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Currents and Residue Exact Sequences

Dedicated to Prof. Atuo Komatu for his 60th birth day

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Contents.

Introduction.

The residue homomorphism and residue exact sequence has been defined by Leray ([12], cf. [2], [13], [16], [21]) for the pair (X, Y), where X is an n-dimensional complex manifold, Y its $(n-1)$-dimensional submanifold. Similar exact sequence

$$\cdots\longrightarrow H^{i-1}(X-A)\xrightarrow{\lambda}H^{i-p}(A)\xrightarrow{\eta}H^i(X)\xrightarrow{\xi^*}H^i(X-A)\longrightarrow\cdots,$$

where X is a Banach manifold, A its closed submanifold with codimension p, was also defined by Eells ([6], cf. [18]).

In this paper, first we study the residue exact sequence for the pair (X, Y) where X is a smooth manifold, Y its r-codimensional closed submanifold, and prove : *If X and Y are both orientable, then a closed current on $X-Y$ is cohomologous to a current on X* (§ 2, Theorem 3). By virtue of this theorem, residue exact sequence for the pair (X, Y) is given as follows : *Denoting* $i^* : H^p(X, \boldsymbol{C}) \longrightarrow H^p(X-Y, \boldsymbol{C})$ *the inclusion map,* $\delta : H^{p-r}(Y, \boldsymbol{C}) \longrightarrow H^p(X, \boldsymbol{C})$ *the map defined by*

$$\delta\langle T\rangle = \langle \delta_Y T\rangle;\ \ \delta_Y(T)[\varphi] = T[\varphi \mid Y],$$

and $res. : H^p(X-Y, \boldsymbol{C}) \longrightarrow H^{p-r+1}(Y, \boldsymbol{C})$ *the map defined by*

$$res.\langle T\rangle = \langle res.\, T\rangle,\ \ res.\, T[\varphi] = T[d\bar{\varphi} \mid Y],\ \ \bar{\varphi} \mid Y = \varphi,$$

the sequence

$$\cdots \longrightarrow H^p(X, \boldsymbol{C}) \xrightarrow{*i} H^p(X-Y, \boldsymbol{C}) \xrightarrow{res.} H^{p-r+1}(Y, \boldsymbol{C}) \xrightarrow{\delta} H^{p+1}(X, \boldsymbol{C}) \longrightarrow \cdots$$

is exact. Moreover, by virtue of Hironaka's resolution theorem ([9]), we can also prove : *If X is real analytic and Y is its real analytic subvariety with codimension r, such that X and each $Y_i - Y_{i+1}$ are all orientable, where Y_i is the i-th mulsiple subvariety of Y, then a closed current on $X-Y$ is cohomologous to a current on X.* (§ 3, Theorem 4). By virtue of this theorem, setting

$$R^p(Y) = \frac{\{\text{closed } (p+r)\text{-current on } X \text{ with carrier in } Y\}}{d\{(p+r-1)\text{-current on } X \text{ with carrier in } Y\}},$$

we have the exact sequence

$$\cdots \longrightarrow H^p(X, \boldsymbol{C}) \xrightarrow{i^*} H^p(X-Y, \boldsymbol{C}) \xrightarrow{res.} R^{p-r+1}(Y) \xrightarrow{\delta} H^{p+1}(X, \boldsymbol{C}) \longrightarrow \cdots,$$

where i^*, $res.$ and δ are defined by the same way as above. (§ 3).

Although the definition of $R^p(Y)$ depends on X, we can prove that $R^p(Y)$ is independent with X (§ 4, Theorem 6, Corollary 1), it is isomorphic to $H^p(Y, \boldsymbol{C})$ for all p if Y is topological non-singular (Theorem 6, Corollary 2). But since we get if Y is given by $z_1 \cdot z_2 = 0$ in $\boldsymbol{C}^2$, then

$$R^0(Y) \simeq \boldsymbol{C} \oplus \boldsymbol{C},$$
$$R^1(Y) \simeq \boldsymbol{C},\ \ R^p(Y) = 0,\ \ p \geqq 2,$$

$R^p(Y)$ is different either of $H^p(Y, \boldsymbol{C})$ or $H^p(\hat{Y}, \boldsymbol{C})$, where $\hat{Y}$ is the non-singular model of Y, in general. If Y is complex analytic, then we can prove (Theorem 6, Corollary 4), $R^0(Y) \simeq \boldsymbol{C}^s$, the s-direct sum of $\boldsymbol{C}$, where s is the number of irreducible components of Y. But since we get $R^0(Y) \simeq \boldsymbol{C} \oplus \boldsymbol{C} \oplus \boldsymbol{C}$ if Y is given by $x_1 \cdot x_2 = 0$ in R^2, this theorem is false for arbitrary real analytic varieties.

