$$

By (26), we have

$$(27)\qquad d_\mu(s_U(y,\ z)s_U(x,\ z)^{-1}s_U(x,\ y))=1.$$

This shows the connected component of the identity of the structure group of the tangent microbundle of X is reduced to the group of the germs of the Lebesgue measure preserving homeomorphisms of R^n.

On the other hand, since the sheaf of germs of (Lebesgue) measurable positive Alexander-Spanier cochains with the operation of $t(x,\ y)$ is fine, if (27) is hold for $s_U(x,\ y)$, then we may set

$$(28)\qquad d_\mu(s_U(x,\ y))=f_{U,x}(s_U(x,\ y)^*f_{U,y})^{-1}.$$

Since there are measurable transformation $\varphi_{U, x}$ of $\boldsymbol{R}^n$ such that

$$d\varphi_{U, x} = f_{U, x},$$

we construct m as follows: Take a measure m_1 on X which satisfies (ii). Then setting

$$m_{1, U, x} = q_U(x, y) m_{1, U, y},$$

we have $q_U(x, y) = r_U(x)^{-1} r_U(y)$. Then to define a measure m_U on U by

$$m_U = h_{U, x}^{-1}(\varphi_{U, x}{}^* r_U(x) m_{1, U, x}),$$

m_U is invariant under the operation of t. Moreover, to set

$$m_U = k_{UV} m_V,$$

$\{k_{UV}\}$ is the orientation class of X (regarding its (Cech) cohomology class to be an element of $H^1(X, \boldsymbol{Z}_2)$). Hence we may consider $m_U = m_V$, because X is orientable.

Note. In the first part of this proof, we need not the orientability of X.

13. ***Theorem 5.*** *Let X be an n-dimensional topological manifold with connection t. Then if $n \neq 4, 5$, the connected component of the identity of the structure group of the tangent microbundle of X is reduced to the group of the gersm of the Lebesgue measure preserving homeomorphisms of $\boldsymbol{R}^n$ if X has a local metric r such that*

(i) $$r(y, z) = r(t(x_0, x_1)y, t(x_0, x_1)z),$$

(ii) $$\int_{\varphi_{U, \lambda(I^n)}} v^n \neq \infty \text{ for some non-degenerated } \varphi_U \text{ for any } U \text{ and } \lambda > 0.$$

Here v^n means the volume element of X with respect to r and $\varphi_{U, \lambda}$ is given by

$$\varphi_{U, \lambda}((t_1, \cdots, t_n)) = \varphi_U((\lambda t_1, \cdots, \lambda t_n)).$$

Proof. By (ii), v^n defines a measure on X which satisfies (25) because X allows the structure of CW-complex if dim. $X \neq 4, 5$. Since this measure is invariant under the operation of t by (i), we have the theorem by lemma 4.

Note. Since a connection $t(x_0, x_1)$ is written as

$$(t(x_0, x_1) | W(\Delta(U)))(y)$$
$$= r_U(x_0, x_1)(h_U^{-1}(h_U(y) + h_U(x_0) - h_U(x_1))),$$

$r_U(x_0, x_1)$ *is a local homeomorphism of X which fix es x_1,*

locally (cf. 4), we obtain the invariant local metric of X if there is a local metric $\rho^U{}_{s, t}$ of $\boldsymbol{R}^n$ such that

$$\rho^U_{s,t}(\xi, \eta) = \rho^U_{s,t}(\varphi_U(s, t)(\xi + \zeta), \varphi_U(s, t)(\eta + \zeta)),$$
$$\varphi_U(s, t) = h_U r_U(x_0, x_1) h_U^{-1}, \ \zeta = h_U(s) - h_U(t),$$

for any U and s, t.

§ 5. Singular integrai operators on CW-complexes.

