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Algebraic Cohomology of Loop Spaces

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Introduction.

It is known that if φ is a closed *1*-form on a smooth (connected) manifold X, then the period of φ, that is the value of φ on γ, a closed path on X, defines a homomorphism from $\pi_1(X)$ into $\boldsymbol{F}$ ($\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$ either φ is real valued or complex valued) and denoting

$$\chi(\langle\varphi\rangle)(\langle\gamma\rangle)=\int_\gamma\varphi,$$

where $\langle\varphi\rangle$ is the de Rham class of φ and $\langle\gamma\rangle$ is the homotopy class of γ, χ is the isomorphism from $H^1(X, \boldsymbol{F})$ onto Hom. $(\pi_1(X), \boldsymbol{F})$. This is, for example, the base of the theory of Picard varieties ([4]). The purpose of this paper is to generalize this relation for higher degree forms.

Roughly speaking, this generalization is done as follows: Denoting $E(X)$ the path space over X (for the convenience, we assume that the end point of a path is the base point), then $E(X)$ is a contractible smooth Banach manifold ([1], [5], [7]). Hence denoting π the projection from $E(X)$ onto X, $\pi^*(\varphi)$ is written as $d\varphi_1$ on $E(X)$ if φ is closed. Then the period of φ_1 with respect to $\Omega(X)$, that is $\varphi_1(\alpha*\eta)-\varphi_1(\alpha)$ is not equal to *0* in general, but $d(\varphi_1(\alpha*\eta)-\varphi_1(\alpha))$ is equal to *0* because $\pi^*(\varphi)$ is invariant under the operation of $\Omega(X)$. Hence we may set

$$\varphi_1(\alpha*\eta)-\varphi_1(\alpha)=d\varphi_2(\alpha,\ \eta)$$

on $E(X)$. Moreover, if the value of $\varphi_1((\alpha*\eta)*\zeta)$ is equal to $\varphi_1(\alpha*(\eta*\zeta))$, then

$$d(\varphi_2(\alpha*\eta_1,\ \eta_2)-\varphi_2(\alpha,\ \eta_1*\eta_2)+\varphi_2(\alpha,\ \eta_1))=0.$$

Repeating this, we can construct $(p-r)$–form $\varphi_r(\alpha,\ \eta_1,\cdots,\eta_{r-1})$ for $r\leqq p$ and we get

$$d(\varphi_p(\alpha*\eta_1,\ \eta_2,\cdots,\eta_p)+\sum_{i=1}^{p-1}(-1)^i\varphi_p(\alpha,\ \eta_1,\cdots,\ \eta_i*\eta_{i+1},\ \cdots,\ \eta_p)+$$

$$+(-1)^p\varphi_p(\alpha,\ \eta_1,\ \cdots,\ \eta_{p-1}))=0.$$

Therefore $\varphi_p(\alpha*\eta_1,\ \eta_1,\cdots,\ \eta_p)+\sum_{i=1}^{p\ 1}(-1)^i\varphi_p(\alpha,\ \eta_1,\cdots,\eta_i*\eta_{i+1},\cdots,\ \eta_p)+(-1)^p\varphi_p(\alpha,\eta_1,\cdots,$ $\eta_{p-1})$ should be constant in α and setting

$$\begin{aligned}&c_p(\eta_1,\ \cdots,\ \eta_p)\\&=\varphi_p(\alpha*\eta_1,\ \eta_2,\cdots,\eta_p)+\sum_{i=1}^{p-1}(-1)^i\varphi_p(\alpha,\ \eta_1,\cdots,\eta_i*\eta_{i+1},\cdots,\eta_p)+\\&\quad+(-1)^p\varphi_p(\alpha,\ \eta_1,\ \cdots,\ \eta_{p-1}),\end{aligned}$$

$c_p(\eta_1,\ \cdots,\ \eta_p)$ must be the period of φ. We remark that this construction is similar the construction of double cochain in harmonic integrals ([10]).

Of course, this discussion needs several justifications. For example, since the multiplication of $\Omega(X)$ is not associative, we cannot expect $\varphi_1((\alpha*\eta)*\zeta)=\varphi_1(\alpha*(\eta*\zeta))$ in general. But since the quotient space $[\Omega(X)]=(X)/H^+(\boldsymbol{I})$, where $H^+(\boldsymbol{I})$ is the group of orientation preserving homeomorphisms of $\boldsymbol{I}=[0,\ 1]$ and $h\in H^+(\boldsymbol{I})$ operates $\Omega(X)$ by $h^*(\alpha)(s)=\alpha(h(s))$, has associative multiplication, the above discussion may be done if we can construct each φ_i to be the forms on $[E(X)]\times[\Omega(X)]\times\cdots\times[\Omega(X)]$, where $[E(X)]=E(X)/H^+(\boldsymbol{I})$. The possibility of this construction is proved in § 4. Then the period of φ is defined to be an element of p-th algebraic cohomology group of $[\Omega(X)]$ with coefficient in $\boldsymbol{F}$ (§ 5). Here the p-th algebraic cohomology group of $[\Omega(X)]$ is defined as follows: Set $C^p([\Omega(X)],\ \boldsymbol{F})$ the group of all (continuous) maps from $[\Omega(X)]\overbrace{\times\cdots\times}^{p}[\Omega(X)]$ into $\boldsymbol{F}$, the coboundary homomorphism $\delta: C^p[(\Omega(X)],\ \boldsymbol{F})\to C^{p+1}([\Omega(X)],\ \boldsymbol{F})$ is given by

$$\begin{aligned}&(\delta c_p)([\eta_1],\ \cdots,\ [\eta_{p+1}])\\&=c_p([\eta_2],\ \cdots,\ [\eta_{p+1}])+\sum_{i=1}^{p}(-1)^ic_p([\eta_1],\ \cdots,\ [\eta_i*\eta_{i+1}],\ \cdots,\ [\eta_{p+1}])+\\&\quad+(-1)^{p+1}c_p([\eta_1],\ \cdots,\ [\eta_p]),\end{aligned}$$

where $[\eta]$ means the class of η in $[\Omega(X)]$, then $B^p([\Omega(X)],\ \boldsymbol{F})/Z^p([\Omega(X)],\ \boldsymbol{F})$ is the p-th algebraic cohomology group $H_\Omega{}^p(X,\ \boldsymbol{F})$ with coefficient in $\boldsymbol{F}$. (In this paper, we denote $\Omega^p(X,\ \boldsymbol{F})$ instead of $C^p([\Omega(X)],\ \boldsymbol{F})$, § 2). Then our main theorem (§ 5, Theorem 4) assert that the above construction of period $\chi(\varphi)$ of φ give the isomorphism from $H^p\ (X,\ \boldsymbol{F})$ onto $H_\Omega{}^p(X,\ \boldsymbol{F})$ for $p\geq 1$. Since we get $H_\Omega{}^0(X,\ \boldsymbol{F})=\boldsymbol{F}$ if X is connected (§ 2, Theorem 1), we have

$$\chi: H^p(X,\ \boldsymbol{F})\simeq H_\Omega{}^p(X,\ \boldsymbol{F})$$

for all p if X is a (connected) smooth manifold. We note that for arbitrary (topological) abelian group G, we can define $H^p{}_\Omega(X, G)$ by the same method for arbitrary topological space X. Then since $H^p(X, G)$ can be defined for any topological space X, the similar isomorphism may be expected for any X and G. But the authour has no proof (or counterexample) for this problem.

The outline of this paper is as follows: In § 1, we treat the properties of $[E(X)]$ and $[\Omega(X)]$. Then we define $H_\Omega{}^p(X, G)$ in § 2. In § 2, we also define the group $H_E{}^p(X, F)$ as the cohomology group with cochain the (continuous) function $f: [E(X)]\times[\Omega(X)]\times\cdots\times[\Omega(X)]\to F$ and the coboundary homomorphism δ is given by

$$\begin{aligned}&(\delta f)([\alpha], [\eta_1], \cdots, [\eta_{p+1}])\\&=f([\alpha*\eta_1], [\eta_2], \cdots, [\eta_{p+1}])+\\&+\sum_{i=1}^{p}(-1)^i f([\alpha], [\eta_1], \cdots, [\eta_i*\eta_{i+1}], \cdots, [\eta_{p+1}])+\\&+(-1)^{p+1}f([\alpha], [\eta_1], \cdots, [\eta_p]).\end{aligned}$$

In § 3, we prove $H_E{}^p(X, F)=0$, $p\geqq 1$ under the assumption of X is a (topological) manifold (Theorem 3). It seems that this may be true for arbitrary X, but the authour has no proof (or counterexample) for this problem. The vanishing of $H_E{}^p(X, F)$ is used in the proof of theorem 4. § 4 is devoted to the study of a class of differential forms on $E(X)$, which includes the π^*-images of the differential forms on X. Then in § 5, we define the period of higher order forms on X and prove the main theorem (Theorem 4).

We note that in § 4 and § 5, we use $E_{2,k,0}(X)$ $(k>dim. X/2)$ instead of $E(X)$, where $E_{2,k,0}(X)$ is given by

$$\begin{aligned}E_{2,k,0}(X)=\{\alpha\mid\alpha: I\to X,\ \alpha \text{ belongs in } k\text{-th Sobolev space and}\\ \alpha(s)=x_0 \text{ if } s>1-\varepsilon \text{ for some } \varepsilon\}.\end{aligned}$$

§ 1. The spaces $[E(X)]$ and $[\Omega(X)]$.