§§ 5–7 of Chapter I are devoted to the applications. They may have been known (cf. [1], [4], [12], [17]). But with the connection of § 4, n^0 14, we note the following. Consider the pair (X, Y), X is a complex manifold, Y its 1-codimensional complex submanifold, $\Omega^p[Y]$ and $\Psi^p[Y]$ the sheaves of germs of analytic p-forms and analytic closed p-forms of X with singularities on Y, we get $\Psi^p[Y] = d\Omega^{p-1}[Y]$ for $p \geqq 2$ and the exact sequence

$$0 \longrightarrow d\Omega^0[Y] \longrightarrow \Psi^1[Y] \longrightarrow \boldsymbol{C}_Y \longrightarrow 0,\ \boldsymbol{C}_Y \text{ is the trivial extension of the constant sheaf of complex numbers on } Y$$

for $p = 1$. Moreover, since $U - Y$ is a Stein manifold if U is a Stein manifold, denoting $\mathfrak{D}^{p,q}[Y]$ and $\mathfrak{G}^{p,q}[Y]$ the sheaves of germs of (p, q)-forms and $\bar{\partial}$-closed (p, q)-forms of X with singularities on Y, we get the following resolution of $\Omega^p[Y]$,

$$0 \longrightarrow \Omega^p[Y] \longrightarrow \mathfrak{D}^{p,0}[Y] \xrightarrow{\bar{\partial}} \mathfrak{D}^{p,1}[Y] \xrightarrow{\bar{\partial}} \cdots.$$

Hence we have

$$H^q(X, \Omega^p[Y]) \simeq H^0(X, \mathfrak{G}^{p,q}[Y])/\bar{\partial} H^0(X, \mathfrak{D}^{p,q-1}[Y])$$
$$= H^0(X - Y, \mathfrak{G}^{p,q})/\bar{\partial} H^0(X - Y, \mathfrak{D}^{p,q-1}) \simeq H^q(X - Y, \Omega^p).$$

Therefore if $X - Y$ is a Stein manifold, we may obtain the residue exact sequence in the category of analytic sheaves (cf. [2], [13], [21]).

In chapter II, we treat the residue exact sequence for the pair (X, Y), where X is a Banach manifold. Since our method in §§ 1, 2 of chapter I can be applicable to the case *codim.* $Y < \infty$, we are mainly interested in the case *dim.* $Y < \infty$. For this purpose, we need to consider the $(\infty - p)$-currents on X. But since X is not locally compact, we consider X to be a (not necessarily closed) submanifold of $\boldsymbol{E}$, a C^∞-smooth Banach space. Then since the metric of $\boldsymbol{E}$ is fixed, we can define the bounded carrier p-forms on X and such forms exist on X, because $\boldsymbol{E}$ is C^∞-smooth (cf. [7]). Then we define an $(\infty - p)$-current of X to be an element of the Dual space of $\mathscr{D}^p(X)$, the space of bounded carrier p-forms on X with Schwartz topology. Using $(\infty - p)$-currents, we define the $(\infty - p)$-de Rham group $\mathscr{H}^{\infty-p}(X)$ as usual. We note that $\mathscr{H}^{\infty-p}(X)$ is not differential structure invariant. For example, if $\boldsymbol{H}$ is a separable real Hilbert space, $\boldsymbol{E}^r$ its r-dimensional subspace, then

$$\mathscr{H}^{\infty-p}(H - E^r) = \{0\},\ \ p \neq r + 1,$$

$$\mathscr{H}^{\infty-r-1}(H-E^r)\simeq C.$$

But $\boldsymbol{H}-\boldsymbol{E}^r$ are diffeomorphic to $\boldsymbol{H}$ for all r ([7]), and $\mathscr{H}^{\infty-p}(\boldsymbol{H})=0$ for all p. We note that with the connection of the expression of functionals on $\boldsymbol{E}$, it seems to be usefull to determin the explicit form of the generator of $\mathscr{H}^{\infty-1}(\boldsymbol{E}-0)$. But since we use Hahn-Banach's theorem to show the existence of such current in this paper, we can not give the explicit form of such current in this stage.

