JOUR. FAC. SCI., SHINSHU UNIV. Vol. 5, pp. 123–139, Dec. 1970

# *A Relative Form of Equivariant K-Theory*

By TOSHIMITSU MATSUDA

Department of Mathematics, Faculty of Science,
Shinshu University

(Received October 30, 1970)

**Introduction.** Let $G$ be a compact Lie group, $T$ a maximal torus of $G$, $W(G)$ the Weyl group of $G$ and $X$ a compact $G$-space. Then the following results on the equivariant $K$-theory will be required from [1] and [2].

**Theorem** ($A$). (i) *We have a ring homomorphism* $R(G)\longrightarrow R(T)$ *(by the restriction map) which is injective.* $R(G)$ *maps (bijectively) onto the ring of invariants of* $R(T)$ *under the action of* $W(G)$.

(ii) *The sequence*

$$0\longrightarrow K_G{}^*(X)\longrightarrow K_T{}^*(X)$$

*is split exact.*

((i) *is obtained from* 4.4 *of* [1] *and* (ii) *from Proposition* (4.9) *of* [2].)

Now the aim of this paper is to prove the following Theorem :

**Theorem** ($B$). *We have the following split exact sequences* :

$$0\longrightarrow K^*{}_G(X)\longrightarrow K^*{}_T(X)\longrightarrow K^*{}_{(G,T)}(X)\longrightarrow 0$$

$$0\longrightarrow K^*{}_G(X)\longrightarrow K^*{}_T(X)^{W(G)}\longrightarrow K^*{}_{(G,T)}(X)^{W(G)}\longrightarrow 0$$

*where* $K^*{}_{(G,T)}(X)$ *is defined in* §1 *and* $K^*{}_T(X)^{W(G)}$ *(respectively,* $K^*{}_{(G,T)}(X)^{W(G)}$*) is an abelian group of invariants of* $K^*{}_T(X)$ *(respectively,* $K^*{}_{(G,T)}(X)$*) under the action of* $W(G)$.

Note : It has been proved by Mr. HARUO MINAMI (to appear) that if $G=U(n)$ and $K^*{}_T(X)$ is torsion free, then $K^*{}_{U(n)}(X)$ and $K^*{}_T(X)^{W(U(n))}$ are isomorphic. So we predict the folloing result :

**Prediction.** If $K^*{}_T(X)$ is torsion free, then $K^*{}_T(X)^{W(G)}$ and $K^*{}_G(X)$ are isomorphic.

Throughout this paper $G$ will denote a compact Lie group, $H$ a closed subgroup of $G$, $X$ a compact $G$-space and $A$ a closed $G$-invariant subspace of $X$.

I am grateful to Professor S. ARAKI for his kind advice.

## § 1 Definition of $K_{(G,H)}(X, A)$

**1.1 Definition.** We define $L_{(G,H)}(X, A)$ to be a category as follows: An object of $L_{(G,H)}(X, A)$ is a pair $(E, F)$ of $G$-vector bundles over $X$, together with an $H$-isomorphism

$$\delta : I\times E|I\times A\cup\{0\}\times X \longrightarrow I\times F|I\times A\cup\{0\}\times X$$

such that $\delta|\{1\}\times A : E|A \longrightarrow F|A$ is a $G$-isomorphism.
The morphism $\varphi : \sigma_0 \longrightarrow \sigma_1$, where $\sigma_i=(E_i, F_i, \delta_i)$ $(i=0, 1)$, is a pair of $G$-homomorphisms $(f, g) : (E_0, F_0) \longrightarrow (E_1, F_1)$ such that the diagram

$$(1.1.1)\qquad \begin{array}{ccc} (I\times E_0)|I\times A\cup\{0\}\times X & \xrightarrow{\delta_0} & (I\times F_0)|I\times A\cup\{0\}\times X \\ \big\downarrow (id_I\times f)|I\times A\cup\{0\}\times X & & \big\downarrow (id_I\times g)|I\times A\cup\{0\}\times X \\ (I\times E_1)|I\times A\cup\{0\}\times X & \xrightarrow{\delta_1} & (I\times F_1)|I\times A\cup\{0\}\times X \end{array}$$

is commutative. From now on, we put $B=I\times A\cup\{0\}\times X$.

An elementary object in $L_{(G,H)}(X, A)$ is an object of the form $(E, E, id)$. If $\sigma_i=(E_i, F_i, \delta_i)$ $(i=0.1)$ are in $L_{(G,H)}(X, A)$, their sum is defined by

$$(1.1.2)\qquad \sigma_0\oplus\sigma_1=(E_0\oplus E_1,\ F_0\oplus F_1,\ \delta_0\oplus\delta_1).$$

Two objects $\sigma_0$ and $\sigma_1$ are homotopic in $L_{(G,H)}(X, A)$, in symbols

$$(1.1.3)\qquad \sigma_0\sim\sigma_1,$$

if there exists an object $\bar{\sigma}=(\bar{E}, \bar{F}, \bar{\delta})$ of $L_{(G,H)}(X\times I, A\times I)$ such that

$$\bar{\sigma}|\{0\}=\sigma_0 \text{ and } \bar{\sigma}|\{1\}=\sigma_1.$$

i, e. $\qquad \bar{E}|X\times\{i\}=E_i,\ \bar{F}|X\times\{i\}=F_i$ and $\bar{\delta}|B\times\{i\}=\delta_i$.

Two objects $\sigma_0$ and $\sigma_1$ are stably homotopic in $L_{(G,H)}(X, A)$, in symbols

$$(1.1.4)\qquad \sigma_0 \underset{s}{\sim} \sigma_1,$$

if there exist elementary objects $\tau_0$ and $\tau_1$ such that

$$\sigma_0\oplus\tau_0\sim\sigma_1\oplus\tau_1.$$

We shall write $[\sigma]$ for the stably homotopic class of $\sigma$. The set of such stably homotopic classes is denoted by $K_{(G,H)}(X, \mathrm{A})$.

If $[\sigma_i]$ $(i=0, 1)$ are in $K_{(G,H)}(X, \mathrm{A})$, their sum is defined by

$$(1.1.5)\qquad [\sigma_0]+[\sigma_1]=[\sigma_0\oplus\sigma_1].$$

Then $K_{(G,H)}(X, A)$ is a semigroup.
Two objects $\sigma_0$ and $\sigma_1$ are isomorphic in $L_{(G,H)}(X, A)$, in symbols

$$\sigma_0 \cong \sigma_1, \tag{1.1.6}$$

if there exists an isomorphism $\varphi : \sigma_0 \longrightarrow \sigma_1$ in $L_{(G,H)}(X, A)$.