14. We assume that X is an n-dimensional CW-complex and we fix the CW-complex structure of X.

We denote by $\mathscr{E}$ and $\mathscr{F}$ vector bundles over X with fibres E and F. Then since a CW-complex is paracompact ([10]), we may assume $\mathscr{E}$ and $\mathscr{F}$ both defined by a locally finite open covering $\{U\}$ of X. The transition functions of $\mathscr{E}$ and $\mathscr{F}$ by this covering are denotedby $\{g_{UV}\}$ and $\{h_{UV}\}$. For the convenience, we assume that the transition functions of $\mathscr{E}$ and $\mathscr{F}$ both operate from the right.

We denote the space of (continuous) linear homomorphisms from E into F by $L(E, F)$ and take a collection of continuous maps $\{k_U\}$, $k_U((x, y), x_1, \cdots, x_n)$: $(U \times U - \Delta(U)) \times W(\Delta_{n-1}(U)) \to L(E, F)$ such that

$$g_{UV}(x) k_V((x, y), x_1, \cdots, x_n) = k_U((x, y), x_1, \cdots, x_n) h_{UV}(y). \tag{29}$$

Difinition. *We call $\{k_U\}$ and $\{k_U'\}$ are equivalent if for some $V(\Delta_{n-1}(U))$,*

$$k_U | (U \times U - \Delta(U)) \times V(\Delta_{n-1}(U))$$
$$= k_U' | (U \times U - \Delta(U)) \times V(\Delta_{n-1}(U)), \ \textit{for all } U,$$

and denote this equivalence class by $\{\bar{k}_U\}$ or simply, $\{k_U\}$.

By (29), if $\{f_U(x)\}$ is a cross-section of $\mathscr{E}$, then $\{f_U(x) k_U((x, y), x_1, \cdots, x_n)\}$ satisfies

$$f_U(x) k_U((x, y), x_1, \cdots, x_n) = f_V(x) k_V((x, y), x_1, \cdots, x_n) h_{VU}(y). \tag{29$'$}$$

Definition. *If $\varphi(I^n)$ is contained in U, then we define the integral of $f_U k_U$ on $\varphi(I^n)$ by*

$$\int_{\varphi(I^n)} f_U(x) k_U((x, y), x_1, \cdots, x_n) = \lim_{|a_{J+1_i} - a_J| \to 0} \sum_J f_U(\varphi(a_J)) k_U((\varphi(a_J), y), \varphi(a_{J+1_1}), \cdots, \varphi(a_{J+1_n})). \tag{30}$$

Since X is a CW-complex, we may set

$$X = \sum_i \varphi_i(I^n), \ \varphi_i(I^n) \cap \varphi_j(I^n) \subset \varphi_i(\partial I^n) \cap \varphi_j(\partial I^n) \ \textit{if } i \neq j.$$

Then we define

$$(31)\qquad \int_X e_U(x)f_U(x)k_U((x,\ y),\ x_1,\ \cdots,\ x_n)$$
$$=\sum_{U\cap\varphi_i(I^n)\neq\phi}\int_{\varphi_i(I^n)} e_U(x)f_U(x)k_U((x,\ y),\ x_1,\ \cdots,\ x_n),$$
$$car.\, e_U(x)\subset U.$$

If $\{V\}$ is a locally finite refinement of $\{U\}$ and $\{e_U(x)\}$ and $\{e_V(x)\}$ are the partition of unities subordinated to $\{U\}$ and $\{V\}$, then we have

$$(32)\qquad \sum_U\int_X e_U(x)f_U(x)k_U((x,\ y),\ x_1,\ \cdots,\ x_n)$$
$$=\sum_V\int_X e_V(x)f_V(x)k_V((x,\ y),\ x_1,\ \cdots,\ x_n),$$
$$f_V(x)=f_U(x)|\,V,\quad V\subset U,$$
$$k_V((x,\ y),\ x_1,\ \cdots,\ x_n)$$
$$=k_U((x,\ y),\ x_1,\ \cdots,\ x_n)|(V\times V-\Delta(V))\times W(\Delta_{n-1}(U)).$$

By (32), setting $f(x)=\{f_U(x)\}$ and $k((x,\ y),\ x_1,\ \cdots,\ x_n)=\{k_U((x,\ y),\ x_1,\ \cdots,\ x_n)$, we define the integral of fk on X by

$$(33)\qquad \int_X f(x)k((x,\ y),\ x_1,\ \cdots,\ x_n)$$
$$=\sum_U\int_X e_U(x)f_U(x)k_U((x,\ y),\ x_1,\ \cdots,\ x_n),$$

where $\{e_U(x)\}$ is a partition of unity subordineted to $\{U\}$.