1. Let X be a (connected) topological space with base point x_0, then we denote by $\Omega(X)$ and $E(X)$ the loop space and path space over X ([7]). For the convenience, in $E(X)$, we assume $\alpha(1)=x_0$, where $\alpha: I\to X$, I is the closed interval $[0, 1]$, is an element of $E(X)$. The projection from $E(X)$ onto X defined by $E(X)\ni\alpha\to\alpha(0)\in X$ is denoted by π. The multiplication of paths α and β is denoted by $\alpha*\beta$ if it is possible. If $0\leqq t\leqq 1$, then we set

$$\alpha_t(s)=\alpha(1-t+ts),\quad \alpha\in E(X),$$

α_t is also an element of $E(X)$ and $\alpha_1=\alpha$, $\alpha_0=e$, where e is given by $e(s)=x_0$,

$0 \leqq s \leqq 1$.

We denote $H^+(I)$ the group of orientation preserving homeomorphisms of I. Then we set

$$h^*(\alpha)(s) = \alpha(h(s)), \quad h \in H^+(I), \quad \alpha \in E(X).$$

Since $h^*(\alpha)$ belongs in $E(X)$ and if $\eta \in \Omega(X)$, then $h^*(\eta)$ also belongs in $\Omega(X)$, $H^+(I)$ operates $\Omega(X)$ and $E(X)$.

Lemma 1. *If $\alpha * \beta$ is possible, then $(h_1^*(\alpha)) * (h_2^*(\beta))$ is also possible and there is an $h \in H^+(I)$ such that*

$$h^*(\alpha * \beta) = (h_1^*(\alpha)) * (h_2^*(\beta)). \tag{1}$$

Proof. Since $h_1^*(\alpha)(1) = \alpha(h_1(1)) = \alpha(1) = \beta(0) = \beta(h_2(0)) = h_2^*(\beta)(0)$, $(h_1^*(\alpha)) * (h_2^*(\beta))$ is possible. Then we have (1) if we set

$$h(s) = \frac{1}{2} h_1(2s), \quad 0 \leqq s \leqq \frac{1}{2}, \quad h(s) = \frac{1}{2}(h_2(2s-1)+1), \quad \frac{1}{2} \leqq s \leqq 1.$$

Note. Similarly, if we set

$$h_1(s) = \frac{1}{2} h^{-1}(2h(s)), \quad 0 \leqq s \leqq s_0 = h^{-1}(\frac{1}{2}),$$

$$h_1(s) = \frac{1}{2}(h^{-1}(2h(s) - 1) + 1), \quad s_0 \leqq s \leqq 1,$$

we obtain

$$h^*(\alpha * \beta) = h_1^*((h^*(\alpha)) * (h^*(\beta)). \tag{1$'$}$$

Lemma 2. *The quotient space $\Omega(X)/H^+(I)$ is a semi-group by the multiplication induced from $\Omega(X)$ and it operates associatively on $E(X)/H^+(I)$.*

Proof. By (1) and (1)$'$, the multiplication in $\Omega(X)$ induces a multiplication of $\Omega(X)/H^+(I)$ and it operates on $E(X)/H^+(I)$.

If $(\alpha * \beta) * \gamma$ is possible, then $\alpha * (\beta * \gamma)$ is also possible and to define $h \in H^+(I)$ by

$$h(s) = 2s, \quad 0 \leqq s \leqq \frac{1}{4}, \quad h(s) = \frac{1}{4} + s, \quad \frac{1}{4} \leqq s \leqq \frac{1}{2},$$

$$h(s) = \frac{1}{2} + \frac{s}{2}, \quad \frac{1}{2} \leqq s \leqq 1,$$

we have

$$h^*((\alpha * \beta) * \gamma) = \alpha * (\beta * \gamma).$$

This proves the associativity.

Definition. *We denote the quotient spaces* $E(X)/H^+(\boldsymbol{I})$ *and* $\Omega(X)/H^+(\boldsymbol{I})$ *by* $[E(X)]$ *and* $[\Omega(X)]$. *The classes of* $\alpha \in E(X)$ *and* $\eta \in \Omega(X)$ *mod.* $H^+(\boldsymbol{I})$ *are denoted by* $[\alpha]$ *and* $[\eta]$. *The multiplications in* $[E(X)]$ *and* $[\Omega(X)]$ *are also denoted by* $*$.

Note. $h^*(\alpha)$ is homotopic to α. Hence there is a homomorphism ρ from $[\Omega(X)]$ onto $\pi_1(X)$ and a continuous map ρ from $[E(X)]$ onto $\widetilde{X}$, where $\widetilde{X}$ is the universal covering space of X.

Proof. By the theorem of Radon -Nikodym ([2]), there is a positive measurable function $m(s)$ on $\boldsymbol{I}$ such that

$$h(s) = \int_0^s m(u)du, \quad 0 \leqq s \leqq 1.$$

Then to define $h_t(s)$ $(0 \leqq t \leqq 1)$ by

$$h_t(s) = \int_0^s (m(u))^t du \Big/ \int_0^1 (m(u))^t du,$$

each h_t belongs in $H^+(\boldsymbol{I})$ and we get $h_1 = h$, $h_0(s) = s$ for all s $(0 \leqq s \leqq 1)$ and $h_t(s)$ is continuous in t and s. Hence $h^*(\alpha)$ is homotopic to α.

2. Definition. *In* $E(X)$, *we set*

$$E_0(X) = \{\alpha \mid \alpha(s) \neq x_0, \ s \neq 1\} \cup \{e\}. \tag{2}$$

By definition, we have

Lemma 3. $E_0(X)$ *has following properties.*

(i). $E_0(X)$ *is contractible by the contraction* $E_0(X) \times \boldsymbol{I} \ni (\alpha, \ t) \to \alpha_t \in E_0(X)$.

(ii). $E_0(X) \cap \Omega(X) = e$.

(iii). *If* $\alpha \in E_0(X)$ *and* $\alpha' \equiv \alpha$ *mod.* $H^+(\boldsymbol{I})$, *then* $\alpha' \in E_0(X)$.

By lemma 3, (iii), we can define the quotient space of $E_0(X)$ by $H^+(\boldsymbol{I})$. It is also denoted by $[E_0(X)]$.

Lemma 4. *If* X *satisfies*

(*). *For any* $x \in X$, *there exists a neighbourhood* $U(x)$ *of* X *such that for any* $y \in U(x)$, *it is possible to determine uniquely a path*

$\gamma = \gamma(s)$, $0 \leqq s \leqq 1$, $\gamma(1) = x$, $\gamma(0) = y$, $\gamma(s) \neq x_0$, $0 < s < 1$, *then there is a homeomorphism* $\iota_{U(x)}$; $U(x) \to E_0(X)$ *such that*

$$\pi \iota_{U(x)}(y) = y, \quad y \in U(x), \tag{3}$$

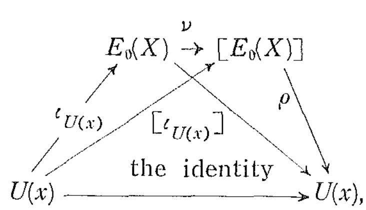
if $x \neq x_0$.

Note. If X is a topological manifold with manifold structure $\{(U, \ h_U)\}$, h_U is the homeomorphism from U onto R^n, $x \in U$, then taking

$$\gamma(s) = h_U^{-1}(sh_U(x) + (1-t)h_U(y)),$$

X satisfies (*). Moreover, if X is a smooth manifold, then $E(X)$ is a smooth Banach manifold ([1], [5]), and we can take $\iota_{U(x)}$ to be a diffeomorphism by taking γ to be the geodesic.

By (3), the induced map $[\iota_{U(x)}] : U(x) \to [E_0(X)]$ is also a homeomorphism. In fact, we have the following commutative diagram.



where ν is the natural map and ρ is the induced map from ρ. If X is smooth, then we can take $\iota_{U(x)}$ and $[\iota_{U(x)}]$ both to be diffeomorphisms.

Definition. *In $\Omega(X)$, we set*

$$(2)' \qquad \Omega_0(X) = \{\eta \mid \eta(s) \neq x_0,\ s \neq 0,\ 1\} \cup \{e\},$$

and denote $[\Omega_0(X)]$ the quotient space $\Omega_0(X)/H^+(\boldsymbol{I})$.