**1.2 Lemma.** *If $\sigma_0 \cong \sigma_1$ then $\sigma_0 \sim \sigma_1$.*

**Proof.** From (1.1.6), there exists an isomorphism $(f, g) : \sigma_0 \longrightarrow \sigma_1$. We define the $G$-vector bundles $\overline{E}$, $\overline{F}$ over $X\times I$ as follows:

$$\overline{E} = E_0\times[0, 1/2]\underset{f}{\cup} E_1\times[1/2, 1] \text{ and } \overline{F} = F_0\times[0, 1/2]\underset{g}{\cup} F_1\times[1/2, 1].$$

Moreover we define an $H$-isomorphism $\overline{\delta} : (I\times\overline{E})|B\times I \longrightarrow (I\times\overline{F})|B\times I$ as follows:

$$\overline{\delta}|B\times[0, 1/2]=\delta_0\times id_{[0,1/2]} \text{ and } \overline{\delta}|B\times[1/2, 1]=\delta_1\times id_{[1/2,1]}.$$

Then $\overline{\sigma}=[\overline{E}, \overline{F}, \overline{\delta}]$ is an object of $L_{(G,H)}(X\times I, A\times I)$ and $\overline{\sigma}|\{i\} = \sigma_i$ $(i = 0, 1)$. Hence $\sigma_0 \sim \sigma_1$.

**1.3 Lemma.** *If $[E, F, \delta]$, $[F, Q, \gamma]$ are in $K_{(G,H)}(X, A)$,* then we have

$$[E, F, \delta] + [F, Q, \gamma] = [E, Q, \gamma\delta]. \tag{1.3.1}$$

**Proof.** We define a $G$-isomorphism $\alpha(t) : F\oplus F \longrightarrow F\oplus F$ by

$$\alpha(t) = \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix} \quad \text{for } t \in [0, 1].$$

Then $(E\oplus F,\ F\oplus Q,\ (id\oplus\gamma)\alpha(t)\ (\delta\oplus id))$ is an object of $L_{(G,H)}(X, A)$, and we have

$$(E\oplus F,\ F\oplus Q,\ \delta\oplus\gamma) \sim (E\oplus F,\ F\oplus Q,\ (id\oplus\gamma)\alpha(1)(\delta\oplus id)). \tag{1.3.2}$$

Now the diagram

$$\begin{array}{ccc} (I\times(E\oplus F))|B & \xrightarrow{(id\oplus\gamma)\alpha(1)(\delta\oplus id)} & (I\times(F\oplus Q))|B \\ \Big\downarrow id & & \Big\downarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ (I\times(E\oplus F))|B & \xrightarrow{\gamma\delta\oplus id} & (I\times(Q\oplus F))|B \end{array} \tag{1.3.3}$$

is commutative. Therefor, the result follows from 1.2, (1.3.2) and (1.3.3).

**1.4 Lemma.** *$K_{(G,H)}(X, A)$ is an abelian group.*

**Proof.** From 1.3, if $[E, F, \delta]$ is in $K_{(G,H)}(X, A)$, we have

$$[E, F, \delta]+[F, E, \delta^{-1}]=[E, E, id]=0. \tag{1.4.1}$$

Hence $K_{(G,H)}(X, A)$ is an abelian group.

**1.5 Definition.** We set

$$K^0{}_{(G,H)}(X, A)=K_{(G,H)}(X, A)$$
$$K^{-1}{}_{(G,H)}(X, A)=K_{(G,H)}(X\times I, A\times I\cup X\times S^0)$$

and inductively

$$K^{-(n+1)}{}_{(G,H)}(X, A)=K^{-n}{}_{(G,H)}(X\times I, A\times I\cup X\times S^0) \text{ for } n=1,2,3,4, ,\cdots.$$

We define $L_G(X, A)$ to be a category as follows: An object of $L_G(X, A)$ is a pair $(E, F)$ of $G$-vector bundles over $X$, together with a $G$-isomorphism over $A$. The morphism $\varphi: \sigma_0 \longrightarrow \sigma_1$, where $\sigma_i=(E_i, F_i, \beta_i)$ $(i=0, 1)$, is a pair of $G$-isomorphisms $(f, g): (E_0, F_0) \longrightarrow (E_1, F_1)$ such that $(g|A)\beta_0=\beta_1(f|A)$. Then we can define the equivalence relation $\sim$, $\underset{s}{\sim}$, $\cong$, and the abelian groups $K_G{}^{-n}(X, A)$ in the same way as 1.1 and 1.5.

**Note**: $K_G(X, A)$, which is defined in this section, and $K_G(X, A)$, which is defined in [3], are isomorphic.

## §2 Properties of the elements of $K_G(\ )$ and $K_{(G,H)}(\ )$.

**2.1 Lemma.** *An element of $K_H{}^{-1}(X, A)$ is represented by an object $(E\times I, E\times I, \beta)$ of $L_H(X\times I, A\times I\cup X\times S^0)$ such that $\beta|X\times\{i\}=id_E$. (Such an object is called a normalized object.)*

**Proof.** If $[\bar{E}, \bar{F}, \beta]$ is in $K_H{}^{-1}(X, A)$, there exist the $H$-isomorphisms

$$p: \bar{E} \longrightarrow (\bar{E}|X\times\{1\})\times I$$
$$q: \bar{F} \longrightarrow (\bar{F}|X\times\{1\})\times I$$

such that $p|X\times\{1\}=id$ and $q|X\times\{1\}=id$. We define an $H$-isomorphism $\beta^*$ by the following composition:

$$\bar{E}|Y \xrightarrow{\beta} \bar{F}|Y \xrightarrow{q|Y} (F\times I)|Y \xrightarrow{(g|Y)^{-1}} (E\times I)|Y \xrightarrow{(p|Y)^{-1}} \bar{E}|Y$$

where $Y=A\times I\cup X\times S^0$, $F=\bar{F}|X\times\{1\}$, $E=\bar{E}|X\times\{1\}$ and $g=\beta|X\times\{1\}\times id_I$. Now the diagrams

$$\begin{array}{ccc} \bar{F}|Y & \xrightarrow{(p^{-1}g^{-1}q)} & \bar{E}|Y \\ \downarrow id & id & \downarrow (q^{-1}gp)|Y \\ \bar{F}|Y & \longrightarrow & \bar{F}|Y \end{array} \qquad \begin{array}{ccc} \bar{E}|Y & \xrightarrow{\beta^*} & \bar{E}|Y \\ \downarrow p|Y & (p\beta^*p^{-1})|Y & \downarrow p|Y \\ (E\times I)|Y & \longrightarrow & (E\times I)|Y \end{array} \tag{2.1.1}$$

are commutative. Therefore, from (2.1.1), we have

$$[\bar{E},\ \bar{F},\ \beta]=[\bar{E},\ \bar{F},\ \beta]+[\bar{F},\ \bar{F},\ id]$$
$$=[\bar{E},\ \bar{F},\ \beta]+[\bar{F},\ \bar{E},\ (p^{-1}g^{-1}q)|Y]$$
$$=[\bar{E},\ \bar{E},\ \beta^{*}]=[E\times I,\ E\times I,\ (p\beta^{*}p^{-1})|Y].$$

Immediately $(E\times I,\ E\times I,\ (p\beta^{*}p^{-1})|Y)$ is a normalized object of $L_{H}(X\times I,\ A\times I\cup X\times S^{0})$.

