On the Number of Representations of an Integer as the Sum of the Square of a Prime and an r-free Integer

Dedicated to the memory of Professor Z. SUETUNA

By SABURÔ UCHIYAMA

Department of Mathematics, Faculty of Science,
Shinshu University
(Received September 28, 1970)

1. Introduction. Let r be any integer greater than 1. An integer is called r-free if it is not divisible by the r th power of any prime.

Th. Estermann [1, § 1] discussed the representation of an integer as the sum of a square and a square-free (i. e. 2-free) integer, giving an asymptotic formula for the number of such representations. A similar problem has been treated by L. Mirsky [3] of representing an integer as the sum of a prime and an r-free integer, where r is an arbitrary integer ≥ 2.

In the present paper we shall consider the number $Q_r(n)$ of representations of an integer n as the sum of the square of a prime and an r-free integer.

For $r=2$ and 3 there are certain irregularities in the distribution of values of $Q_r(n)$. Indeed, one observes at once that

$$Q_2(n) \leq 1 \quad \text{for} \quad n \equiv 1 \pmod 4$$

and

$$Q_3(n) \leq 1 \quad \text{for} \quad n \equiv 1 \pmod 8.$$

For the sake of convenience we define the symbol $c_r(n)$ for integers $r \geq 2$ and $n \geq 1$ by setting for $r = 2$

$$c_2(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

and for $r \geq 3$

$$c_r(n) = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

Also, li x denotes the logarithmic integral, so that

$$\text{li } x = \text{li } 2 + \int_2^x \frac{dt}{\log t} \qquad (x > 1).$$

We shall prove the following

Theorem. *Let r be an arbitrary integer ≥ 2 and H any positive real number. Then we have*

$$Q_r(n) = A_r(n) \text{ li } n^{1/2} + O\left(\frac{n^{1/2}}{\log^H n}\right) \qquad (n \to \infty)$$

with

$$A_r(n) = \left(1 - \frac{c_r(n)}{2^{r-2}}\right) \prod_{p \nmid 2n} \left(1 - \frac{1 + \left(\frac{n}{p}\right)}{p^{r-1}(p-1)}\right),$$

where the O-constant depends at most on r and H. In particular, every sufficiently large integer can be represented as the sum of the square of a prime and an r-free integer, provided that $r \geq 4$.

2. Preliminaries. Letters d, m, n will be used to denote positive integers and p to denote prime numbers. As usual, $\varphi(m)$ denotes the Euler totient function and $\mu(m)$ the Möbius function. Also, we denote by $v(m)$ the number of distinct prime divisors of m and by $\tau(m)$ the number of positive divisors of m. We have $2^{v(m)} \leq \tau(m)$ with equality for square-free m. $\left(\frac{n}{p}\right)$ is the Legendre symbol for quadratic residuals.

The following facts are well known (cf. [2, Theorems 328, 315, 318]):

$$\frac{1}{\varphi(m)} = O\left(\frac{\log\log 3m}{m}\right),$$

$$\tau(m) = O(m^\varepsilon) \text{ for any fixed } \varepsilon > 0,$$

$$\sum_{m \leq x} \tau(m) = x \log x + O(x) \text{ for } x \geq 1.$$

Furthermore we need two auxiliary results which will be formulated as lemmas.

Lemma 1. *Denote by $\pi(x, k, l)$ the number of primes $p \leq x$, $p \equiv l \pmod{k}$, where $k \geq 1$, l are integers with $(k, l) = 1$. Let B and E be arbitrarily large but fixed positive real numbers. Then we have*

$$\pi(x, k, l) = \frac{1}{\varphi(k)} \text{ li } x + O\left(\frac{x}{\log^B x}\right) \qquad (x \to \infty)$$

uniformly for $1 \leq k \leq \log^E x$, $(l, \ k) = 1$, *where the* O-*constant depends on* B *and* E.

This is a slightly weakened form of the well-known Siegel-Walfisz theorem (cf. [5, Chap. IV, Theorem 8.3]).