For any $\alpha \in E(X)$, we set

$$\alpha^{-1}(x_0) = \bigcup_{i=1}^{2k-1} [s_i,\ s_{i+1}],\quad 0 \leqq s_1 \leqq \cdots \leqq s_{2k} = 1,$$
$$s_{2j} \neq s_{2j+1},\quad j = 1,\ \cdots,\ k-1,$$

and define $\alpha_0 \in E_0(X)$, $\eta_1, \cdots, \eta_{k-1} \in \Omega(X)$ and $\eta_{1,0}, \cdots, \eta_{k-1,0} \in \Omega_0(X)$ by

$$\alpha_0(s) = \alpha(s_1 s),\quad s_1 \neq 0,$$
$$\eta_j(s) = \alpha(s_{2j-1} + (s_{2j+1} - s_{2j-1})s),$$
$$\eta_{j,0}(s) = \alpha(s_{2j} + (s_{2j+1} - s_{2j})s),\quad j = 1,\ \cdots,\ k-1.$$

Then we get

$$(4) \qquad [\alpha] = [\alpha_0]*[\eta_1]* \cdots *[\eta_{k-1}],\quad s_1 \neq 0,$$
$$[\alpha] = [\eta_1]*\cdots*[\eta_{k-1}],\quad s_1 = 0,\quad [\alpha] = [\alpha_0],\quad s_1 = 1.$$
$$(4)' \qquad [\eta_j] = [\eta_{j,0}],\quad s_{2j-1} = s_{2j},\quad [\eta_j] = [e]*[\eta_{j,0}],\quad s_{2j-1} \neq s_{2j}.$$

Therefore $[E(X)]$ is generated by $[E_0(X)]$ and $[\Omega_0(X)]$ and we have

$$(5) \qquad [E(X)] = [E_0(X)] \cup ([E_0(X)]*[\Omega(X)]) \cup [\Omega(X)].$$

For the convenience, we introduce the notation ϕ such that

$$\phi*[\alpha]=[\alpha]*\phi=[\alpha],$$

then (5) is rewritten

$$(5)' \qquad [E(X)]\cup\{\phi\}=([E(X)]\cup\{\phi\}*[(\Omega(X)]\cup\{\phi\}).$$

3. Lemma 5. *Let f be a continuous map from $E(X)$ to Y (resp from $\Omega(X)$ to Y) such that*

$$(6) \qquad f(h^*(\alpha))=f(\alpha),\quad h\in H^+(\boldsymbol{I}),$$

then

$$(7) \qquad f(\alpha*e*\beta)=f(\alpha*\beta),$$

where Y is a topological space and either α or β may be equal to ϕ.

Proof. Shince the method is similar, we assume $\alpha\neq\phi$, $\beta\neq\phi$. Then by (6), we can take

$$(\alpha*e*\beta)(s)=\alpha(4s),\ \ 0\leqq s\leqq\frac{1}{4},\ \ (\alpha*e*\beta)(s)=x_0,\ \frac{1}{4}\leqq s\leqq\frac{1}{2},$$

$$(\alpha*e\beta)(s)=\beta(2s-1),\ \ \frac{1}{2}\leqq s\leqq 1.$$

$$(\alpha*\beta)(s)=\alpha(2s),\quad 0\leqq s\leqq\frac{1}{2},\quad (\alpha*\beta)(s)=\beta(2s-1),\ \ \frac{1}{2}\leqq s\leqq 1.$$

We define $h_t\in H^+(\boldsymbol{I})(1\leqq t\leqq 2)$ by

$$h_t(s)=ts,\ \ 0\leqq s\leqq\frac{1}{4},\ \ h_t(s)=(2-t)s-\frac{1}{2}+\frac{t}{2},\ \ \frac{1}{4}\leqq s\leqq\frac{1}{2}$$

$$h_t(s)=s,\ \ \frac{1}{2}\leqq s\leqq 1\,.$$

Then by (6), we have

$$f(h_t^*(\alpha*e*\beta))=f(\alpha*e*\beta).\ \ 1\leqq t<2.$$

On the other hand, by the definition of h_t and the continuity of f, we have

$$\lim_{t\to 2}.\ f(h_t^*(\alpha*e*\beta))=f(\alpha*\beta).$$

Therefore we have (7).

As usual, for $\alpha \in E(X)$, we set α^{-1} by $\alpha^{-1}(s) = \alpha(1-s)$, then we have

Lemma 6. *If a continuous man* $f: E(X) \to Y$ *satisfies* (6) *and*

$$f(\alpha * \alpha^{-1} * \beta) = f(\beta), \tag{8}$$

then f *is written as* $\pi^* g$, $g: X \to Y$ *if and only if*

$$f(\alpha * \eta) = f(\alpha), \quad \text{for any } \eta \in \Omega(X). \tag{9}$$

Proof. Since $\pi^* g$ satisfies (6), (8) and (9), we only need to prove sufficiency. If $\alpha(0) = \beta(0) = x \in X$, then $\alpha^{-1} * \beta = \eta$ is defined and belongs in $\Omega(X)$. Then by (8) and (9), we obtain

$$f(\alpha) = f(\alpha * \eta) = f(\alpha * \alpha^{-1} * \beta) = f(\beta).$$

Therefore the value of f at α is determined only by $\alpha(0)$. Hence we get

$$f = \pi^* g, \quad g(x) = f(\alpha), \quad \alpha(0) = x.$$

This proves the lemma.

§ 2. The groups $H_E^p(X, G)$ and $H_\Omega^p(X, G)$.

4. **Definition.** *Let* $\boldsymbol{G}$ *be a topological abelian group, then the set of all continuous maps* $f: E(X) \times \overbrace{\Omega(X) \times \cdots \times \Omega(X)}^{p} \to \boldsymbol{G}$ *which satisfies*

$$\text{(i)} \quad f(h_0^*(\alpha), h_1^*(\eta_1), \cdots, h_p^*(\eta_p)) = f(\alpha, \eta_1, \cdots, \eta_p),$$
$$h_i \in H^+(\boldsymbol{I}), \quad i = 0, 1, \cdots, p,$$
$$\alpha \in E(X), \quad \eta_j \in \Omega(X), \quad j = 1, \cdots, p,$$
$$\text{(ii)} \quad f(\alpha * \alpha^{-1} * \beta, \eta_1, \cdots, \eta_p) = f(\beta, \eta_1, \cdots, \eta_p),$$

is denoted by $E^p(X, \boldsymbol{G})$.

Defnition. *Let* $\boldsymbol{G}$ *be a topological abelian group, then the set of all continuous maps* $g: \overbrace{\Omega(X) \times \cdots \times \Omega(X)}^{p} \to \boldsymbol{G}$ *which satisfies*

$$\text{(i)}' \quad g(h_1^*(\eta_1), \cdots, h_p^*(\eta_p)) = g(\eta_1, \cdots, \eta_p),$$
$$h_i \in H^+(\boldsymbol{I}), \quad i = 1, \cdots, p, \quad \eta_j \in \Omega(X), \quad f = 1, \cdots, p,$$

is denoted by $\Omega^p(X, \boldsymbol{G})$ *for* $p \geqq 1$. *If* $p = 0$, *then we set*

$$\Omega^0(X, \boldsymbol{G}) = \boldsymbol{G}.$$

Note. If X is a smooth manifold, then $\Omega(X)$ and $E(X)$ are also smooth

(Banach) manifolds ([1], [5]). Hence if $\boldsymbol{G}$ is a Lie group, we can define the smooth maps $f: E(X)\times\Omega(X)\times\cdots\times\Omega(X)\to\boldsymbol{G}$ and $g:\Omega(X)\times\cdots\times\Omega(X)\to\boldsymbol{G}$. Then using smooth maps, we can define similar sets. They are also denoted by $E^p(X, \boldsymbol{G})$ and $\Omega^p(X, \boldsymbol{G})$.

By definition, $E^p(X, \boldsymbol{G})$ and $\Omega^p(X, \boldsymbol{G})$ are modules and setting

(10) $$\iota(g)(\alpha, \eta_1, \cdots, \eta_p)=g(\eta_1, \cdots, \eta_p),$$

$\iota:\Omega^p(X, \boldsymbol{G})\to E^p(X, \boldsymbol{G})$ is an into isomorphism for all $p\geqq 0$.

Lemma 7. *$\Omega(X)$ operates associatively on $E^p(X, \boldsymbol{G})$ by the operation*

(11) $$f^\eta(\alpha, \eta_1, \cdots, \eta_p)=f(\alpha*\eta, \eta, \cdots, \eta_p).$$

Proof. if $h\in H^+(\boldsymbol{I})$, then to define $h'\in H^+(\boldsymbol{I})$ by

$$h'(s)=\frac{1}{2}h(2s),\ 0\leqq s\leqq\frac{1}{2},\quad h'(s)=s,\ \frac{1}{2}\leqq s\leqq 1,$$

we have $h^*(\alpha)*\eta=h'^*(\alpha*\eta)$. Hence f satisfies (i).

Since we know by lemma 2,

$$([\alpha]*[\alpha^{-1}]*[\beta])*[\eta]=[\alpha]*[\alpha^{-1}]*([\beta]*[\eta]),$$

f satisfies (ii).

If $h\in H^+(\boldsymbol{I})$, then to define $h''\in H^+(\boldsymbol{I})$ by

$$h''(s)=s,\ 0\leqq s\leqq\frac{1}{2},\quad h''(s)=\frac{1}{2}(h(2s-1)+1),\quad \frac{1}{2}\leqq s\leqq 1,$$

we get $\alpha*h^*(\eta)=h''^*(\alpha*\eta)$. Therefore we have

(12) $$f^\eta=f^{h^*(\eta)},\quad h\in H^+(\boldsymbol{I}).$$

On the other hand, since we know

(13) $$(f^\eta)^\zeta=f^{\eta*\zeta},$$

we have the associativity by lemma 2.