**2.2 Lemma.** *An element of* $K^{-1}{}_{(G,H)}(A)$ *is represented by an object* $(E\times I, E\times I, \delta)$ *of* $L_{(G,H)}(A\times I,\ A\times S^{0})$ *such that* $\delta|I\times A\times\{1\}=id_{E}$. *(Such an object is called a normalized object.)*

**Proof.** This follows in the same way as that of 2.1.

**2.3 Lemma.** (i) *Let $E$ be a $G$-vector bundle over $X$, then there exists a complementary $G$-vector bundle of $E$.*

(ii) *Let $E$ be an $H$-vector bundle over $X$, then there exist an $H$-vector bundle $E'$ over $X$ and a $G$-vector bundle $F$ over $X$ such that $E\oplus E'$ and $F$ are $H$-isomorphic.*

**Proof.** (i) (cf. 2.4 Existence of complementary bundles of [3]).
(ii). From (i), there exist an $H$-vector bundle $E''$ over $X$ and an $H$-module $N$ such that $E\oplus E''$ and $X\times N$ are $H$-isomorphic. From Corollary 1.1.4 of [3], there exist an $H$-module $N'$ and a $G$-module $M$ such that $N\oplus N'$ and $M$ are $H$-isomorphic. Hence the result follows by defining $E'=E''\oplus(X\times N')$ and $F=X\times M$.

**2.4 Lemma.** (i) *An element of* $K_{H}{}^{-1}(X,\ A)$ *is represented by a normalized object* $(E\times I,\ E\times I,\ \beta)$ *of* $L_{H}(X\times I,\ A\times I\cup X\times S^{0})$ *such that $E$ has a $G$-vector bundle structure over $X$.*

(ii) *An element of* $K^{-1}{}_{(G,H)}(A)$ *is represented by a normalized object* $(E\times I,\ E\times I,\ \delta)$ *of* $L_{(G,H)}(A\times I,\ A\times S^{0})$ *such that $E$ is a restriction of a $G$-vector bundle over $X$ to $A$.*

**Proof.** This is clear from 2.3.

**2.5 Lemma.** *Let $M$ be a $G$-module and*

$$f:(A\times I\cup X\times\{0\})\times M\longrightarrow(A\times I\cup X\times\{0\})\times M$$

*a $G$-isomorphism. Then $f$ is extendable to a $G$-isomorphism $f^{*}$ over $X\times I$.*

**Proof.** From Lemma 2.2.1 of [3], there exists a $G$-invariant neighbourhood $U$ $(U\supset A)$, and $f|A\times I$ is extendable to a $G$-isomorphism $f'$ over $U\times\{0\}\cup A\times I$. Since $X$ is a compact $G$-space and $G$ is a compact Lie group, so there exists a $G$-map $\varphi: X\longrightarrow I$ such that $\varphi|U^{c}=0$ and $\varphi|A=1$. Therefore $f^{*}$, which is defined by $f^{*}(x,\ t,\ m)=f'(x,\ t\varphi(x),\ m)$ for $(x,\ t,\ m)\in X\times I\times M$, is the required extention.

**2.6 Lemma.** *Let $(E,\ E,\ \beta)$ be an object of $L_{G}(X,\ A)$. If $[E,\ E,\ \beta]=0$, there exist a $G$-vector bundle $P$ over $X$ and a $G$-isomorphism $\beta^{*}: E\oplus P\longrightarrow E\oplus P$ such* *that*

$$\beta^*|A = \beta \oplus id_P.$$

**Proof.** If $[E, E, \beta]=0$, from (1.1.4), we have

$$(E\oplus P, \ E\oplus P, \ \beta\oplus id_P) \sim (Q, \ Q, \ id)$$

for some $G$-vector bundle $P$, $Q$ over $X$. From (1.1.3), there exists an object $\bar{\sigma}=(\bar{E}, \bar{F}, \bar{\beta})$ of $L_G(X\times I, \ A\times I)$ such that

$$\bar{\sigma}|\{0\} = (E\oplus P, \ E\oplus P, \ \beta\oplus id_P)$$
$$\bar{\sigma}|\{1\} = (Q, \ Q, \ id).$$

Now $\bar{E}|X\times\{0\}=\bar{F}|X\times\{0\}$, so there exist the $G$-isomorphisms

$$f : \bar{E} \longrightarrow (E\oplus P)\times I$$
$$g : \bar{F} \longrightarrow (E\oplus P)\times I$$

such that

$$f|X\times\{0\} = g|X\times\{0\} = id_{E\oplus P}.$$

We define a $G$-isomorphism $\tilde{\beta}^* : (E\oplus P)\times I|A\times I \longrightarrow (E\oplus P)\times I|A\times I$ by the following composition :

$$(E\oplus P)\times I|A\times I \xrightarrow{f^{-1}|A\times I} \bar{E}|A\times I \xrightarrow{\bar{\beta}} \bar{F}|A\times I \xrightarrow{g|A\times I} (E\oplus P)\times I|A\times I$$

Then we have

(2.6.1)
$$\tilde{\beta}^*|A\times\{0\} = \beta\oplus id_P$$
$$\tilde{\beta}^*|A\times\{1\} = (gf^{-1}|X\times\{1\})|A\times\{1\}.$$

Now, from 2.3 (i), we can regard $E\oplus P$ as a trivial $G$-vector bundle over $X$. Therefore the result follows from (2.6.1) and 2.5.

**2.7 Lemma.** Let $(E, F, \beta)$ *be an object of* $L_G(X, A)$. *If* $[E, F, \beta]=0$, *there exist a G-vector bundle P over X and a G-isomorphism* $\beta^* : E\oplus P \longrightarrow F\oplus P$ *such that*

$$\beta^*|A = \beta\oplus id_P.$$

**Proof.** If $[E, F, \beta]=0$, we have

$$(E\oplus P', \ F\oplus P', \ \beta\oplus id_{P'}) \sim (Q', \ Q', \ id)$$

for some $G$-vector bundles $P'$, $Q'$ over $X$. So there exist the $G$-isomorphisms $f: E\oplus P' \longrightarrow Q'$ and $g: F\oplus P' \longrightarrow Q'$. The diagram

(2.7.1)
$$\begin{array}{ccc} (E\oplus P')|A & \xrightarrow{f|A} & Q'|A \\ \downarrow \beta\oplus id_{P'} & & \downarrow \beta' = (g|A)(\beta\oplus id_{P'})(f|A)^{-1} \\ (F\oplus P')|A & \xrightarrow{g|A} & Q'|A \end{array}$$

is commutative.