By definition, we have

Theorem 6. *If* $\int_X f(x)k((x,\ y),\ x_1,\cdots,\ x_n)$ *exists, then it isa cross-section of* $\mathscr{F}$.

We denote the spaces of cross-sections of $\mathscr{E}$ and $\mathscr{F}$ by $\Gamma(\mathscr{E})$ and $\Gamma(\mathscr{F})$, then by theorem 6, we can define the map $I(k)$, $I(k):\Gamma(\mathscr{E})\to\Gamma(\mathscr{F})$ by

$$(34)\qquad I(k)(f)=\int_X f(x)k((x,\ y),\ x_1,\ \cdots,\ x_n).$$

By definiton, if $I(k)(f)$ and $I(k)(g)$ both exist, then $I(k)\ (\alpha f+\beta g)$ exists for any scalar α, β and we have

$$(35)\qquad I(k)(\alpha f+\beta g)=\alpha I(k)(f)+\beta I(k)(g).$$

Note. Similarly, we define $I_A(k):\Gamma(\mathscr{E})\to\Gamma(\mathscr{F})$ by

$$(34)'\qquad I_A(k)(f)=\int_X A_n(f(x)k((x,\ y),\ x_1,\ \cdots,\ x_n),$$

where A_n means the alternation in $x_1, \cdots, x_n$.

15. On X, we fix a (real or complex valued) n-cochain $q(x_{11}, \cdots, x_n)$ and set

(36) $$k_U((x, y), x_1, \cdots, x_n) = k_U(x, y)q(x_1, \cdots, x_n).$$

Here $\{k_U(x, y)\}$ satisfies

(29)″ $$g_{UV}(x)k_V(x, y) = k_U(x, y)h_{UV}(y).$$

(29)″ shows

Lemma 6. *Setting $k = \{k_U(x, y)\}$ and define $p_i : X \times X \to X$, $i = 1, 2$ by*

$$p_1((x, y)) = x, \quad p_2((x, y)) = y,$$

k is a bundle map from $p_1^*(\mathscr{E})|(X \times X - \Delta(X))$ into $p_2^*(\mathscr{F})|(X \times X - \Delta(X))$.

On the other hand, to define $I(k)_q : \Gamma(\mathscr{E}) \to \Gamma(\mathscr{F})$ by

(34)″ $$I(k)_q(f) = \int_X f(x)k(x, y)q(x_1, \cdots, x_n),$$

we have

Lemma 6′. *$I(k)_q$ is defined for any bunydle map*
$k : p_1^*(\mathscr{E})|(X \times X - \Delta(X)) \to p_2^*(\mathscr{F})|(X \times X - \Delta(X))$.

By definition, we obtain

Theorem 7. *If X is compact, q is absolutely and uniformly integrable on X and k is a bundle map from $p_1^*(\mathscr{E})$ into $p_2^*(\mathscr{F})$ (defined on $X \times X$), then $I(k)_q$ is defined for any continuous section f of $\mathscr{E}$ and it is a compact operator regarding $\Gamma(\mathscr{E})$ and $\Gamma(\mathscr{F})$ to be the Banach spaces by the uniform convergence topology.*

By theorem 7, to treat $I(k)_q$, it should be useful to treat k *mod.* $Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F}))$. Here $Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F}))$ is the space of bundle maps from $p_1^*(\mathscr{E})$ into $p_2^*(\mathscr{F})$ defined on $X \times X$ and it is considered to be a subspace of the bundle maps from $p_1^*(\mathscr{E})|(X \times X - \Delta(X))$ into $p_2^*(\mathscr{F})|(X \times X - \Delta(X))$.