By lemma 2, we also obtain

Lemma 8. *If $f\in E^p(X, \boldsymbol{G})$ (resp. $g\in\Omega^p(X, \boldsymbol{G})$), then for any i $(1\leqq i\leqq p)$, we have*

(14) $$\begin{aligned}&f(\alpha, \eta_1, \cdots, \eta_{i-1}, (\zeta_1*\zeta_2)*\zeta_3, \eta_{i+1}, \cdots, \eta_p)\\&=f(\alpha, \eta_1, \cdots, \eta_{i-1}, \zeta_1*(\zeta_2*\zeta_3), \eta_{i+1}, \cdots, \eta_p),\end{aligned}$$

(14)' $$g(\eta_1, \cdots, \eta_{i-1}, (\zeta_1 * \zeta_2) * \zeta_3, \eta_{i+1}, \cdots, \eta_p)$$
$$= g(\eta_1, \cdots, \eta_{i-1}, \zeta^1 * (\zeta_2 * \zeta_3), \eta_{i+1}, \cdots, \eta_p).$$

5. **Definition.** *We define the homomorphisms* $\delta : E^p(X, G) \to E^{p+1}(X, G)$ *and* $\delta : \Omega^p(X, G) \to \Omega^{p+1}(X, G)$ *by*

(15) $$(\delta f)(\alpha, \eta_1, \cdots, \eta_{p+1})$$
$$= f^{\eta_1}(\alpha, \eta_2, \cdots, \eta_{p+1}) + \sum_{i=1}^{p}(-1)^i f(\alpha, \eta_1, \cdots, \eta_i * \eta_{i+1}, \cdots, \eta_{p+1}) +$$
$$+ (-1)^{p+1} f(\alpha, \eta_1, \cdots, \eta_p), \quad p \geqq 1,$$
$$(\delta f)(\alpha, \eta_1) = f^{\eta_1}(\alpha) - f(\alpha),$$

(15)' $$(\delta g)(\eta_1, \cdots, \eta_{p+1})$$
$$= g(\eta_2, \cdots, \eta_{p+1}) + \sum_{i=1}^{p}(-1)^i g(\eta_1, \cdots, \eta_i * \eta_{i+1}, \cdots, \eta_{p+1}) +$$
$$+ (-1)^{p+1} g(\eta_1, \cdots, \eta_p), \quad p \geqq 1,$$
$$(\delta g)(\eta_1) = 0.$$

By definition, we have the following commutative diagram.

$$\begin{array}{ccc} E^p(X, G) & \xrightarrow{\delta} & E^{p+1}(X, G) \\ \iota \downarrow & & \downarrow \iota \\ \Omega^p(X, G) & \xrightarrow{\delta} & \Omega^{p+1}(X, G). \end{array}$$

By lemma 7 and lemma 8, we have

Lemma 9. $\delta(\delta f)$ *and* $\delta(\delta g)$ *are equal to 0 for any* $f \in E^p(X, G)$ *and* $g \in \Omega^p(X, G)$, $p \geqq 0$.

By lemma 9, we can define

Definition. *For each* $p \geqq 0$, we set

(16) $$H_E{}^p(X, G)$$
$$= \text{ker.}\ [\delta : E^p(X, G) \to E^{p+1}(X, G)] / \delta E^{p-1}(X, G), \quad p \geqq 1,$$
$$H_E{}^0(X, G) = \textit{ker.}\ [\delta : E^0(X, G) \to E^1(X, G)].$$

(16)' $$H_\Omega{}^p(X, G)$$
$$= \textit{ker.}\ [\delta : \Omega^p(X, G) \to \Omega^{p+1}(X, G)] / \delta\Omega^{p-1}(X, G), \quad p \geqq 1, ,$$
$$H_\Omega{}^0(X, G) = \textit{ker.}\ [\delta : \Omega^0(X, G) \to \Omega^1(X, G)].$$

Theorem 1. *We have*

(17)i $$H_E{}^0(X, G) \simeq C(X, G),$$

(17)ii $$H_\Omega{}^0(X, G) \simeq G,$$

(17)iii $$H_\Omega^1(X, G) \simeq \mathrm{Hom.}([\Omega(X)], G),$$

where $C(X, G)$ is the module of continuous (or smooth) maps from X to G.

Proof. (17)ii follows from the definition.

Since we have

$$\delta g(\eta_1, \eta_2) = g(\eta_2) - g(\eta_1 * \eta_2) + g(\eta_1),$$

we obtain (17)iii because $\delta\Omega^0(X, G) = 0$.

Since $\delta f(\alpha, \eta)$ is equal to 0 if and only if $f^\eta(\alpha) = f(\alpha)$ for all $\eta \in \Omega(X)$, we have (17)i by lemma 6.

By theorem 1, we have

$$H_\Omega^0(X, G) \simeq H^0(X, G),$$

$$\rho^*(H_\Omega^1(X, G)) = H^1(X, G),$$

where ρ^* is the homomorphism induced from ρ.

§ 3. Calculation of $H_E^p(X, G)$, $p \geqq 1$.

6. Definition. *For an element g of $\Omega^p(X, G)$, we define an operation $g^\sharp\eta$ of $\eta \in \Omega(X)$ by*

(18)
$$\begin{aligned}&(g^\sharp\eta)(\zeta_1, \cdots, \zeta_p)\\ &= g(\eta*\zeta_1, \cdots, \zeta_p) + \sum_{i=1}^{p-1}(-1)^i g(\eta, \zeta_1, \cdots, \zeta_i*\zeta_{i+1}, \cdots, \zeta_p) +\\ &+(-1)^p g(\eta, \zeta_1, \cdots, \zeta_{p-1}), \quad p \geqq 1,\\ &g^\sharp\eta = g, \quad p = 0.\end{aligned}$$

We donote $\Omega_\sharp{}^p(X, G)$ if we consider $\Omega^p(X, G)$ together with the above operation of $\Omega(X)$.

Definition. *We define the homomorphisms* $\delta_\sharp : E^q(X, \Omega_\sharp{}^p(X, G))\ E^{q+1}(X, \Omega_\sharp{}^p(X, G))$ and $\delta_\sharp : \Omega^q(X, \Omega_\sharp{}^p(X, G)) \to \Omega^{q+1}(X, \Omega_\sharp{}^p(X, G))$ by

(19)
$$\begin{aligned}&(\delta_\sharp f)(\alpha, \eta_1, \cdots, \eta_{q+1})\\ &= f^{\eta_1}(\alpha, \eta_2, \cdots, \eta_{q+1}) + \sum_{i=1}^{q}(-1)^q f(\alpha, \eta_1, \cdots, \eta_i*\eta_{i+1}, \cdots, \eta_{q+1}) +\\ &+ (-1)^{q+1} f(\alpha, \eta_1, \cdots, \eta_q)^\sharp\eta_{q+1}, \quad q \geqq 1,\\ &(\delta_\sharp f)(\alpha, \eta_1) = f^{\eta_1}(\alpha) - f(\alpha)_\sharp\eta_1,\end{aligned}$$

(19)′
$$\begin{aligned}&(\delta_\sharp g)(\eta_1, \cdots, \eta_q{}^+{}_1)\\ &= g(\eta_2, \cdots, \eta_{q+1}) + \sum_{i=1}^{q}(-1)^i g(\eta_1, \cdots, \eta_i*\eta_{i+1}, \cdots, \eta_{q+1}) +\end{aligned}$$

$$+(-1)^{q+1}g(\eta_1, \cdots, \eta_q)^{\#}\eta_{q+1}, \quad q \geqq 1,$$
$$(\delta^{\#}g)(\eta_1) = g - g^{\#}\eta_1.$$

We define the homomor phisms $j : E^q(X, \Omega_{\#}{}^p(X, \boldsymbol{G})) \to E^{q+p}(X, \boldsymbol{G})$ and $j : \Omega^q(X, \Omega_{\#}{}^p(X, \boldsymbol{G})) \to \Omega^{q+p}(X, \boldsymbol{G})$ by

$$(20) \qquad (jf)(\alpha, \eta_1, \cdots, \eta_q, \eta_{q+1}, \cdots, \eta_{q+p})$$
$$= f(\alpha, \eta_1, \cdots, \eta_q)(\eta_{q+1}, \cdots, \eta_{q+p}),$$
$$(20)' \qquad (jg)(\eta_1, \cdots, \eta_q, \eta_{q+1}, \cdots, \eta_{q+p}) = g(\eta_1, \cdots, \eta_q)(\eta_{q+1}, \cdots, \eta_{q+p}).$$

Then we have

$$(21) \qquad \delta(jf)(\alpha, \eta_1, \cdots, \eta_q, \eta_{q+1}, \eta_{q+2}, \cdots, \eta_{q+p+1})$$
$$= (\delta_{\#}f)(\alpha, \eta_1, \cdots, \eta_q, \eta_{q+1})(\eta_{q+2}, \cdots, \eta_{q+p+1}),$$
$$(21)' \qquad \delta(jg)(\eta_1, \cdots, \eta_q, \eta_{q+1}, \eta_{q+2}, \cdots, \eta_{q+p+1})$$
$$= (\delta_{\#}g)(\eta_1, \cdots, \eta_q, \eta_{q+1})(\eta_{q+2}, \cdots, \eta_{q+p+1}).$$

By (21) and (21)′, we obtain

Lemma 10. *For any* $f \in E^q(X, \Omega_{\#}{}^p(X, \boldsymbol{G}))$ *and* $g \in \Omega^q(X, \Omega_{\#}{}^p(X, \boldsymbol{G}))$, *we have* $\delta_{\#}(\delta_{\#}f) = 0$ *and* $\delta_{\#}(\delta_{\#}g) = 0$.

