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Note on Quadratic Extensions of Rings

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Throughout the present paper, we assume that Δ is a ring with identity 1, Γ a subring with the same identity whose center contains a field of characteristic $\neq 2$. Further $[\Delta:\Gamma]_r=2$ if there exists a right Γ -free basis for Δ. Then the followings are well known.

i) If Δ, Γ are fields, then Δ/Γ is a Galois extension.

ii) (Jacobson) If Δ is a division ring and Γ is a finite dimensional central division ring, then Δ/Γ is a Galois extension [3].

iii) (Tominaga) If Δ is a simple ring and Γ is a finite dimensional central simple algebra, then Δ/Γ is a Galois extension [6].

Let $\Delta=Q[X]/(X^2)$ where Q is the field of rational numbers. Then $\Delta=Q\oplus\alpha Q$ is a free Q-module of rank 2 where α is the residue class of X modulo (X^2). Then Δ/Q is non-Galois. For if σ is an arbitrary Q-automorphism of Δ, then $\sigma(\alpha)=\alpha q$ for some $q\in Q$ since $\alpha^2=0$. Let $\mathfrak{M}$ be a maximal ideal of Δ containing αQ, $\beta=q_0+\alpha q_1$ $(q_0, q_1\in Q)$ an arbitrary element of Δ. Then $\sigma(\beta)-\beta=\alpha(qq_1-q_1)\in\mathfrak{M}$. Hence Δ/Q is non-Galois by Th. *1.3*, (*f*) of [2].

In this paper, we shall give

iv) Necessary and sufficient conditions for Δ/Γ to be a σ-Galois extension for some automorphism σ of Δ of order 2 when Δ, Γ are commutative rings and $\Delta=\Gamma+\alpha\Gamma$ is Γ projective such that $\{1\otimes 1,\ \alpha\otimes 1\}$ is a $\Gamma_{\mathfrak{m}}$-free basis of $\Delta_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of Γ.

In the subsequent studies, we assume that Δ, Γ are commutaive.

Let $\Delta=\Gamma+\alpha\Gamma$ for some $\alpha\in\Delta$. Then we may assume that $\alpha^2\in\Gamma$. For, if $\alpha^2+\alpha\gamma_1+\gamma_0=0$ for some γ_0, $\gamma_1\in\Gamma$, $\{1,\ \beta=\alpha+\gamma_1/2\}$ forms a system of Γ -generator for Δ and $\beta^2\in\Gamma$.

Hence, in what follows, a system of Γ -generator (or a Γ -free basis) $\{1,\ \alpha\}$ for Δ, we mean such one that $\alpha^2\in\Gamma$.

Lemma 1 *Let $\Delta=\Gamma+\alpha\Gamma$ be Γ -projective such that $\{1\otimes 1,\ \alpha\otimes 1\}$ is a free $\Gamma_{\mathfrak{m}}$ -basis for $\Delta_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of Γ where $\Gamma_{\mathfrak{m}}$ is a localization of Γ at*

$\mathfrak{m}$ *and* $\Delta_{\mathfrak{m}} = \Delta \otimes_{\Gamma} \Gamma_{\mathfrak{m}}$. *Then the correspondence* $\sigma : \gamma_0 + \alpha\cdot\gamma_1 \to \gamma_0 - \alpha\cdot\gamma_1$ $(\gamma_0, \gamma_1 \in \Gamma)$ *of* Δ *is an automorphism of* Δ *(of order 2) and* $\Delta^{\sigma} = \Gamma$.

Proof. Let $\mathfrak{m}$ be an arbitrary maximal ideal of Γ. Then the map $\Pi\iota_{\mathfrak{m}} : \Delta \to \Pi_{\mathfrak{m}}\Delta_{\mathfrak{m}}$ defined by $\delta \to (\cdots, \delta \otimes 1, \cdots)$* is a $\Gamma_{\mathfrak{m}}$-monomorphism. Now the map $\bar{\sigma}$ of $\Pi_{\mathfrak{m}}\Delta_{\mathfrak{m}}$ defined by $(\cdots, (\gamma_0 + \alpha\cdot\gamma_1) \otimes 1, \cdots) \to (\cdots, (\gamma_0 - \alpha\cdot\gamma_1) \otimes 1, \cdots)$ is a $\Gamma_{\mathfrak{m}}$-automorphism.