From 2.6 and (2.7.1), there exists a $G$-vector bundle $P''$ over $X$ such that $\beta'\oplus id_{P''}$ is extendable to a $G$-isomorphism $\beta^*$ over $X$. Thus $\beta^*$ is the required extension.

**2.8 Lemma.** *Let $(E, E, \delta)$ be an object of $L_{(G,H)}(X, A)$. If $[E, E, \delta]=0$, there exists an object $((E\oplus P)\times I, (E\oplus P)\times I, \bar{\delta})$ of $L_{(G,H)}(X, A)$ such that*

$$\bar{\delta}|B\times\{0\} = \delta\oplus id_P$$
$$\bar{\delta}|B\times\{1\} \text{ is a } G\text{-isomorphism.}$$

**Proof.** This can be proved in the same way as in the proof of 2.6.

**2.9 Lemma.** *Let $(E, F, \delta)$ be an object of $L_{(G,H)}(X, A)$. If $[E, F, \delta]=0$, there exist a $G$-vector bundle $P$ over $X$ and an object $(Q\times I, Q\times I, \bar{\delta})$ of $L_{(G,H)}(X\times I, A\times I)$ such that*

$$(E\oplus P, F\oplus P, \delta\oplus id_P) \cong (Q, Q, \bar{\delta}|B\times\{0\})$$
$$\bar{\delta}|B\times 1 \text{ is a } G\text{-isomorphism.}$$

**Proof.** This follows from 2.8 in the same way as followed 2.7 from 2.6.

## § 3 Exact sequences.

**3.1 Definition.** We define the homomorphisms $u$, $v$, $i^*$, $j^*$ as follows:

$u: K_{(G,H)}(X, A) \longrightarrow K_G(X, A)$ by $u([E, F, \delta]) = [E, F, \delta|\{1\}\times A]$,
$v: K_G(X, A) \longrightarrow K_H(X, A)$ induced by an inclusion $H\subset G$,
$j^*: K_{(G,H)}(X, A) \longrightarrow K_{(G,H)}(X)$ by $j^*([E, F, \delta]) = [E, F, \delta|\{0\}\times X,]$
$i^*: K_{(G,H)}(X) \longrightarrow K_{(G,H)}(A)$ by $i^*([E, F, \alpha]) = [E|A, F|A, \alpha|A]$.

Moreover we can define the following homomorphisms:

$\Lambda^n u\ K^{-n}{}_{(G,H)}(X, A) \longrightarrow K^{-n}{}_G(X, A)$,
$\Lambda^n v: K^{-n}{}_G(X, A) \longrightarrow K^{-n}{}_H(X, A)$,
$\Lambda^n j^*: K^{-n}{}_{(G,H)}(X, A) \longrightarrow K^{-n}{}_{(G,H)}(X)$,
$\Lambda^n i^*: K^{-n}{}_{(G,H)}(X) \longrightarrow K^{-n}{}_{(G,H)}(A)$.

From 2.4, an element of $K^{-1}{}_H(X, A)$ is represented by a normalized element

$(E\times I,\ E\times I,\ \alpha)$ of $L_H(X\times I,\ A\times I\cup X\times S^0)$ such that $E$ has a $G$-vector bundle structure. So we define a (boundary) homomorphism

$$\partial : K^{-1}{}_H(X,\ A)\longrightarrow K_{(G,H)}(X,\ A)$$

by

$$\partial([E\times I,\ E\times I,\ \alpha])=[E,\ E,\ \alpha|I\times A\cup\{0\}\times X].$$

From 2.4, an element of $K^{-1}{}_{(G,H)}(A)$ is represented by a normalized object $(E\times I,\ E\times I,\ \delta)$ of $L_{(G,H)}\ (A\times I,\ A\times S^0)$ such that $E$ is a restriction of a $G$-vector bundle $F$ over $X$ to $A$. Now $\delta$ is an $H$-isomorphism over $I\times A\times S^0\cup\{0\}\times A\times I$, so we can regard $\delta$ as an $H$-isomorphism over $I\times A$ by an identification $I\times A\times S^0\cup 0\times A\times I\equiv I\times A$. Then we have $\delta|\{0\}\times A=id$ and $\delta|\{1\}\times A$ is a $G$-isomorphism. We define an $H$-isomorphism $\gamma : I\times F|B\longrightarrow I\times F|B$ by

$$\gamma|I\times A=\delta \text{ and } \gamma|\{0\}\times X=id.$$

Then we define a (boundary) homomorphism

$$\Delta : K^{-1}{}_{(G,H)}(A)\longrightarrow K_{(G,H)}(X,\ A)$$

by

$$\Delta([E\times I,\ E\times I,\ \delta])=[F,\ F,\ \gamma].$$

Moreover we can define the following boundary homomorphisms:

$$\Lambda^n\partial : K_H{}^{-(n+1)}(X,\ A)\longrightarrow K^{-n}{}_{(G,H)}(X,\ A)$$
$$\Lambda^n\Delta : K^{-(n+1)}{}_{(G,H)}(A)\longrightarrow K^{-n}{}_{(G,H)}(X,\ A).$$

**3.2 Theorem.** *The sequence*

$$K_{(G,H)}(X,\ A)\xrightarrow{u}K_G(X,\ A)\xrightarrow{v}K_H(X,\ A)$$

*is exact.*

**Proof.** It is clear that Image $u\subset$ Kernel $v$, so it is sufficient to prove that Image $u\supset$Kernel $v$. Let $[E,\ F,\ \beta]$ be an element of $K_G(X,\ A)$ such that $v([E,\ F,\ \beta])=0$. From 2.7, there exist an $H$-vector bundle $P$ over $X$ and an $H$-isomorphism $\beta^* : E\oplus P\longrightarrow F\oplus P$ such that $\beta^*|A=\beta\oplus id_P$. We define an $H$-isomorphism $\delta : (I\times(E\oplus P))|B\longrightarrow(I\times(F\oplus P))|B$ by $\delta|\{0\}\times X=\beta^*$ and $\delta|I\times A=id_I\times(\beta\oplus id_P)$. Now, from 2.3 (ii), we can regard $P$ as a $G$-vector bundle over $X$. Therefore we have

$$u([E\oplus P,\ F\oplus P,\ \delta])=[E\oplus P,\ F\oplus P,\ \delta|\{1\}\times A]$$
$$=[E\oplus P,\ F\oplus P,\ \beta\oplus id_P]=[E,\ F,\ \beta].$$

**3.3 Theorem.** *The sequence*

$$K^{-1}{}_H(X, A) \xrightarrow{\partial} K_{(G,H)}(X, A) \xrightarrow{u} K_G(X, A)$$