Corollary. $[\Omega(X)]$ *operates on* $\Omega_{\#}{}^p(X, \boldsymbol{G})$ *as a semigroup, that is, we have*

$$(22) \qquad (g^{\#}\eta_1)^{\#}\eta_2 = g^{\#}(\eta_1 * \eta_2),$$

fos any $g \in \Omega^p(X, \boldsymbol{G})$ *and* $\eta_1, \eta_2 \in \Omega(X)$.

Proof. By lemma 10, we have

$$0 = (\delta^{\#}(\delta^{\#}g))(\eta_1, \eta_2)$$
$$= (\delta^{\#}g)(\eta_2) - (\delta^{\#}g)(\eta_1 * \eta_2) + ((\delta^{\#}g)^{\#}\eta_2)(\eta_1)$$
$$= g^{\#}(\eta_1 * \eta_2) - (g^{\#}\eta_1)^{\#}\eta_2.$$

This shows (22).

By lemma 10, we can define the groups $\mathrm{H}_{E\#}{}^q(X, \Omega^q(X, \boldsymbol{G}))$ and $H^{\Omega}{}_{\#}{}^q(X, \Omega^p(X, \boldsymbol{G}))$ by

$$(23) \qquad \mathrm{H}_{E\#}{}^q(X, \Omega^p(X, \boldsymbol{G}))$$
$$= \frac{ker.\ [\delta_{\#} : E^q(X, \Omega_{\#}{}^p(X, \boldsymbol{G})) \to E^{q+1}(X, \Omega_{\#}{}^p(X, \boldsymbol{G}))]}{\delta_{\#}E^{q-1}(X, \Omega_{\#}{}^p(X, \boldsymbol{G}))}, \quad q \geqq 1,$$
$$H_{E\#}{}^0(X, \Omega^p(X, \boldsymbol{G}))$$
$$= ker.\ [\delta^{\#} : E^0(X, \Omega_{\#}{}^p(X, \boldsymbol{G})) \to E^1(X, \Omega_{\#}{}^p(X, \boldsymbol{G}))],$$
$$(23)' \qquad H_{\Omega\#}{}^q(X, \Omega^p(X, \boldsymbol{G}))$$

$$=\frac{ker.\ [\delta_{\#}: \Omega^q(X,\ \Omega_{\#}{}^p(X,\ G)) \to \Omega^{q+1}(X,\ \Omega_{\#}{}^p(X,\ G))]}{\delta_{\#}\Omega^{q-1}(X,\ \Omega_{\#}{}^p(X,\ G))},\quad q\geqq 1,$$

$$H_{\Omega\#}{}^0(X,\ \Omega^p(X,\ G))$$
$$= ker.\ [\delta_{\#}: \Omega^0(X,\ \Omega_{\#}{}^p(X,,\ G)) \to \Omega^1(X,\ \Omega_{\#}{}^p(X,\ G))].$$

Then by (21) and (21)′, we have

Theorem 2 ([**3**]). *If* $q\geqq 1$, *we get for all* $p\geqq 0$,

(24) $$H_{E\#}{}^q(X,\ \Omega^p(X,\ G)) \simeq H_E{}^{q+p}(X,\ G),$$

(24)′ $$H_{\Omega\#}{}^q(X,\ \Omega^p(X,\ G)) \simeq H_\Omega{}^{q+p}(X,\ G).$$

7. Since $H_E{}^p(X,\ G) \simeq H_{E\#}{}^1(X,\ \Omega^{p-1}(X,\ G))$ if $p\geqq 1$ by theorem 2, to calculate $H_E{}^p(X,\ G)$, it is sufficient to calculate $H_{E\#}{}^1(X,\ \Omega^q(X,\ G))$, $q\geqq 0$. If $f\in E^1(X,\ \Omega_{\#}{}^q(X,\ G))$ satisfies $\delta_{\#}f=0$, then we have

$$f(\alpha * e,\ \eta) = f(\alpha\ e*\eta) - f(\alpha,\ e)_{\#}\eta$$
$$= (\delta g)(\alpha,\ \eta),$$

where $g(\alpha)=g(\alpha,\ e)$ and $g(\alpha*\eta)=f(\alpha,\ e*\eta)$ if we fix $\alpha\in E(X)$. Then since we know $f(\alpha*e,\ \eta)=f(\alpha,\ \eta)$ by lemma 5, for a fixed $\alpha\in E(X)$, we get

(25) $$f(\alpha,\ \eta)=(\delta g)(\alpha,\ \eta),\ \text{where } g(\alpha*\eta)=f(\alpha,\ e*\eta).$$

Although g is defined only on $\alpha*\Omega(X)$ in general, if X satisfies the condition (*) of lemma 4, then by the condition (ii) of n⁰ 4 and (5), we can define $g=g_U$ to be continuous (or smooth) on $\iota_U(U(x))*\Omega(X)$, where $U(x)$ is a neighborhood of $x\in X$ (in X) and $x=\alpha(0)$. Then since we know by (ii) of n⁰ 4 that the value of f is determined by the value on $\cup_{x\in\omega}{}_{U(x)}(U(x))*\Omega(X)$, we may assume that for a suitable covering $\{U\}$ of $E(X)$ such that

$$U*\Omega(X)\subset U,\ H^p(U,\ Z)\simeq H^p(\Omega(X),\ Z),\ p\geqq 0,$$

we obtain $\{g_U\}$, $g_U\in E^1(U,\ \Omega_{\#}{}^p(X,\ G))$ such that

$$\delta_{\#}g_U = f\,|\,U.$$

Then since $\delta_{\#}(g_U-g_V)$ is equal to 0 on $U\cap V$, $\{h_{UV}\}$, $h_{UV}=g_U-g_V$ on $U\cap V$ defines a 1-codycle with coefficients in $\mathscr{B}E\,(\Omega_{\#}{}^p(X,\ G))$ on $E(X)$. Here $\mathscr{B}E^0(\Omega_{\#}{}^p(X,\ G))$ means the sheaf of germs of the continuous (or smooth) maps from $E(X)$ to $ker.\ \delta_{\#}$ in $E^0(X,\ \Omega_{\#}{}^p(X,\ G))$. Then since we know

(26) $$H^1(E(X),\ \mathscr{B}E^0(\Omega_{\#}{}^p(X,\ G)))$$

$\simeq$ {the set of equivalence classes of topological (or differentiable) *ker*. $[\delta_\# : \Omega_\#^p(X, G) \to \Omega_\#^{p+1}(X, G)]$-bundle over $E(X)$},

we obtain

$$H^1(E(X), \mathscr{B}E^0(\Omega_\#^p(X, G))) = 0, \tag{26$'$}$$

for any p, because $E(X)$ is contractible.

By (26)$'$, there exists a refinement $\{U'\}$ of $\{U\}$ and $\{h_{U'}\}$, $h_{U'} : U' \to \Omega_\#^p(X, G)$, $\delta_\# h_{U'} = 0$ such that

$$h_{UV} | U' \cap V' = h_{U'} - h_{V'}. \tag{27$'$}$$

Then to define $g \in E^0(X, \Omega_\#^p(X, G))$ by $g|U' = g_U|U' - h_{U'}$, we get

$$f = \delta_\# g. \tag{27}$$

Hence we have

Theorem 3. *$H_E{}^p(X, G)$ vanishes for $p \geqq 1$ if X satisfies the condition* (*) *of lemma 4. Especially, if X is a topological manifold, then $H_E{}^p(X, G)$ vanishes for $p \geqq 1$.*

Note. Since $E(X)$ is not C^∞-smooth ([1]), $h_{U'}$ of (27)$'$ is not smooth although h_{UV} is smooth. But setting

$$E_{2,k}(X) = \{\alpha | \alpha : I \to X \text{ and } \alpha \text{ belongs in } k\text{-th Sobolev space over } I, \; \alpha(1) = x_0, \}$$

we know that $E_{2,k}(X)$ is C^∞-smooth ([1]) and $E_{2,k}(X)$ is contained in $E(X)$ if $k > dim.\ X/2$ (8). Then we set

$$\Omega_{2,k}(X) = \Omega(X) \cap E_{2,k}(X).$$

Although $\Omega_{2,k}(X)$ does not operates on $E_{2,k}(X)$, setting

$$E_{2,k,0}(X) = \{\alpha | \alpha \in E_{2,k}(X), \; \alpha(s) = x_0 \text{ if } s > 1\text{-}\varepsilon \text{ for some } \varepsilon\},$$
$$\Omega_{2,k,0}(X) = \Omega_{2,k}(X) \cap E_{2,k,0}(X),$$