Since there exists a closed non compact p-dimensional submanifold Y contained in U for any open set U of X and for any integer p, we can not define the current T_Y defined by

$$T_Y[\varphi]=\int_Y\varphi$$

for arbitrary closed (orientable) submanifold of X. But if Y is orientable and satisfies

(*) $\quad Y\cap B$ is compact for any bounded closed set of X,

then we can define the current T_Y. Moreover, we prove : *If X is a Banach manifold such that there exists a series of finite dimensional closed submanifolds X^r which satisfies*

(i) $\quad X^r\subset X^{r+1}$, dim. $X^r<$ dim. X^{r+1}, $\overline{\bigcup_r X^r}=X$,

(ii) $\quad$ each X^r is orientable and satisfies (*),

Y its closed submanifold such that $Y\cap X^r$ is a closed orientable submanifold of X^r for each r, and Y satisfies (*) if dim. $Y<\infty$, then a closed current of $X-Y$ is cohomologous to a current of X (§3, Theorem 17). Similar theorem is also true for the pair (X, Y) if X is a Banach analytic manifold ([5]) and Y its sub-analytic variety. By virtue of this theorem, we get the exat sequences

$$\cdots\longrightarrow\mathscr{H}^{\infty-p}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-p}(X-Y)\xrightarrow{res.}H^{r-p+1}(Y,\boldsymbol{C})\xrightarrow{\pi^*}\mathscr{H}^{\infty-p+1}(Y)\longrightarrow\cdots$$

dim. $Y=r$, Y is a submanifold,

$$\cdots\longrightarrow\mathscr{H}^{\infty-p}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-p}(X-Y)\xrightarrow{res.}\mathscr{H}^{\infty-p+1}(Y)\xrightarrow{\pi^*}\mathscr{H}^{\infty-p+1}(X)\longrightarrow\cdots$$

dim. $Y=\infty$, Y is a submanifold,

$$\cdots\longrightarrow\mathscr{H}^{\infty-p}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-p}(X=Y)\xrightarrow{res.}R^{r-p+1}(Y)\xrightarrow{\pi^*}\mathscr{H}^{\infty-p+1}(X)\longrightarrow\cdots$$

dim. $Y=r$, Y is a subvariety,

$$\cdots \longrightarrow \mathscr{H}^{\infty-p}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-p}(X-Y) \xrightarrow{res.} R^{\infty-p+1}(Y) \xrightarrow{\pi^*} \mathscr{H}^{\infty-p+1}(Y) \longrightarrow \cdots$$

dim. $Y=\infty$, Y is a subvariety,

similarly as the finite dimensional cases (§ 3, Theorem 18). Here $R^{\infty-p}(Y)$ is defined similarly as $R^{r-p}(Y)$. We note that denoting bounded carrier cohomology group of X by $H^p_b(X, \boldsymbol{C})$, $\mathscr{H}^{\infty-p}(X)$ is the dual space of $H^p_b(X, \boldsymbol{C})$ (§ 2, Theorem 16), and under suitable condition, $H^p_b(X, \boldsymbol{C})$ is the dual space of $\mathscr{H}^{\infty-p}(X)$. On the other hand, if X is closed and bounded in $\boldsymbol{E}$, then we get $H^p_b(X, \boldsymbol{C}) = H^p(X, \boldsymbol{C})$, the Čech cohomology group of X. Hence from the above exact sequences, we obtain the exact sequence

$$\cdots \longrightarrow H^{p-1}(X, \boldsymbol{C}) \longrightarrow H^{p-1}(X-Y, \boldsymbol{C}) \longrightarrow H^p_b(Y, \boldsymbol{C}) \longrightarrow H^p(X, \boldsymbol{C}) \longrightarrow \cdots$$