*is exact.*

**Proof.** It is clear that Image $\partial \subset$ Kernel $u$, so it is sufficient to prove that Image $\partial \supset$ Kernel $u$. Let $[E, F, \delta]$ be an element of $K_{(G,H)}(X, A)$ such that $u([E, F, \delta])=0$. From 2.7, there exist a $G$-vector bundle $P$ over $X$ and a $G$-isomorphism $\beta : E\oplus P \longrightarrow F\oplus P$ such that $\beta|A=(\delta|\{1\}\times A)\oplus id_P$. We define an $H$-isomorphism

$$\alpha : (I\times(E\oplus P))|Y \longrightarrow (I\times(E\oplus P))|Y,$$

where $Y=I\times A\cup S^0\times X$, by the following composition :

$$(I\times(E\oplus P))|Y \xrightarrow{\delta^*} (I\times(F\oplus P))|Y \xrightarrow{(id_I\times\beta^{-1})|Y} (I\times(E\oplus P))|Y$$

where $\delta^*$ is defined by $\delta^*|\{1\}\times X=\beta$ and $\delta^*|I\times A\cup\{0\}\times X=\delta\oplus id_P$.
Then $(I\times(E\oplus P), I\times(E\oplus P), \alpha)$ is a normalized object of $L_H(I\times X, I\times A\cup S^0\times X)$. Now the diagram

$$\begin{array}{ccc} (I\times(E\oplus P))|B & \xrightarrow{\alpha|B} & (I\times(E\oplus P))|B \\ \downarrow id & & \downarrow (id_I\times\beta)|B \\ (I\times(E\oplus P))|B & \xrightarrow{\delta\oplus i_P} & (I\times(F\oplus P))|B \end{array} \qquad (3.3.2)$$

is commutative. So, from (3.3.2), we have

$$\partial([I\times(E\oplus P), I\times(E\oplus P), \alpha])=[E\oplus P, E\oplus P, \alpha|B]$$
$$=[E\oplus P, F\oplus P, \delta\oplus id_P]=[E, F, \delta].$$

**3.4 Theorem.** *The sequence*

$$K^{-1}{}_G(X, A) \xrightarrow{A^1v} K^{-1}{}_H(X, A) \xrightarrow{\partial} K_{(G,H)}(X, A)$$

*is exact.*

**Proof.** It is clear that Image $A^1v\subset$ Kernel $\partial$, so it is sufficient to prove that Image $A^1v\supset$ Kernel $\partial$. Let $(I\times E, I\times E, \alpha)$ be a normalized object of $L_H(I\times X, I\times A\cup S^0\times X)$ such that $\partial([I\times E, I\times E, \alpha])=[E, E, \alpha|\{0\}\times X\cup I\times A]=0$. From 2.8, there exist a $G$-vector bundle $P$ over $X$ and an object $((E\oplus P)\times I, (E\oplus P)\times I, \bar{\delta})$ of $L_{(G,H)}(X\times I, A\times I)$ such that $\bar{\delta}|B\times\{1\}$ is a *$G$-isomorphism and* $\bar{\delta}|B\times\{0\}=(\alpha|\{0\}\times X\cup I\times A)\oplus id_P$. Now, from 2.3, we can regard $E\oplus P$ as a trivial $G$-vector bundle over $X$, Since $\bar{\delta}|\{1\}\times A\times I$ is a $G$-isomorphism and $\bar{\delta}|\{1\}\times A\times\{0\}=id$, so from 2.5, there exists a $G$-isomorphism

$$\delta^* : (I\times(E\oplus P)\times I)|\{1\}\times X\times I \longrightarrow (I\times(E\oplus P)\times I)|\{1\}\times X\times I$$

such that $\delta^*|\{1\}\times A\times I=\bar{\delta}|1\times A\times I$ and $\delta^*|\{1\}\times X\times\{0\}=id$. We define an $H$-isomorphism

$$\bar{\alpha} : (I\times(E\oplus P)\times I)|(I\times A\cup S^0\times X)\times I \longrightarrow (I\times(E\oplus P)\times I)|(I\times A\cup S^0\times X)\times I$$

by

$$\bar{\alpha}|(I\times A\cup\{0\}\times X)\times I=\bar{\delta}|(I\times A\cup\{0\}\times X)\times I$$
$$\bar{\alpha}|\{1\}\times X\times I=\delta^*|\{1\}\times X\times I.$$

Then $(I\times(E\oplus P)\times I,\ I\times(E\oplus P)\times I,\ \bar{\alpha})$ is an object of $L_H(I\times X\times I,\ (I\times A\cup S^0\times X)\times I)$ and $\bar{\alpha}|(I\times A\cup S^0\times X)\times\{0\}=\alpha\oplus id_P$. So we have

$$\begin{aligned}&\Lambda^1 v([I\times(E\oplus P),\ I\times(E\oplus P),\ \bar{\alpha}|(I\times A\cup S^0\times X)\times\{1\}])\\&=[I\times(E\oplus P),\ I\times(E\oplus P),\ \alpha\oplus id_P]\\&=[I\times E,\ I\times E,\ \alpha].\end{aligned}$$

**3.5 Theorem.** *The sequence*

$$\cdots\longrightarrow K^{-n}{}_G(X,A)\xrightarrow{\Lambda^n v}K^{-n}{}_H(X,A)\xrightarrow{\Lambda^{(n-1)}\partial}K^{-(n-1)}{}_{(G,H)}(X,A)\xrightarrow{\Lambda^{(n-1)}u}\cdots$$
$$\cdots\cdots\xrightarrow{\partial}K_{(G,H)}(X,A)\xrightarrow{u}K_G(X,A)\xrightarrow{v}K_H(X,A).$$

*is exact.*

**Proof.** It follows from 3.2, 3.3 and 3.4.

**3.6 Theorem.** *The sequence*

$$\cdots\cdots\longrightarrow K^{-n}{}_{(G,H)}(X)\xrightarrow{\Lambda^n i^*}K^{-n}{}_{(G,H)}(A)\xrightarrow{\Lambda^{(n-1)}\Delta}K^{-(n-1)}{}_{(G,H)}(X,A)\xrightarrow{\Delta^{(n-1)}j^*}\cdots\cdots$$
$$\cdots\cdots\xrightarrow{\Delta}K_{(G,H)}(X,A)\xrightarrow{j^*}K_{(G,H)}(X)\xrightarrow{i^*}K_{(G,H)}(A).$$

*is exact.*

**Proof.** This can be proved by the same methods as in the proof of 3.5.

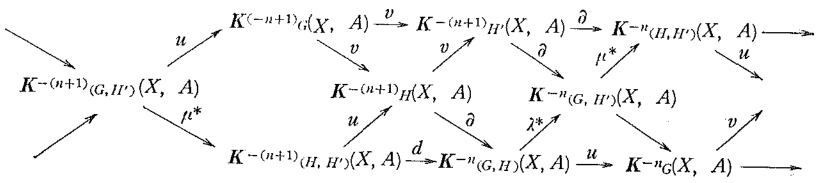
**3.7 Definition.** Let $H'$ be a clsed subgroup of $H$, then we define the natural homomorphisms

$$\Lambda^n\mu^* : K^{-n}{}_{(G,H')}(X,A)\longrightarrow K^{-n}{}_{(H,H')}(X,A) \text{ induced by } (H,H')\subset(G,H'),$$
$$\Lambda^n\lambda^* : K^{-n}{}_{(G,H)}(X,A)\longrightarrow K^{-n}{}_{(G,H')}(X,A) \text{ induced by } (G,H')\subset(G,H).$$