$\Omega_{2,k,0}(X)$ operates on $E_{2,k,0}(X)$. Moreover, denoting $H^{+,k}(I)$ the group of orientation preserving C^k-diffeomorphisms of I, $H^{+,k}(I)$ operates on $E_{2,k,0}(X)$ and we obtain same results as in §1 and §2 for $E_{2,k,0}(X)$ and $\Omega_{2,k,0}(X)$ except lemma 6. But lemma 6 is also true because if $\alpha(0) = \beta(0)$, then there exists β_n such that

$$\lim_{n\to\infty} \beta_n = \beta \text{ in } E_{2,k,0}(X), \quad \alpha^{-1}*\beta_n \text{ belongs in } E_{2,k,0}(X) \text{ for all } n.$$

Then since $E_{2,k,0}(X)$ is a C^∞-smooth (Fréchet) manifold and contractible, we can take each $h_{U'}$ of (27)' to be smooth if each h_{UV} is smooth ([1], [9]). Hence in (27), we can take g to be smooth if f is smooth. Therefore we obtain

Theorem 3'. *If X is smooth and $f \in E^p{}_{2,k,0}(X, G)$ is a smooth map and $\delta f = 0$, then there exists a smooth map $g \in E^{p-1}{}_{2,k,0}(X, G)$ such that $f = \delta g$ if $q \geqq 1$ and G is a Lie group. Here $E^p{}_{2,k,0}(X)$ means the set of continuous maps from $E_{2,k,0}(X) \times \overbrace{\Omega_{2,k,0}(X) \times \cdots \times \Omega_{2,k,0}(X)}^{p}$ into G which satisfies the condition* (i), (ii) *of n^0 4 where $H^+(I)$ is changed by $H^{+,k}(I)$.*

§4. The module $A^{p,q}(X)$.

8. In the rest, we assume X to be a paracompact arcwise connected smooth manifold. For the simplicity, we denote $E(X)$, $\Omega(X)$, $E^p(X, G)$, $\Omega^p(X, G)$, $[E(X)]$, $[\Omega(X)]$ and $H^+(I)$ instead of $[E_{2,k,0}(X)]$, $[\Omega_{2,k,0}(X)]$, $E^p{}_{2,k,0}(X, G)$, $\Omega^p{}_{2,k,0}(X, G)$, $[E_{2,k,0}(X)]$, $[\Omega_{2,k,0}(X)]$ and $H^{+,k}(I)$. Here $\Omega^p{}_{2,k,0}(X, G)$ is defined similarly as $E^p{}_{2,k,0}(X, G)$, $[E_{2,k,0}(X)]$ and $[\Omega_{2,k,0}(X)]$ are the quotient spaces of $E_{2,k,0}(X)$ and $\Omega_{2,k,0}(X)$ *mod.* $H^{+,k}(I)$.

Since $E(X)$ is a smooth (Fréchet) manifold, we denote the group of (real or complex valued) p-forms on $E(X)$ by $C^p(E(X))$. Denoting the cotangent bundle of $E(X)$ by $T^*(E(X))$, we know that $C^p(E(X)) = \Gamma(E(X), \Lambda^p(T^*(E(X))))$. Since $E(X)$ is contractible, we can define a homomorphism $k: C^p(E(X)) \to C^{p-1}(E(X))$ by

$$(28) \qquad k\varphi = P_*\int_0^1 i\left(\frac{\partial}{\partial t}\right)(F^*\varphi)dt, \quad p = 1,$$

$$\langle u_1 \wedge \cdots \wedge u_{p-1}, k\varphi\rangle = P^*\int_0^1 \langle P^*u_1 \wedge \cdots \wedge P^*u_{p-1}, \ i\left(\frac{\partial}{\partial t}\right)(F^*\varphi)\rangle dt, \quad p > 1,$$

where u_i means a vector field on $E(X)$, $F: E(X) \times I \to E(X)$ is the contraction of $E(X)$ given by $F(\alpha, t) = \alpha_t$ and $P: E(X) \times I \to E(X)$ is the projection ([6]). Then we know that

$$(dk + kd)\varphi = \varphi, \quad p \geqq 1.$$

We denote the induced bundle from $\Lambda^p T^*(E(X))$ on $E(X) \times \Omega(X) \times \cdots \times \Omega(X)$ also by $\Lambda^p T^*(E(X))$. Then a cross-section of $\Lambda^p T^*(E(X))$ on $E(X) \times \Omega(X) \times \cdots \times \Omega(X)$ is a p-form on $E(X) \times \Omega(X) \times \cdots \times \Omega(X)$ and denoting d_1 the exterior differentiation in $E(X)$-direction, we get

$$d_1(\Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X), \ \Lambda^p T^*(E(X)))$$

$$\subset \Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^{p+1}T^*(E(X))).$$

Moreover, we can define k for the elements of $\Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega)X),\ \Lambda^p T^*(E(X)))$ because we know

$$\langle v,\ \varphi\rangle = 0,\ \varphi \in \Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^p T^*(E(X))),$$
$$\text{if } v \notin \Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^p T(E(X))),$$

where $T(E(X))$ is the tangent bundle of $E(X)$, and we have

$$k(\Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^p T^*(E(X)))$$
$$\subset \Gamma(E(X) \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^{p-1}T^*(E(X))),$$
$$(d_1 k + k d_1)\varphi = \varphi,\ p \geqq 1.$$

Definition. *The set of all* φ *such that* φ *belongs in* $\Gamma(E(X) \times \overbrace{\Omega(X) \times \cdots \times \Omega(X)}^{q},$ $\Lambda^p T^*(E(X))$ *and satisfiet the following conditions* (i) *and* (ii) *is denoted by* $A^{p,q}(X)$.

(i) $\varphi(h_0{}^*(\alpha),\ h_1{}^*(\eta_1),\ \cdots,\ h_q{}^*(\eta_q)) = \varphi(\alpha,\ \eta_1,\ \cdots,\ \eta_q),$
$h_i \in H^+(\boldsymbol{I}),\ 0 \leqq i \leqq q,\ \alpha \in E(X),\ \eta_j \in \Omega(X),\ 1 \leqq j \leqq q,$

(ii) $(\alpha * \alpha^{-1} * \beta,\ \eta_1,\ \cdots,\ \eta_q) = (\beta,\ \eta_1,\ \cdots,\ \eta_q)$ if $\alpha^{-1} * \beta \in E(X).$

By definition, $A^{p,q}(X)$ is a module, and denoting $\pi : E(X) \times \Omega(X) \times \cdots \times \Omega(X) \rightarrow X \times \Omega(X) \times \cdots \times \Omega(X)$ the projection induced by $\pi : E(X) \rightarrow \times$, $\pi^*(\Gamma(X \times \Omega(X) \times \cdots \times \Omega(X),\ \Lambda^p T^*(X))$ is contained in $A^{p,q}(X)$. Here $T^*(X)$ is the cotangent bundle of X and $\Lambda^p T^*(X)$ is the induced bundle from $\Lambda^p T^*(X)$ on $X \times \Omega(X) \times \cdots \times \Omega(X)$.

Lemma 6′. $\varphi \in A^{p,q}(X)$ *belongs in* π^**-image if and only if*

(9)′ $$\varphi(\alpha * \eta,\ \eta_1,\ \cdots,\ \eta_q) = \varphi(\alpha,\ \eta_1,\ \cdots,\ \eta_q),\ \text{for any } \eta \in \Omega(X),$$

9. Lemma 11. *if* φ *belongs in* $A^{p,q}(X)$ *and* $h \in H^+(\boldsymbol{I})$, *then*

(29) $$\varphi((h^*\alpha)_t(s),\ \eta_1,\ \cdots,\ \eta_q)$$
$$= \varphi(\alpha(h(1-t) + (1 - h(1-t))s),\ \eta_1,\ \cdots,\ \eta_q),\ t > 0,$$

where $h_t \in H^+(\boldsymbol{I})$ *is given by*

$$h_t(s) = \frac{h(1-t+ts) - h(1-t)}{1 - h(1-t)},\ \ t > 0.$$

Proof. By the definition, we have $h_t{}^*(\alpha(h(1-t) + (1 - h(1-t))s)) = (h^*\alpha)_t(s)$. Hence we have the lemma by (i).

Lemma 12. k *maps* $A^{p,q}(X)$ *into* $A^{p-1,q}(X)$.

Proof. By the definition of k, we have by lemma 11,

$$\langle (u_1\wedge\cdots\wedge u_{p-1})(\alpha),\ k\varphi(h^*(\alpha(s)),\ h_1(\eta_1),\ \cdots,\ h_q(\eta_q))\rangle$$

$$=\lim_{\varepsilon\to 0}\int_\varepsilon^1 \langle (P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\alpha,\ t),\ i(\frac{\partial}{\partial t})(\varphi((h^*\alpha)_t(s),\ h_1(\eta_1),\ \cdots,\ h_q(\eta_q)))\rangle dt$$

$$=\lim_{\varepsilon\to 0}\int_\varepsilon^1 \langle (P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\alpha,\ t),$$

$$i(\frac{\partial}{\partial t})(\varphi(\alpha(h(1-t)+(1-h(1-t))s),\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$

$$=\lim_{\varepsilon\to 0}\int_\varepsilon^1 \langle (P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\alpha,\ t),$$

$$i(\frac{\partial}{\partial(1-h(1-t))})(\varphi(\alpha(h(1-t)+(1-h(1-t))s),\eta_1,\ \cdots$$

$$\cdots,\ \eta_q))\rangle d(1-h(1-t))$$

$$=\lim_{\varepsilon'\to 0}\int_{\varepsilon'}^1 \langle (P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\alpha,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\alpha_t,\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$

$$=\langle (u_1\wedge\cdots\wedge u_{p-1})(\alpha),\ k\varphi(\alpha,\ \eta_1,\ \cdots,\ \eta_q)\rangle,$$

where $\varepsilon'=1-h(1-\varepsilon)$. Hence k satisfies (i) if $p\geqq 2$. The proof for $p=1$ is done similarly.

