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A Note on Complex Characters of Finite Groups

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Let G be a finite group, P a p-Sylow subgroup of G where p is a prime number, and χ and θ complex characters of G. For a natural number m, we set $\chi^{\langle m\rangle}(g)=\chi(g^m)$ $(g\in G)$. Then $\chi^{\langle m\rangle}$ is a class function on G. If every irreducible character of G is an induced character of a linear character of a normal subgroup of G, then we call G an M^*-group.

The purpose of this paper is to prove the following

Theorem. [1)] *If every subgroup of P is an M^*-group, then $\chi^{\langle p\rangle}$ is contained in the character ring of G.*

Corollary. *If P is abelian, then $\chi^{\langle p\rangle}$ is contained in the character ring of G.*

We shall need some lemmas to prove this theorem.

Lemma 1. *The inner product $(\chi^{\langle m\rangle}, \theta)$ of $\chi^{\langle m\rangle}$ and θ is a rational number.*

Proof. For g, $h\in G$, let $g\sim h$ if $\langle g\rangle=\langle h\rangle$. It is evident that "$\sim$" is an equivalence relation and $g\sim h$ if and only if $g=h^i$ for some i with $(i, |\langle g\rangle|)=1$. If C_g is an equivalence class of g with respect to $\sim$, $\sum_{h\in C_g}\chi(h^m)\theta(h)$ is invariant by all elements of the Galois group of $Q_{|\langle g\rangle|}$ over Q. [2)] Hence, the assertion is clear.

Lemma 2. *Let G be an M^*-group. If χ is irreducible and θ is linear, then $(\chi^{\langle m\rangle}, \theta)$ is an integer.*

Proof. Since G is an M^*-group, there exist a normal subgroup H of G and a linear character η of H such that χ is an induced character or η. Let $G=\bigcup_{i=1}^{e} Ha_i$ be the right coset decomposition of G modulo H. Then

$$
\begin{aligned}
(\chi^{\langle m\rangle},\ \theta) &= \frac{1}{|H|}\sum_{g\in G}\eta(g^m)\overline{\theta(g)} \\
&= \frac{1}{|H|}\sum_{i=1}^{e}\sum_{h\in H}\eta((ha_i)^m)\,\overline{\theta(ha_i)} \\
&= \sum_{i=1}^{e}\{\frac{1}{|H|}\sum_{h\in H}(\eta\eta^{a_i}\cdots\eta^{a_i^{m-1}})(h)\overline{\theta(h)}\}\eta(a_i^{m})\overline{\theta(a_i)}
\end{aligned}
$$

1) Cf. [1, (3.5)].

2) Q denotes the rational number field and $Q_{|\langle g\rangle|}$ the field of $|\langle g\rangle|$th roots of unity over Q.

$$=\sum_{i=1}^{e}(\eta\eta^{a_i}\cdots\eta^{a_i^{m-1}},\ \theta_H)_H\eta(a_i^m)\overline{\theta(a_i)}.$$

Therefore, $(\chi^{\langle m\rangle},\theta)$ is an algebraic integer, and so an integer by Lemma 1.

Lemma 3 *If* χ *is irreducible,* θ *is linear and* $(|G|, m)=1$, *then* $(\chi^{\langle m\rangle}, \theta)=0$ *or* 1.

Proof. Since $|G|$ and m are relatively prime, $g\rightarrow g^m$ is a permutation on G and $(\chi^{\langle m\rangle}, \theta)=(\chi, \theta^k)$ where $mk\equiv 1 \bmod |G|$. Since θ^k is an irreducible character, we obtain the assertion.

Proof of Theorem. By Brauer induction theorem and Frobenius reciprocity theorem, it suffices to prove that for an irreducible character ω and a linear character η of an elementary subgroup H of G, $(\omega^{\langle p\rangle}, \eta)$ is an integer. If H is p-elementary, then by [1, (6.3)], and Lemma 2, $(\omega^{\langle p\rangle}, \eta)$ is an integer. If H is q-elementary $(q\neq p)$, then by Lemma 3, $(\omega^{\langle p\rangle}, \eta)$ is an integer.

Reference

[1] W. FEIT: *Characters of finite groups*, Benjamin, New York, 1967.