if X is closed and bounded in $\boldsymbol{E}$. In § 4, we prove the existence of residue exct sequence for the pair (X, Y) where X is a Banach manifold and need not be satisfy (i), (ii), Y its finite dimensional closed orientable submanifold and satisfies (*). We note that it seems to be natural to denote $\mathscr{H}^{\infty+r-p}(X)$ instead of $\mathscr{H}^{\infty-p}(X)$ if *dim.* $Y=\infty$ and *codim.* $Y=r$, r is a positive integer or ∞. Then the residue exact sequence is rewritten as

$$\cdots \longrightarrow \mathscr{H}^{\infty+r-p}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty+r-p}(X-Y) \xrightarrow{res.} \mathscr{H}^{\infty-p+1}(Y) \xrightarrow{\pi^*} \mathscr{H}^{\infty+r-p+1}(X) \longrightarrow \cdots.$$

In § 5, we state some applications of residue exact sequences. But they are quite similar as the finite dimensional case.

Chapter I. Residue exact sequences of finite dimensional manifolds.

§ 1. Local properties of differential forms with singularities.

1. **Definition.** *Let Y be a closed subset of X, a smooth manifold, such that $\overline{X-Y}=X$, U an open set of X, then a differential form (or a current) on $U-Y$ is called a differential form (or a current) on U with singularities on Y.*

We consider the following sheaves on X.

$\mathfrak{D}^p$: *the sheaf of germs of C^∞-class p-forms on X.*

$\mathfrak{G}^p$: *the sheaf of germs of C^∞-class closed p-forms on X.*

$\mathfrak{D}^p[Y]$: *the sheaf of germs of C^∞-class p-forms with singularities on Y on X.*

$\mathfrak{G}^p[Y]$: *the sheaf of germs of C^∞-class closed p-forms with singularities on Y on X.*

The corresponding sheaves of currents of these sheaves are denoted by $\mathfrak{D}'^p$, $\mathfrak{G}'^p$, $\mathfrak{D}'^p[Y]$ and $\mathfrak{G}'^p[Y]$. The stalks of these sheaves at x are denoted by $\mathfrak{D}^p_x$, $\cdots\cdots$ $\mathfrak{G}'^p[Y]_x$.

In the rest, we consider only complex valued forms and in this §, we assume Y to be a submanifold of X with codimension r. Hence we may assume Y is defined on U, a coordinate neighborhood of X with local coordinates $\{x^U{}_1, \cdots x^U{}_n\} = \{x_1, \cdots, x_n\}$, by $x_1 = \cdots = x_r = 0$. We use the following notations if $r \geqq 2$.

$$\omega^{r-1} = \frac{1}{\sigma^{r-1}} \frac{(\sum_{i=1}^{r} x_i \overset{\vee}{dx_i})}{(\sqrt{x_1^2 + \cdots + x_r^2})^r}, \quad \sigma^{r-1} = \frac{2\pi^{r/2}}{\Gamma(r/2)},$$

$$\overset{\vee}{dx_i} = (-1)^i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_r.$$

We often denote by ω instead of ω^{r-1}, and denote ω_u and $dx^U{}_i$ if there need to clarify the coordinate system by which ω is defined.

Lemma 1. *If $r \geqq 2$ and φ is a closed $(r-1)$-form on U with singularities on Y, then setting*

$$c(a_{r+1}, \cdots, a_n, \varepsilon) = \int_{\substack{x^2{}_1 + \cdots + x^2{}_r = \varepsilon \\ x_{r+1} = a_{r+1}, \cdots, x_n = a_n}} \varphi,$$

$c(a_{r+1}, \cdots, a_n, \varepsilon)$ does not depend on $a_{r+1}, \cdots, a_n$ and ε.

Proof. $c(a_{r+1}, \cdots, a_n, \varepsilon)$ does not depend on ε because φ is closed. On the other hand, we have by Stokes' theorem,

$$0 = \int_{\substack{x_1^2 + \cdots + x_r^2 = \varepsilon \\ x_{r+j} = a_{r+j},\ j \neq i,\ a \leqq x_{r+i} \leqq b}} d\varphi$$

$$= \int_{\substack{x_1^2 + \cdots + x_r^2 = \varepsilon \\ x_{r+j} = a_{r+j},\ j \neq i,\ x_{r+i} = b}} \varphi - \int_{\substack{x_1^2 + \cdots + x_r^2 = \varepsilon \\ x_{r+j} = a_{r+j},\ j \neq i,\ x_{r+i} = a}} \varphi.$$

Hence $c(a_{r+1}, \cdots, a_n,)$ does not depend on $a_{r+1}, \cdots, a_n$.