To show k satisfies (ii), we set $\gamma=\alpha*\alpha^{-1}*\beta$ and assume

$$\gamma(s)=\alpha(4s),\quad 0\leqq s\leqq\frac{1}{4},\quad \gamma(s)=\alpha(2-4s),\quad \frac{1}{4}\leqq s\leqq\frac{1}{2},$$

$$\gamma(s)=\beta(2s-1),\quad \frac{1}{2}\leqq s\leqq 1.$$

Then we get by (i) and (ii),

$$\varphi(\gamma_t,\ \eta_1,\ \cdots,\ \eta_q)=\varphi(\beta_{2t},\ \eta_1,\ \cdots,\ \eta_q),\quad 0\leqq t\leqq\frac{1}{2}, \tag{30}$$

$$\varphi(\gamma_t,\ \eta_1,\ \cdots,\ \eta_q)=\varphi(\beta*(\alpha^{-1})_{4t-2},\ \eta_1,\ \cdots,\ \eta_q),\quad \frac{1}{2}\leqq t\leqq\frac{3}{4},$$

$$\varphi(\gamma_t,\ \eta_1,\ \cdots,\ \eta_q)=\varphi(\beta*(\alpha^{-1})_{4-4t},\ \eta_1,\ \cdots,\ \eta_q),\quad \frac{3}{4}\leqq t\leqq 1.$$

Hence we have

$$\langle (u_1\wedge\cdots\wedge u_{p-1})(\gamma),\ k\varphi(\gamma,\ \eta_1,\ \cdots,\ \eta_q)\rangle$$

$$=\int_0^1 \langle (P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\gamma_t,\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$

$$=\int_0^{\frac{1}{2}} \langle P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ \mathrm{t}),\ i(\frac{\partial}{\partial t})(\varphi(\beta_{2t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt +$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{4t-2},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt+$$
$$+\int_{\frac{3}{4}}^{1}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{4-4t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt.$$

Then since we have

$$\int_0^{\frac{1}{2}}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta_{2t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$
$$=\int_0^{1}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta_{t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt,$$
$$\int_{\frac{1}{2}}^{\frac{3}{4}}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{4t-2},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$
$$=\int_0^{1}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt,$$
$$\int_{\frac{3}{4}}^{1}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{4-4t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt$$
$$=-\int_0^{1}\langle(P^*u_1\wedge\cdots\wedge P^*u_{p-1})(\gamma,\ t),\ i(\frac{\partial}{\partial t})(\varphi(\beta*(\alpha^{-1})_{t},\ \eta_1,\ \cdots,\ \eta_q))\rangle dt,$$

we obtain

$$\langle(u_1\wedge\cdots\wedge u_{p-1})(\gamma),\ k\varphi(\gamma,\ \eta_1,\ \cdots,\ \eta_q)\rangle$$
$$=\langle(u_1\wedge\cdots\wedge u_{p-1})(\gamma),\ k\varphi(\beta,\ \eta_1,\ \cdots,\ \eta_q)\rangle.$$

This shows k satisfies (ii) for $p\geqq 2$. The proof for $p=1$ is done similarly.

Corollary. *If $\varphi\in A^{p,q}(X)$ is d_1-closed, then we can write*

(31) $$\varphi=d\psi,\quad \psi\in A^{p1,q}(X),\quad p\geqq 1.$$

10. We can define the homomorphism $\delta: A^{p,q}(X)\to A^{p,q+1}(X)$ by

(15)″ $$(\delta f)(\alpha,\ \eta_1,\ \cdots,\ \eta_{q+1})$$
$$=f^{\eta_1}(\alpha,\ \eta_2,\ \cdots,\ \eta_{q+1})+\sum_{i=1}^{p}(-1)^i f(\alpha,\ \eta_1,\ \cdots,\ \eta,\ {}_i*\eta_{i+1},\eta_{p+1})+$$
$$+(-1)^{p+1}f(\alpha,\ \eta_1,\ \cdots,\ \eta_p),\quad p\geqq 1,$$
$$(\delta f)(\alpha,\eta)=f^{\eta_1}(\alpha)-f(\alpha),$$

where $f^{\eta}(\alpha,\ \eta_1,\ \cdots,\ \eta_q)=f(\alpha*\eta,\ \eta_1,\ \cdots,\ \eta_q)$. Then we can define the cohomology groups $H_E{}^{p,q}(X,\ F)$ by

$$H_E{}^{q,p}(X,\ \boldsymbol{F})=ker.\ [\delta: A^{p,q}(X)\to A^{p,q+1}(X)/\delta A^{p,q-1}(X)],\quad q\geqq 1,$$
$$H_E{}^{0,p}(X,\ \boldsymbol{F})=ker.\ [\delta: A^{p,0}(X)\to A^{p,1}(X)].$$

Here $\boldsymbol{F}=\boldsymbol{R}$ if real valued forms are considered and $\boldsymbol{F}=\boldsymbol{C}$ if complex valued forms are considered.

Lemma 13. *Denoting typical fibre of* $T^*(E(X))$ *by* $\mathscr{T}^*$, *we have*

$$A^{p,q}(X)=E^q(X,\ \Lambda^p\mathscr{T}^*)). \tag{32}$$

Proof. Since $T^*(E(X))$ is trivial, a cross-section of $\Lambda^pT^*(E(X))$ is a (smooth or continuous) function on $E(X)\times\Omega(X)\times\cdots\times\Omega(X)$ with values in $\Lambda^p(^*\mathscr{T})$. Hence we have the lemma by the definitions of $A^{p,q}(X)$ and $E^q(X,\ \Lambda^p(\mathscr{T}^*))$.

Corollary. *For each* $p\geqq 0$, $q\geqq 0$, *we have*

$$H_E{}^{q,p}(X,\ \boldsymbol{F})=H_E{}^q(X,\ \Lambda^p(\mathscr{T}^*)). \tag{33}$$

By this corollary, theorem 1 and theorem 3, we obtain

Theorem 3″. *For each* $p\geqq 0$, *we get*

$$H_E{}^{0,p}(X,\ \boldsymbol{F})\simeq C^p(X), \tag{17)i'}$$

$$H_E{}^{q,p}(X,\ \boldsymbol{F})=0,\ \ q\geqq 1. \tag{34}$$

Note 1. Since $\Lambda^0(\mathscr{T}^*)=\boldsymbol{F}$, we have for all q

$$H_E{}^{q,0}(X,\ \boldsymbol{F})=H_E{}^q(X,\ \boldsymbol{F}). \tag{33)'}$$

Note 2. In theorem 3″, we may consider each $A^{p,q}(X)$ is consisted by smooth forms by theorem 3′.

§5. Period of higher order forms.

11. Let φ be a closed p-form on X, then $\pi^*(\varphi)$ belongs in $A^{p,0}(X)$ and we have

$$\pi^*(\varphi)=d_1(k\pi^*(\varphi)).$$

By lemma 12, $k\pi^*(\varphi)$ belongs in $A^{p-1,0}(X)$. Then by lemma 6′, if $\delta k\pi^*(\varphi)=0$, that is $k\pi^*(\varphi)$ is invariant under the operation of $\Omega(X)$, $k\pi^*(\varphi)$ comes from $C^{p-1}(X)$. Therefore φ is exact on X. But since we know

$$d_1(\phi^\eta)=(d_1\phi)^\eta,\ \ \eta\in\Omega(X), \tag{35}$$

for any differential form on $E(X)\times\Omega(X)\times\cdots\times\Omega(X)$, we get

$$d_1(\delta k\pi^*(\varphi))=\delta(d_1k\pi^*(\varphi))=0. \tag{36}$$

In general, by (35), we get

$$(35)' \qquad d_1(\delta\psi) = \delta(d_1\psi).$$

Hence setting

$$\varphi_0 = \pi^*(\varphi),\ \ \varphi_1 = k\varphi_0,\ \cdots,\ \varphi_{2r} = \delta\varphi_{2r-1},\ \ \varphi_{2r+1} = k\varphi_{2r},\ \cdots,$$

we have

$$(37) \qquad \varphi_{2r} \in A^{p-r,r}(X),\ \ \varphi_{2r+1} \in A^{p-r-1,r}(X),$$

$$(36)' \qquad d_1\varphi_{2r} = 0, \quad \delta\varphi_{2r} = 0,$$

because $d_1\varphi_{2r} = d_1\delta k\delta\varphi_{2r-2}$.