If $\gamma_0 + \alpha\cdot\gamma_1 = \gamma_0' + \alpha\cdot\gamma_1'(\gamma_i, \gamma_i' \in \Gamma)$, $\bar{\sigma}\Pi\iota_{\mathfrak{m}}((\gamma_0 - \gamma_0') + \alpha(\gamma_1 - \gamma_1')) = \bar{\sigma}(\cdots, ((\gamma_0 - \gamma_0') + \alpha(\gamma_1 - \gamma_1')) \otimes 1, \cdots) = (\cdots, (\gamma_0 - \gamma_0') - \alpha(\gamma_1 - \gamma_1')) \otimes 1, \cdots) = \Pi\iota_{\mathfrak{m}}((\gamma_0 - \gamma_0') - \alpha(\gamma_1 - \gamma_1')) = 0$. Hence $\gamma_0 - \gamma_0' - \alpha(\gamma_1 - \gamma_1') = 0$. This means that σ is well defined, and hence the following diagram is commutative

$$\begin{array}{ccccc} 0 & \longrightarrow & \Delta & \xrightarrow{\Pi\iota_{\mathfrak{m}}} & \Pi_{\mathfrak{m}}\Delta_{\mathfrak{m}} \quad \text{(exact)} \\ & & \downarrow \sigma & & \downarrow \bar{\sigma} \\ 0 & \longrightarrow & \Delta & \xrightarrow{\Pi\iota_{\mathfrak{m}}} & \Pi_{\mathfrak{m}}\Delta_{\mathfrak{m}} \quad \text{(exact)} \end{array}$$

Hence σ is a Γ -automorphism of Δ as a Γ -module.

Now $\sigma((\gamma_0 + \alpha\cdot\gamma_1)(\gamma_0' + \alpha\cdot\gamma_1')) = \sigma(\gamma_0\gamma_0' + \alpha^2\gamma_1\gamma_1' + \alpha(\gamma_1\gamma_0' + \gamma_0\gamma_1')) = \gamma_0\gamma_0' + \alpha^2\gamma_1\gamma_1' - \alpha(\gamma_1\gamma_0' + \gamma_0\gamma_1') = (\gamma_0 - \alpha\gamma_1)(\gamma_0' - \alpha\cdot\gamma_1') = \sigma(\gamma_0 + \alpha\cdot\gamma_1)\sigma(\gamma_0' + \alpha\gamma_1')$ yields at once σ is an automorphism of Δ of order 2.

Proposition 1. *Let* Γ *be a local ring,* Δ *a free* Γ *-module of rank 2 with a* Γ *-free basis* $\{1, \alpha\}$.

a) Δ/Γ *is a* σ*-Galois extension where* σ *is an automorphism of* Δ *of order 2 if and only if* $\alpha^2 \in U(\Gamma)$, *the group of units of* Γ.

b) $\alpha^2 \in J(\Gamma)$, *the Jacobson radical of* Γ, *if and only if* $J(\Delta) \supsetneqq J(\Gamma)\Delta$. *Moreover, if this is the case,* $J(\Delta) = J(\Gamma) \oplus \alpha\Gamma$, *and* Δ *is local.*

Proof. **a)** Let Δ/Γ be a σ-Galois extension. Then, since Δ/Γ is separable and $\Delta \cong \Gamma[X]/(X^2 - \alpha^2)$, we can see that $X^2 - \alpha^2$ is a separable polynomial. Hence $\alpha^2 \in U(\Gamma)$ by Cor. 2.4 of [4].

Conversely, let σ be the map defined by $\gamma_0 + \alpha\cdot\gamma_1 \to \gamma_0 - \alpha\cdot\gamma_1$ $(\gamma_0, \gamma_1 \in \Gamma)$. Then σ is an automorphism of Δ with $\Delta^{\sigma} = \Gamma$. Now since $\alpha^2 \in U(\Gamma)$, $-(\alpha^{-1})\sigma(\alpha)\cdot 1/2 + \alpha^{-1}\cdot\alpha/2 = 1$, $-(\alpha^{-1})\sigma(\alpha)\sigma(1/2) + \alpha^{-1}\sigma(\alpha/2) = 0$ show that Δ/Γ is a σ-Galois extension.

b) Let $\mathfrak{M}$ be a maximal ideal of Δ. If $\mathfrak{M} \not\supseteq J(\Gamma)$, $\Delta = \mathfrak{M} + J(\Gamma)\Delta$. Hence, by Nakayama's Lemma, we have a contradiction $\Delta = \mathfrak{M}$. Thus $J(\Gamma) \subseteq \cap\, \mathfrak{M} = J(\Delta)$, and hence $J(\Gamma)\Delta = J(\Gamma) \oplus \alpha J(\Gamma) \subseteq J(\Delta)$. Now, assume that $\alpha^2 \in J(\Gamma)$. Then $\alpha^2 \in J(\Delta) \subseteq \mathfrak{M}$ for each maximal ideal $\mathfrak{M}$ of Δ yields $\alpha \in \mathfrak{M}$ and so $\alpha \in J(\Delta)$. Consequently, we have $J(\Gamma)\Delta = J(\Gamma) \oplus \alpha J(\Gamma) \subsetneqq J(\Gamma) \oplus \alpha\Gamma \subseteq J(\Delta)$. On the other hand, if $\gamma_0 + \alpha\cdot\gamma_1 \in J(\Delta)$, then $\gamma_0 \in J(\Gamma)$ since $\alpha\cdot\gamma_1 \in J(\Delta)$. Thus we obtain $J(\Delta) = J(\Gamma) \oplus \alpha\Gamma$. Next, let $\gamma_0 + \alpha\cdot\gamma_1$ and $\gamma_0' + \alpha\cdot\gamma_1'$ be non units of Δ. Then γ_0, γ_0' are so in Γ, and hence $\gamma_0 + \gamma_0' \in J(\Gamma)$, since Γ is local. From this, we obtain $(\gamma_0 + \alpha\gamma_1) + (\gamma_0' + \alpha\gamma_1') = (\gamma_0$