Let $\Lambda^n d$ be a boundary homomorphism which defined by the following composition :

$$\Lambda^n d : K^{-(n+1)}_{(H,H')}(X,\ A) \xrightarrow{\Lambda^{(n+1)}u} K^{-(n+1)}_{H}(X,\ A) \xrightarrow{\Lambda^n \partial} K^{-n}_{(G,H)}(X,\ A).$$

**3.8 Lemma.** *The following diagram is commutative.*




**Proof.** From the definitions of homomorphisms, this is clear.

**3.9 Theorem.** *The sequence*

$$\cdots\longrightarrow K^{-(n+1)}_{(H,H')}(X,A) \xrightarrow{\Lambda^n d} K^{-n}_{(G,H)}(X,A) \xrightarrow{\Lambda^n \lambda^*} K^{-n}_{(G,H')}(X,A) \xrightarrow{\Lambda^n \mu^*}\cdots$$
$$\cdots\xrightarrow{d} K_{(G,H)}(X,A) \xrightarrow{\lambda^*} K_{(G,H')}(X,A) \xrightarrow{\mu^*} K_{(H,H')}(X,A).$$

*is exact.*

**Proof.** From 3.5 and 3.8, this is clear.

**3.10 Theorem.** *The following diagrams are commutative, and each row and each colum are exact.*

$$\begin{array}{ccccccccccc}
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow & & \downarrow \\
\xrightarrow{\lambda^*} & & K^{-(n+2)}_{(G,H')}(A) & \xrightarrow{\mu^*} & K^{-(n+2)}_{(H,H')}(A) & \xrightarrow{d} & K^{-(n+1)}_{(G,H)}(A) & \dashrightarrow & K_{(G,H)}(A) & \xrightarrow{\lambda^*} & K_{(G,H')}(A)\ \cdots \\
 & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
\xrightarrow{\lambda^*} & & K^{-(n+1)}_{(G,H')}(X,A) & \xrightarrow{\mu^*} & K^{-(n+1)}_{(H,H')}(X,A) & \xrightarrow{d} & K^{-n}_{(G,H)}(X,A) & \dashrightarrow & K_{(G,H)}(X,A) & \xrightarrow{\lambda^*} & K_{(G,H')}(X,A)\ \cdots \\
 & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\
\xrightarrow{\lambda^*} & & K^{-(n+1)}_{(G,H')}(X) & \xrightarrow{\mu^*} & K^{-(n+1)}_{(H,H')}(X) & \xrightarrow{d} & K^{-n}_{(G,H)}(X) & \dashrightarrow & K_{(G,H)}(X) & \xrightarrow{\lambda^*} & K_{(G,H')}(X)\ \cdots \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\xrightarrow{\lambda^*} & & K^{-(n+1)}_{(G,H')}(A) & \xrightarrow{\mu^*} & K^{-(n+1)}_{(H,H')}(A) & \xrightarrow{d} & K^{-n}_{(G,H)}(A) & \dashrightarrow & K_{(G,H)}(A) & \xrightarrow{\lambda^*} & K_{(G,H')}(A)\ \cdots \\
 & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta}
\end{array}$$

**Proof.** From the above arguments, this is clear.

## §4 $(C(G/H),\ G/H)$ coefficient $K$-theory.

**4.1 Theorem.** *We obtain the following isomorphism:*

$$K_{(G,H)}(X,\ A) \cong K_G((C(G/H),\ G/H)\times(X,\ A)),$$

*where $C(G/H)$ is a cone over $G/H$.*

The proof of the Theorem will be broken down into a series of Lemmas.

**4.2** Let $Y$ be an $H$-space. Let $G\underset{H}{\times}Y$ denote the identification space obtained from $G\times Y$ by the equivalence reration:

$$(g_1,\ y_1)\sim(g_2,\ y_2) \text{ if and only if } g_2=g_1h^{-1} \text{ and } y_2=hy_1 \text{ for some } h\in H.$$

Then $G \underset{H}{\times} Y$ admits a $G$-space structure: we define

$$g(g_1, y) = (gg_1, y)$$

and note that

$$g(g_1h^{-1}, hy) = (gg_1h^{-1}, hy) \sim (gg_1, y) = g(g_1, y).$$

Let $E$ be an $H$-vector bundle over $Y$, then $G \underset{H}{\times} E$ admits a $G$-vector bundle structure over $G \underset{H}{\times} Y$.

If $Y$ is a $G$-space, $f : (G/H) \times Y \longrightarrow G \underset{H}{\times} Y$, which defined by

$$f(gH, y) = [g, g^{-1}y], \tag{4.2.1}$$

is a $G$-homeomorphism.

Let $E$, $F$ be $H$-vector bundle over $Y$ and $\alpha : E \longrightarrow F$ an $H$-isomorphism. Then $\bar{\alpha} : G \underset{H}{\times} E \longrightarrow G \underset{H}{\times} F$, which defined by

$$\bar{\alpha}[(g, e)] = [g, \alpha(e)], \tag{4.2.2}$$

is a $G$-isomorphism.

If $Y$ is a $G$-space and $E$, $F$ are $G$-vector bundles over $Y$, then $\tilde{\alpha} : (G/H) \times E \longrightarrow (G/H) \times F$, which defined by

$$\tilde{\alpha}(gH, e) = (gH, g\alpha(g^{-1}e)), \tag{4.2.3}$$

is a $G$-isomorphism. Moreover we have

$$f^*(G \underset{H}{\times} E) = (G/H) \times E \text{ and } f^*(\bar{\alpha}) = \tilde{\alpha}. \tag{4.2.4}$$

**4.3 Lemma.** *Let* $l_1 : K_H(X, A) \longrightarrow K_G((G/H) \times X, (G/H) \times A)$ *be a following composition*:

$$K_H(X, A) \xrightarrow{l'_1} K_G(G \underset{H}{\times} X, G \underset{H}{\times} A) \xrightarrow{f^*} K_G((G/H) \times X, (G/H) \times A),$$

*where* $l'_1$ *is defined by* $l'_1(E, F, \alpha) = [G \underset{H}{\times} E, G \underset{H}{\times} F, \bar{\alpha}]$. *Then* $l_1$ *is an isomorphism.*