By (37), φ_{2p} belongs in $A^{0,p}(X)$ and by (36)$'$, $d_1\varphi_{2p}$ is equal to 0. Hence $\varphi_{2p}(\alpha, \eta_1, \cdots, \eta_p)$ is constant in α. Since $A^{0,p}(X) = E^p(X, \boldsymbol{F})$, there is an into isomorphism $\iota : \Omega^p(X, \boldsymbol{F}) \to E^p(X, \boldsymbol{F})$ and φ_{2p} belongs in ι-image. Then since the diagram

$$\begin{array}{ccc} E^p(X, \boldsymbol{F}) & \xrightarrow{\delta} & E^{p+1}(X, \boldsymbol{F}) \\ \iota\downarrow & & \iota\downarrow \\ \Omega^p(X, \boldsymbol{F}) & \xrightarrow{\delta} & \Omega^{p+1}(X, \boldsymbol{F}) \end{array}$$

is commutative by the definition of δ, denoting $\iota^{-1}(\varphi_{2p}) = \psi_{2p}$, we obtain by (36)$'$

$$(38) \qquad \delta\psi_{2p} = 0.$$

Hence ψ_{2p} defines an element $\langle\psi_{2p}\rangle$ of $H_\Omega{}^p(X, \boldsymbol{F})$. Moreover, since φ is exact on X implies $\delta k\pi^*(\varphi) = 0$, $\langle\psi_{2p}\rangle$ is determined by the de Rham class $\langle\varphi\rangle$ of φ. Hence we can define a homomorphism $\chi : H^p(X, \boldsymbol{F}) \to H_\Omega{}^p(X, \boldsymbol{F})$ by

$$(39) \qquad \chi(\langle\varphi\rangle) = \langle\psi_{2p}\rangle.$$

Definition. *We call* $\chi(\langle\varphi\rangle)$ *the period of* φ.

12. Theorem 4. χ *is an isomorphism.*

Proof. Let $\langle c_{2p}\rangle$ be an element of $H_\Omega{}^p(X, \boldsymbol{F})$ with representation c_{2p}, then since $d_1\iota(c_{2p})$ is equal to 0, we can construct a series $\omega_{2p-1}, \omega_{2p-2}, \cdots, \omega_0$ by

$$\iota(c_{2p}) = \delta\omega_{2p-1},\ \cdots,\ d_1\omega_{2p-2r+1} = \omega_{2p-2r},$$
$$\delta\omega_{2p-2r-1} = \omega_{2p-2r},\ \cdots,$$

because $\delta\omega_{2p-2r} = \delta d_1\omega_{2p-2r+1} = d_1\delta\omega_{2p-2r+1} = d_1\omega_{2p-2r} = 0$ and by (34) if $\delta\omega_{2p-2r}$ is equal to 0, then ω_{2p-2r} is written as $\delta\omega_{2p-2r-1}$ $(r < p$, cf. note 2 of n^{0}10). Then we get

$$\omega_{2p-2r+1} \in A^{r-1,p-r}, \quad \omega_{2p-2r} \in A^{r,p-r},$$

Therefore ω_0 belongs in $A^{p,0}$ and since

(40) $$d_1\omega_0 = 0, \quad \delta\omega_0 = 0,$$

ω_0 is written as $\pi^*(\omega)$. Although $\omega_{2p-2r-1}$ is not determined uniquely by ω_{2p-2r}, if $\delta\omega'_{2p-2r-1} = \omega_{2p-2r}$, then $\omega'_{2p-2r-1} = \omega_{2p-2r-1} + \delta\xi$ by theorem 3''. Hence by (35)', $\omega'_{2p-2r-3}$ is taken as $\omega_{2p-2r-3} + d_1\xi$. Therefore the de Rham class of ω is determined uniquely by the cohomology class of c_{2p}. Moreover, by the definitions of χ and ω, we get

$$\chi(\langle\omega\rangle) = \langle c_{2p}\rangle.$$

Hence χ is onto. Moreover, since we can take ω_{2p-1} to satisfy $d_1\omega_{2p-1} = 0$ if $\langle c_{2p}\rangle = 0$, the correspondence $\tilde{\omega}(\langle c_{2p}\rangle) = \langle\omega\rangle$ defines a homomorphism $\tilde{\omega} : H_\Omega{}^p(X, \boldsymbol{F}) \to H^p(X, \boldsymbol{F})$, $p \geqq 1$. Then by the definitions χ and $\tilde{\omega}$, we obtain

$$\tilde{\omega}\chi(\langle\varphi\rangle) = \langle\varphi\rangle, \quad \chi\tilde{\omega}(\langle c_{2p}\rangle) = \langle c_{2p}\rangle.$$

Therefore χ and $\tilde{\omega}$ are both isomorphisms and we have the theorem.

Corollary 1. *If X is a paracompact arcwise connected smooth manifold, then*

(41) $$H^p(X, \boldsymbol{F}) \simeq H_\Omega{}^p(X, \boldsymbol{F})$$

for all $p \geqq 0$.

Proof. If $p=0$, then we obtain the corollary by $(17)_{\text{ii}}$. For $p \geqq 1$, the corollary follows from theorem 4.

Corollary 2. *If f is a homomorphism from $[\Omega(X)]$ to $\boldsymbol{F}$, then f is induced from a homomorphism from $\pi_1(X)$ to $\boldsymbol{F}$ if X is a paracompact arcwise connected smooth manifold.*

Proof. Since we know

$$H^1(X, \boldsymbol{F}) = \text{Hom.}\ (\pi_1(X), \boldsymbol{F}),$$

we get the corollary by $(17)_{\text{iii}}$ and the above corollary 1.

Note 1. Since $\Omega_{2,k,0}(X)$ is different from $\Omega(X)$, $H_{\Omega 2,k,0}{}^p(X, \boldsymbol{F})$, the cohomology group constructed by $\Omega^p{}_{2,k,0}(X, \boldsymbol{F})$, may be different from usual $H_\Omega{}^p(X, \boldsymbol{F})$. But since $\Omega_{2,k,0}(X)$ and $E_{2,k,0}(X)$ are dense subsets of $\Omega(X)$ and $E(X)$ and denoting $i^* : \Omega^p(X, \boldsymbol{F}) \to \Omega^p{}_{2,k,0}(X, \boldsymbol{F})$ the map induced from the inclusion, we obtain

Lemma 14. *There is a homomorphism $i^\# : H^p(X, \boldsymbol{F}) \to H_{\Omega 2,k,0}{}^p(X, \boldsymbol{F})$ for all p.*

Then since χ is defined as the map from $H^p(X, \boldsymbol{F})$ to $H_\Omega{}^p(X, \boldsymbol{F})$, we obtain the following commutative diagram by the definition of $i_\#$.

$$\begin{array}{ccc} H^p(X, \boldsymbol{F}) & \xrightarrow{\chi} & H^p(X, \boldsymbol{F}) \\ & \underset{\sim}{\overset{\chi}{\nwarrow\!\searrow}} \qquad i^\# & \downarrow \\ \omega & & H_{\Omega_2,k,0}{}^p(X, \boldsymbol{F}). \end{array}$$

Therefore χ is also an (into) isomorphism in this case.

Note 2. Since we know

$$H_\Omega{}^p(X, \boldsymbol{F}) \simeq H_{\Omega_\#}{}^1(X, \Omega^{p-1}(X, \boldsymbol{F})), \quad p \geqq 1,$$

by theorem 2, we have

$$H^p(X, \boldsymbol{F}) \simeq H_{\Omega_\#}{}^1(X, \Omega^{p-1}(X, \boldsymbol{F})), \quad p \geqq 1, \tag{42}$$

if X is a compact arcwise connected smooth manifold. We note that a representation f of an element $\langle f \rangle$ of $H_{\Omega_\#}{}^1(X, \Omega^{p-1}(X, \boldsymbol{F}))$ satisfies

$$f(\eta_1 * \eta_2) = f(\eta_1)_\# \eta_2 + f(\eta_2),$$

and the class of f is equal to 0 if f is written as

$$f(\eta) = g - g^\# \eta,$$

where g is an element of $\Omega_\#{}^{p-1}(X, \boldsymbol{F})$.

References

[1]. EELS, J. JR. : A setting for global analysis, Bull. Amer. Math. Soc., 72 (1966), 751–807.

[2]. HALMOS, P. R. : *Measure theory*, Princeton, 1950.

[3]. HOCHSCHILD, G. : On the cohomology groups of an associative algebra, Ann. of Math., 46 (1945,) 58–67.

[4]. IGUSA, J. : On Picard varieties attached to algebraic varieties, Amer. Jour. Math., 74 (1952), 1–22.

[5]. LANG, S. : *Introduction to differential manifolds*, New York, 1962.

[6]. NICKERSON, H. K. -SPENCER, D. C. -STEENROD, N. E. : *Advanced calculus*, Princeton, 1959.

[7]. SERRE, J. P. : Homologie singulière des espaces fibrès, Ann. of Math., 54 (1951), 425–505.

[8]. SOBOLEV, S. L. : *Applications of functional analysis in mathematical physics*, (English transl.) Providence, 1964.

[9]. STEENROD, N. E. : *The topology of fibre bundlds*, Princeton, 1951.

[10]. WEIL, A. : Sur les theorèmes de de Rham, Comm. Math. Helv., 26 (1952), 119–145.