Lemma 7. *There is a (cannonical) isomorphism*
$\tau_{21} : p_1^*(\mathscr{E})|U(\Delta(X)) \to p_2^*(\mathscr{E})|U(\Delta(X))$, where $U(\Delta(X))$ is a suitable neighborhood of $\Delta(X)$ in $X \times X$.

Since we know

$$k \text{ mod. } Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F})) = ek \text{ mod. } Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F})),$$

where e is a continuous function on $X \times X$ such that

$$0 \leq e(x, y) \leq 1, \quad e(x, y) = 0, \quad (x, y) \notin U(\Delta(X)),$$
$$e(x, y) = 1, \quad (x, y) \in V(\Delta(X)), \quad \overline{V(\Delta(X))} \subset U(\Delta(X)),$$

and for ek, we can multiply τ_{21} for any
k *mod.* $Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F}))$. We denote it by $(k \text{ mod. } Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F}))\tau_{21}$.
Then we have

Lemma 8. *We can take as the representation of* $(k \text{ mod. } Hom(p_1^*(\mathscr{E}), p_2^*(\mathscr{F}))\tau_{21}$ *a bundle map* $k' : p_2^*(\mathscr{E})|(X\times X - \Delta(X)) \to p_2^*(\mathscr{F})|(X\times X - \Delta(X))$ *and the class of* k' *mod.* $Hom(p_2^*(\mathscr{E}), p_2^*(\mathscr{F}))$ *is unique. Here* $Hom(p_2^*(\mathscr{E}), p_2^*(\mathscr{F}))$ *means the space of bundle maps from* $p_2^*(\mathscr{E})$ *into* $p_2^*(\mathscr{F})$ *on* $X\times X$.

Definition. *We set*

$$\sigma(k) = k' \text{ mod. } Hom(p_2^*(\mathscr{E}), p_2^*(\mathscr{F})), \tag{37}$$

and call $\sigma(k)$ *the symbol of* k. *Here* k' *is determined for* k *by lemma 8.*

Definition. *For* $I(k)_q$, *we set*

$$\sigma(I(k)_q) = \sigma(k) \tag{38}$$

and call the symbol of $I(k)_q$ *(cf. [6], [11]).*

16. We take v as q in (36), where v is the volume element of X with respect to a (local) metric r of X.

By the definitions of v and integral, we obtain

Theorem 7. *If* X *is compact and* v *is absolutely and uniformly integrable on* X *and* $k(x, y)$ *satisfies for some* $M > 0$ *and* $(x, y) \in U(\Delta(X))$

$$||k(x, y)|| \leqq Mr(x, y)^{1-n}, \tag{39}$$

then $I(k)_v$ *is defined on* $\Gamma(\mathscr{E})$. *Here* $||\kappa||$ *means the norm of* κ *in* $L(E, F)$.

Similarly, we have

Theorem 7'. *If* X *is a compact (topological) manifold and allow the structure of CW-complex and* V *is given by (24), then* $I(k)_V$ *is defined on* $\Gamma(\mathscr{E})$ *if* $k(x, y)$ *satisfies*

$$||k(x, y)|| \leqq M \max_{(x, y)\in U\times U}(|h_U(x) - h_U(y)|^{1-n}), \tag{39'}$$

for some $M > 0$ *and* $(x, y) \in U(\Delta(X))$.

Note. Theorem 7' is true although we use $A_n(V)$ instead of V.

If X is a (topological) manifold, then $U(\Delta(X))$ allow the structure (of the tatal space) of (the tangent) microbundle τ, and denoting the projection of τ by p, we get

$$p^*(\mathscr{E}) = p_2^*(\mathscr{E})|U(\Delta(X)),\ p^*(\mathscr{F}) = p_2^*(\mathscr{F})|U(\Delta(X)). \tag{40}$$

Moreover, $U(\Delta(X)) - \Delta(X)$ is regarded to be the associated Thom complex of τ.

Hence we obtain

Theorem 8. *If X is a manifold, then $\sigma(k)$ is the bundle map from $p^*(\mathscr{E})$ into $p^*(\mathscr{F})$ on the associated Thom complex of the tangent microbundle of X (cf.* [5]).

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