**Proof.** This follows directly from Proposition 1.1.3 of [3].

**4.4 Lemma.** *We define* $l_2 : K_G(X, A) \longrightarrow K_G(C(G/H) \times X, . C(G/H) \times A)$ *by*

$$l_2([E, F, \alpha]) = [C(G/H) \times E, C(G/H) \times F, id \times \alpha].$$

*Then* $l_2$ *is an isomorphism.*

**Proof.** Since $C(G/H)$ is $G$-contractible, so the result follows at once.

**4.5 Definition.** We define $l_3 : K_{(G,H)}(X, A) \longrightarrow K_G((C(G/H), G/H)\times(X, A))$ as follows: Let $[E, F, \delta]$ be an element of $K_{(G,H)}(X, A)$. From 1.1, we can construct a $G$-isomorphism

$$\tilde{\delta} : (G/H)\times((I\times E)|B) \longrightarrow (G/H)\times(I\times F)|B)$$

as (4.2.3). Now $\tilde{\delta}(gH, (1, e)) = (gH, g\delta(1, g^{-1}e))$ and $\tilde{\delta}|1\times A$ is a $G$-isomorphism, so we have

$$\tilde{\delta}(gH, (1, e)) = (gH, \delta(1, e)). \tag{4.5.1}$$

From (4.5.1), we can regard $\tilde{\delta}$ as a $G$-isomorphism

$$\tilde{\delta} : C(G/\tilde{H})\times E|C(G/H)\times A\cup(G/H)\times X \longrightarrow C(G/H)\times F|C(G/H)\times A\cup(G/H)\times X.$$

So we define $l_3$ by

$$l_3([E, F, \delta]) = (C(G/H)\times E, C(G/H)\times F, \tilde{\delta}].$$

**4.6 Lemma.** *We obtain the following exact sequence:*

$$\cdots \xrightarrow{\partial_1} K_G((C(G/H), G/H)\times(X, A)) \xrightarrow{j_1^*} K_G(C(G/H)\times X, C(G/H)\times A)$$
$$\xrightarrow{i_1^*} K_G((G/H)\times X, (G/H)\times A).$$

**Proof.** For a triple $(C(G/H)\times X, C(G/H)\times A\cup(G/H)\times X, C(G/H)\times A)$, we have the exact sequence:

$$\cdots \xrightarrow{\partial} K_G((C(G/H), G/H)\times(X, A) \xrightarrow{j^*} K_G(C(G/H)\times X, C(G/H)\times A)$$
$$\xrightarrow{i^*} K_G(C(G/H)\times A\cup(G/H)\times X, C(G/H)\times A).$$

Now, from the Excision Theorem, we have the following isomorphism $\bar{i}^*$:

$$K_G(C(G/H)\times A\cup(G/H)\times X, C(G/H)\times A) \xrightarrow{\bar{i}^*} K_G((G/H)\times X, (G/H)\times A).$$

So the result follows by defining $\partial_1 = \partial(\bar{i}^*)^{-1}$, $i_1^* = i^*\bar{i}^*$ and $j_1^* = j^*$.

**4.7 Lemma.** *The diagram*

$$\begin{array}{ccc} K_G(X, A) & \xrightarrow{\quad v \quad} & K_H(X, A) \\ \downarrow l_2 & & \downarrow l_1 \\ K_G(C(G/H)\times X, C(G/H)\times A) & \xrightarrow{i_1^*} & K_G((G/H)\times X, (G/H)\times A) \end{array}$$

*is commutative.*

**Proof.** Let $[E, F, \alpha]$ be an element of $K_G(X, A)$. Then we have

$$\begin{aligned} l_1 v([E, F, \alpha]) &= l_1([E, F, \alpha]) \\ &= [f^*(G \times_H E),\ f^*(G \times_H F),\ f^*(\bar{\alpha})] \\ &= [(G/H)\times E,\ (G/H)\times F,\ id\times\alpha]. \end{aligned}$$

and

$$\begin{aligned} i_1{}^* l_2([E, F, \alpha]) &= i_1{}^*([C(G/H)\times E,\ C(G/H)\times F,\ id\times\alpha]) \\ &= [(G/H)\times E,\ (G/H)\times F,\ id\times\alpha]. \end{aligned}$$

Therefore $l_1 v = i_1{}^* l_2$.

**4.8 Lemma.** *The diagram*

$$\begin{array}{ccc} K_{(G,H)}(X, A) & \xrightarrow{\quad u \quad} & K_G(X, A) \\ \downarrow l_2 & & \downarrow l_1 \\ K_G((C(G/H), G/H)\times(X, A)) & \xrightarrow{\ j_1{}^*\ } & K_G(C(G/H)\times X,\ C(G/H)\times A) \end{array}$$

*is commutative.*

**Proof.** Let $[E, F, \delta]$ be an element of $K_{(G,H)}(X, A)$. Then we have

(4.8.1) $$\begin{aligned} l_2 u(E, F, \delta) &= l_2([E, F, \delta|\{1\}\times A]) \\ &= [C(G/H)\times E,\ C(G/H)\times F,\ id\times(\delta|\{1\}\times A]). \end{aligned}$$

and

(4.8.2) $$\begin{aligned} j_1{}^* l_3([E, F, \delta]) &= i_1{}^*([C(G/H)\times E,\ C(G/H)\times F,\ \tilde{\delta}]) \\ &= [C(G/H)\times E,\ C(G/H)\times F,\ \tilde{\delta}|C(G/H)\times A]. \end{aligned}$$

Now we define a $G$-isomorphism $\gamma : C(G/H)\times(E|A)\times I \longrightarrow C(G/H)\times(F|A)\times I$ by

$$\gamma(([t, gH], e, s) = ([t, gH],\ g\delta'((1-t)s+t,\ g^{-1}e),\ s),$$

where $\delta'$ is defined by $\delta(t, e) = (t, \delta'(t, e))$. Then we have

$$\gamma|(s=0) = \tilde{\delta}|C(G/H)\times A \ \text{ and } \ \gamma|(s=1) = id\times(\delta|\{1\}\times A).$$

Therefore, from (4.8.1) and (4.8.2), we have $l_2 u = j_1{}^* l_3$.