* Cf. Corollaire I, Chap. II, §3 of [1].

$+\gamma_0')+\alpha(\gamma_1+\gamma_1')\in J(\Gamma)\oplus\alpha\Gamma\subsetneqq J(\Delta)$. Thus $J(\Delta)$ is the set of non units of Δ. Conversely, let $J(\Delta)\supsetneqq J(\Gamma)\Delta$. If $\alpha^2\notin J(\Gamma)$, then $\alpha^2\in U(\Gamma)$, since Γ is local. Then Δ/Γ is Galois by **a**). But this yields a contradiction $J(\Delta)=J(\Gamma)\Delta$ by Prop. *7.8* of [*5*].

Corollary 1. *Let Γ be local, Δ a free Γ-module of rank 2. Then the followings are equivalent.*

a) *Δ/Γ is a σ-Galois extension where σ is an automorphism of Δ of order 2.*

b) *If $\{1,\alpha\}$ is a Γ -free basis for Δ, then $\alpha^2\in U(\Gamma)$.*

c) $J(\Delta)=J(\Gamma)\Delta$.

Proof. **a**) $\rightleftarrows$ **b**), **c**) $\to$ **b**) are direct consequences of Proposition *1*.

a) $\to$ **c**). Since $\Delta/J(\Gamma)\Delta$ is a σ-Galois extension over $\Gamma/J(\Gamma)\Delta\cap\Gamma$, if we note that $J(\Gamma)\Delta\cap\Gamma=J(\Gamma)$, $\Delta/J(\Gamma)\Delta$ is a -Galois extension over a field $\Gamma\oplus J(\Gamma)\Delta/J(\Gamma)\Delta$. Hence $\Delta/J(\Gamma)\Delta$ is semi-simple. Thus $J(\Delta)=J(\Gamma)\Delta$.

Theorem 1. *Let $\Delta=\Gamma+\alpha\Gamma$ be Γ-projective such that $\{1\otimes 1,\alpha\otimes 1\}$ is a $\Gamma_{\mathfrak{m}}$-basis of $\Delta_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of Γ. Then Δ/Γ is a σ-Galois extension where σ is an automorphism of Δ of order 2, if and only if, $\alpha^2\in U(\Gamma)$.*

Proof. By Lemma *1*, there exists an automorphism σ of Δ (of order *2*) defined by $\sigma(\gamma_0+\alpha\cdot\gamma_1)=\gamma_0-\alpha\cdot\gamma_1$ and $\Delta^\sigma=\Gamma$. Hence, if $\alpha^2\in U(\Gamma)$, we can easily see that the existence of a σ-Galois coordinate system of Δ/Γ as similar methods as that of Proposition *1*. Conversely, if Δ/Γ is Galois, then Δ/Γ is separable. As is shown in Prop. *2.3* of [*4*], Δ/Γ is separable if and only if $\Delta_{\mathfrak{m}}/\Gamma_{\mathfrak{m}}$ is separable for each maximal ideal $\mathfrak{m}$ of Γ. Since Δ/Γ is σ-Galois, so is $\Delta_{\mathfrak{m}}/\Gamma_{\mathfrak{m}}$. Hence $\alpha\otimes 1\in\Delta_{\mathfrak{m}}=\Delta\otimes_\Gamma\Gamma_{\mathfrak{m}}$ is contained in $U(\Delta_{\mathfrak{m}})$ by Proposition *1*. Thus $\alpha^2\otimes 1=1\otimes\alpha^2\in U(\Gamma_{\mathfrak{m}})$. This shows that $\alpha^2\in U(\Gamma)$, since $\alpha\in\Gamma-\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of Γ.

Corollary 2. *Let $\Delta=\Gamma+\alpha\Gamma$ be a free Γ-module of rank 2. Then Δ/Γ is a σ-Galois extension where σ is an automorphism of Δ of order 2 if and only if $\alpha^2\in U(\Gamma)$.*

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