Lemma 2. *Let* r *be an integer* ≥ 2, d *a square-free integer* ≥ 1, *and* a *an integer with* $(a, \ d) = 1$. *Suppose that* a *is a quadratic residue* (mod d^r). *Then, the number* $s(d)$ *of the incongruent solutions* z *of the congruence*

$$z^2 \equiv a \ (\text{mod} \ d^r)$$

is given by

$$s(d) = 2^{v(d)}$$

for $r = 2$ *and*

$$s(d) = \begin{cases} 2^{v(d)} & \text{if } d \text{ is odd,} \\ 2^{v(d)+1} & \text{if } d \text{ is even,} \end{cases}$$

for $r \geq 3$.

This result can be easily verified by appealing e. g. to [4, Theorem 47]. Note that the integer a is a quadratic residue (mod d^r) if and only if a is a quadratic residue (mod p) for all prime factors p of d, and further $a \equiv 1$ (mod 4 or 8) when d is divisible by 2 according as $r = 2$ or $r \geq 3$.

3. Proof of the theorem. Our proof will be carried out along the lines parallel to those in [3].

We write $Q_r'(n)$ for the number of ways of representing an integer n in the form $n = p^2 + m$, where p is a prime with $(p, \ n) = 1$ and m is an r-free integer. Since

$$\sum_{d^r | m} \mu(d) = \begin{cases} 1 & \text{if } m \text{ is } r\text{-free,} \\ 0 & \text{otheirwise,} \end{cases}$$

we have, indicating by $\sum_d^*$ the summaton over d for which n is a quadratic residue (mod d^r),

$$Q_r'(n) = \sum_{\substack{p \leq n^{1/2} \\ (p, n) = 1}} \sum_{d^r | n - p^2} \mu(d)$$

$$= \sum_{\substack{d \leq n^{1/r} \\ (d, n) = 1}}^* \mu(d) \sum_{\substack{p \leq n^{1/2} \\ (p, n) = 1 \\ p^2 \equiv n(d^r)}} 1$$

$$= \sum_{d \leq x}^* \mu(d) \sum_p 1 + \sum_{x < d \leq n^{1/2r}}^* \mu(d) \sum_p 1 + \sum_{n^{1/2r} < d \leq n^{1/r}}^* \mu(d) \sum_p 1$$

$$=\sum_1 + \sum_2 + \sum_3,$$

say, where $x = \log^H n$.

To evaluate $\sum_1$ we use Lemma 1 with $B = 3H$ and $E = rH$ and Lemma 2. Let a_i $(i = 1, 2, \cdots, s(d))$ be the incongruent solutions z of the congruence

$$z^2 \equiv n \pmod{d^r},$$

n being assumed to be a quadratic residue (mod d^r). Obviously, $(d, n) = 1$ implies $(a_i, d) = 1$ $(i = 1, 2, \cdots, s(d))$. We have then

$$\sum_1 = \sum_{\substack{d \leq x \\ (d,n)=1}}^* \mu(d) \sum_{\substack{p \leq n^{1/2} \\ (p,n)=1 \\ p^2 \equiv n(d^r)}} 1$$

$$= \sum_{\substack{d \leq x \\ (d,n)=1}}^* \mu(d) \sum_{i=1}^{s(d)} \pi(n^{1/2}, d^r, a_i) + O\left(xv(n)\right)$$

$$= \sum_{\substack{d \leq x \\ (d,n)=1}}^* \mu(d) \left(\frac{s(d)}{\varphi(d^r)} \operatorname{li} n^{1/2} + O\left(\frac{s(d)n^{1/2}}{\log^B n} \right) \right) + O(xv(n))$$

$$= \sum_{(d,n)=1}^* \frac{\mu(d)s(d)}{\varphi(d^r)} \operatorname{li} n^{1/2} + O\left(\sum_{d>x} \frac{s(d)}{\varphi(d^r)} \frac{n^{1/2}}{\log n} \right)$$