**4.9 Lemma.** *The diagram*

$$\begin{array}{ccc} K_H{}^{-1}(X, A) & \longrightarrow & K_{(G,H)}(X, A) \\ \downarrow l_1 & & \downarrow l_3 \\ K_G{}^{-1}((G/H)\times X,\ (G/H)\times A) & \xrightarrow{\ \partial_1\ } & K_G((C(G/H), G/H)\times(X, A)) \end{array}$$

*is commutative.*

**Proof.** Let $x=[I\times E,\ I\times E,\ \alpha]$ be an element of $K_H{}^{-1}(X,\ A)$. Then we have

$$(\bar{i}^*)^{-1}l_1(x)=(\bar{i}^*)^{-1}([G/H)\times I\times E,\ (G/H)\times I\times E,\ \tilde{\alpha}])$$
$$=[C(G/H)\times I\times(E|A)\cup(G/H)\times I\times E,\ C(G/H)\times I\times(E|A)\cup(G/H)\times I\times E,\tilde{\alpha}'],$$

where $\tilde{\alpha}'$ is defined by

$$\tilde{\alpha}'([t,\ gH],\ s,\ e)=([t,\ gH],\ s,\ g\alpha'((1-t)s+t,\ g^{-1}e)),$$

where $\alpha'$ is defined by $\alpha(s,\ e)=(t,\ \alpha'(s,\ e))$. Since $(\bar{i}^*)^{-1}l_1(x)$ is also a normalized object, we have

$$\partial_1 l_1(x)=[C(G/H)\times E,\ C(G/H)\times E,\ \beta],$$

where $\beta=\tilde{\alpha}^{\cdot}|C(G/H)\times\{0\}\times A\cup(G/H)\times\{0\}\times X$. On the other hand we have

$$l_3\partial(x)=[C(G/H)\times E,\ C(G/H)\times E,\ \tilde{\delta},]$$

where $\tilde{\delta}$ is defined by $\tilde{\delta}([t,\ gH],\ e)=([t,\ gH],\ g\alpha(t,\ g^{-1}e))$. Then, from the definitions of $\beta$ and $\tilde{\delta}$, $\beta=\tilde{\delta}$. So we have $l_3\partial=\partial_1 l_1$.

**4. 10 Proof of 4. 1 Theorem.**

From the above Lemmas, the following diagram is commutative and each row is exact.

$$\begin{array}{ccccc}
\cdots\xrightarrow{v} K_H{}^{-1}(X,\ A) & \xrightarrow{\partial} & K_{(G,H)}(X,\ A) & \xrightarrow{u} & \\
\downarrow l_1 & & \downarrow l_3 & & \\
\cdots\longrightarrow K_G{}^{-1}((G/H)\times X,\ (G/H)\times A & \longrightarrow & K_G((C(G/H),\ G/H)\times(X,A)) & \longrightarrow &
\end{array}$$

$$\begin{array}{ccc}
K_G(X,\ A) & \xrightarrow{v} & K_H(X,\ A). \\
\downarrow l_2 & & \downarrow l_1 \\
K_G(C(G/H)\times X,\ C(G/H)\times A) & \xrightarrow{i_1{}^*} & K_G((G/H)\times X,\ (G/H)\times A).
\end{array}$$

Therefore the result follows from Five Lemma.

**4. 11 Corollary.** *We obtain the following isomorphism:*

$$K^{-n}{}_{(G,H)}(X,\ A)\cong K^{-(n+2)}{}_{(G,H)}(X,\ A)\ (\textit{Complex case})$$

**Proof.** From 4. 1 Theorem, this is clear.

## § 5 Wyel group operations.

**5. 1** Let $G$ be a compact connected Lie group, $T$ a maximal torus of $G$ and $W(G)=N(T)/T$ the Wyel group. Let $E$ be a $T$-vector bundle over $X$. For each $n\in N(T)$, $n^*E$ admits a $T$-vector bundle structure (we regard $n$ as a continuous map $n: X\longrightarrow X$ by its action on $X$): we define $h:(n^*E)_x\longrightarrow(n^*E)_{hx}$ by $nhn^{-1}$:

$E_{nx} \longrightarrow E_{nhx}$ for all $h \in T$. If $n$ is in $T$, $n^*E$ and $E$ are isomorphic by $T$-isomorphism $n^{-1}$. If $E$ *is* a $G$-vector bundle, $n^*E$ admits a $G$-vector bundle structure, and $n^*E$ and $E$ are isomorphic by $G$-rsomorphism $n^{-1}$. So the following operation is well defined :

(5. 1. 1)
$$K_T(X) \times W(G) \longrightarrow K_T(X)$$
$$([E,\ F],\ [n]) \longrightarrow [n^*E,\ n^*E].$$

Let $E$, $F$, be $G$-vector bundle over $X$ and $\alpha : E \longrightarrow F$ a $T$-isomorphism. In general the diagram

$$\begin{array}{ccc} n^*E & \xrightarrow{\ n^*\alpha\ } & n^*F \\ \downarrow{\scriptstyle n^{-1}} & & \downarrow{\scriptstyle n^{-1}} \\ E & \xrightarrow{\ \alpha\ } & F \end{array}$$

is not commutative, but if $n$ is in $T$, the diagram is commutative. So the following operation is well defined :

(5. 1. 2)
$$K_{(G,T)}(X) \times W(G) \longrightarrow K_{(G,T)}(X)$$
$$([E,\ F,\ \alpha],\ [n]) \longrightarrow [n^*E,\ n^*F,\ n^*\alpha].$$

Similary, we can define the following operations :

(5. 1. 3)
$$K_T{}^*(X) \times W(G) \longrightarrow K_T{}^*(X)$$
$$K^*{}_{(G,T)}(X) \times W(G) \longrightarrow K^*{}_{(G,T)}(X).$$

Let $K_T{}^*(X)^{W(G)}$ (respectively $K^*{}_{(G,T)}(X)^{W(G)}$) be an abelian group of invariants of $K_T{}^*(X)$ (respectively $K^*{}_{(G,T)}(X)$) under the action of $W(G)$. Then we have

(5. 1. 4)
$$v(K_G{}^*(X)) \subset K_T{}^*(X)^{W(G)}$$
$$\partial(K_T{}^*(X)^{W(G)}) \subset K^*{}_{(G,T)}(X)^{W(G)},$$

and the commutative diagram

(5. 1. 5)
$$\begin{array}{ccccccc} K_G{}^*(X) & \xrightarrow{\ v\ } & K_T{}^*(X) & \xrightarrow{\ \partial\ } & K^*{}_{(G,T)}(X) & \xrightarrow{\ u\ } & K_G{}^*(X) \\ & \searrow{\scriptstyle v} & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} & \nearrow{\scriptstyle u} & \\ & & K_T{}^*(X) & \xrightarrow{\ \partial\ } & K^*{}_{(G,T)}(X) & & \end{array}$$

for all $w \in W(G)$.

**5. 2 Proof of Maine Theorem (B).**

By 3. 5 Theorem, Theorem (A), 4. 11 Corollary and (5. 1. 5), the proof will be carried out directly. **Note :** From 4. 11 Corollary, the exact seqences of 3. 5, 3. 6

and 3.9 are extendable to the right side.

## References

[1]. M. F. ATIYAH and F. HIRZEBRUCH, Vector bundles and homogeneous spaces, Proc. Symp. Pure Math. 3, Differential Geometry, 1961, 7–38.

[2]. M. F. ATIYAH, Bott periodicity and the index of elliptic operators, Quart. J. of Math. (Oxford) (2) 19 (1968), 113–140.

[3]. M. F. ATIYAH and G. B. SEGAL, Equivariant $K$-theory, Lectures at Oxford univ. 1965, preprint of Univ. of Warwick.