$$+ O\left(\sum_{d \leq x} s(d) \frac{n^{1/2}}{\log^B n} \right) + O(xv(n))$$

$$= \sum_{(d,n)=1}^* \frac{\mu(d)s(d)}{\varphi(d^r)} \operatorname{li} n^{1/2} + O\left(\frac{n^{1/2}}{\log^H n} \right),$$

where

$$\sum_{\substack{d \text{ odd} \\ (d,n)=1}}^* \frac{\mu(d)s(d)}{\varphi(d^r)} = \prod_{p \nmid 2n} \left(1 - \frac{1 + \left(\dfrac{n}{p}\right)}{p^{r-1}(p-1)} \right)$$

and

$$\sum_{\substack{d \text{ even} \\ (d,n)=1}}^* \frac{\mu(d)s(d)}{\varphi(d^r)} = -\frac{c_r}{2^{r-2}} \prod_{p \nmid 2n} \left(1 - \frac{1 + \left(\dfrac{n}{p}\right)}{p^{r-1}(p-1)} \right)$$

with $c_2 = 1$, $c_r = 2$ $(r \geq 3)$.

Next, we find easily

$$\left|\sum_2\right| \leq \sum_{x<d\leq n^{1/2}} \sum_{\substack{p\leq n^{1/2} \\ p^2\equiv n(d^r)}} 1 = O\left(\sum_{d>x} \frac{s(d)n^{1/2}}{d^r}\right)$$

$$= O\left(\frac{n^{1/2}\log x}{x^{r-1}}\right) = O\left(\frac{n^{1/2}}{\log^H n}\right).$$

Finally, we have for $r = 2$

$$\sum_3 = \sum_{n^{1/4}<d\leq n^{1/3}}^{*} \mu(d)\sum_p 1 + \sum_{n^{1/3}<d\leq n^{1/2}}^{*} \mu(d)\sum_p 1$$

$$= \sum_{3,1} + \sum_{3,2},$$

say, where

$$\left|\sum_{3,1}\right| \leq \sum_{n^{1/4}<d\leq n^{1/3}} \sum_{\substack{p\leq d^r \\ p^2\equiv n(d^r)}} 1 \leq \sum_{d\leq n^{1/3}} s(d)$$

$$= O(n^{1/3} \log n)$$

and, arguing just as in [1, p. 654],

$$\left|\sum_{3,2}\right| \leq 2\tau(n) \sum_{m<n^{1/3}} 1 = O(n^{(1/3)+\varepsilon}),$$

and for $r \geq 3$

$$\left|\sum_3\right| \leq \sum_{n^{1/2r}<d\leq n^{1/r}} \sum_{\substack{p\leq d^r \\ p^2\equiv n(d^r)}} 1 \leq \sum_{d\leq n^{1/r}} s(d')$$

$$= O(n^{1/r} \log n),$$

so that in either case

$$\sum_3 = O(n^{(1/3)+\varepsilon}) = O\left(\frac{n^{1/2}}{\log^H n}\right).$$

Gathering up these results, we thus obtain the asymptotic formula for $Q_r(n)$ enunciated in the theorem, since

$$Q_r(n) = Q_r'(n) + O(v(n)).$$

We could, of course, somewhat simplify our argument in the proof of the theorem, if use were made of some recent results, in place of Lemma 1, from the theory of the large sieve, but yielding substantially nothing more.

References

[1] Th. Estermann : Einige Sätze über quadratfreie Zahlen. Math. Annalen, **105** (1931), 653–662.

[2] G. H. Hardy and E. M. Wright : *An Introduction to the Theory of Numbers.* 4 th ed. Oxford Univ. Press, London, 1965.

[3] L. Mirsky : The number of representations of an integer as the sum of a prime and a k-free integer. Amer. Math. Monthly, **56** (1949), 17–19.

[4] T. Nagell : *Introduction to Number Theory.* Almqvist & Wiksell, Stockholm, 1951.

[5] K. Prachar : *Primzahlverteilung.* Springer-Verl., Berlin-Göttingen-Heidelberg, 1957.