Corollary. A closed $(r-1)$-form φ on U with singularities on Y is written

(1) $$\varphi = c\omega^{r-1} + d\psi, \quad c = \int_{\substack{x^2{}_1 + \cdots + x^2{}_r = \varepsilon \\ x_{r+1} = a_{r+1}, \cdots, x_n = a_n}} \varphi.$$

By this corollary and de Rham's theorem, we have

Lemma 2. *Denoting $\boldsymbol{C}$ the complex number field, we get if $r \geqq 2$,*

(2) $$\mathfrak{G}^p[Y]_x = d\mathfrak{D}^{p-1}[Y]_x, \quad p \neq 0, \ s-1, \quad x \in Y,$$

$$\mathfrak{G}^{r-1}[Y]_x = \boldsymbol{C}\omega^{r-1} \oplus d\mathfrak{D}^{r-2}[Y]_x, \quad x \in Y.$$

The map from $\mathfrak{G}^{r-1}[Y]_x$ to $C \oplus C\omega^{r-1}$ is denoted by $res._x^U = res._x$ or $res.$.

Note. If $r=1$, then we may set $U-Y=U^+ \cup U^-$, $U^+=\{(x_1, \cdots, x_n) | x_1 > 0\}$, $U^-=\{(x_1, \cdots, x_n) | x_1 < 0\}$, and we obtain

$$\mathfrak{G}^0[Y]_x = \{\varphi_x | \varphi_x \text{ is the class of } \varphi,\ \varphi = c_1 \text{ on } U^+,\ \varphi = c_2 \text{ on } U^-\}.$$

Hence we get

$$(2)' \qquad \mathfrak{G}^p[Y]_x = d\mathfrak{D}^{p-1}[Y]_x,\ p \neq 0,\ x \in Y,$$
$$\mathfrak{G}^0[Y]_x = C \oplus \mathfrak{G}^0_x.$$

Here an element of $\mathfrak{G}^0_x$ is considered to be the class of $\varphi=(c, c)$, and the map from $\mathfrak{G}^0[Y]_x$ to C (which is also denoted by $res.$) is given by

$$res.\ \varphi = c_1 - c_2,\ \varphi = (c_1, c_2).$$

2. If U is a coodinate neighborhood system of X, then we get

$$(3) \qquad x^U{}_i - x^U{}_i(p) = \sum_j c^{VU}{}_{ij}(p) x^V{}_j + O(x^2), \text{ on } U \cap V,$$
$$p \in U \cap V,\ x^V{}_j(p) = 0.$$

We know that setting $g_{UV} = (c^{UV}{}_{ij})$, $\{g_{UV}\}$ is the transition function of the tangent bundle of X. Therefore we may assume that if $U \cap Y \neq \varphi$, then the local coordinate $\{x^U{}_i\}$ of X at U satisfies

$$(3)' \qquad x^U{}_i - x^U{}_i(p) = \sum_j c^{UV}{}_{ij}(p) x^V{}_j + 0(x^2), \text{ on } U \cap V,\ Y \cap U \cap V \neq \varphi,$$
$$(c^{UV}{}_{ij}(p)) \in O(r) \oplus GL(n-r).$$

We know that in (3)$'$, $(c^{UV}{}_{ij})$ belongs in $SO(r) \oplus GL(n-r)$ if and only if the normal bundle of Y is orientable. In other word, if X and Y are both orientable. or both unorientable, then $(c^{UV}{}_{ij})$ belongs in $SO(r) \oplus GL(n-r)$.

Lemma 3. *If the normal bundle of Y in X is orientable and the local coordinates $\{x_i\} = \{x^U{}_i\}$ of X at U, $U \cap Y \neq \varphi$, are taken to be satisfy* (3)$'$, *then we have*

$$\omega_U = \omega_V + d\psi.$$

Proof. By assumption, we have

$$\omega_U = \frac{1}{\sigma^{r-1}} \frac{\left\{(\sum_{j=1}^{r} x^V{}_i \check{d} x^V{}_i) + O((|x^V|_r)^2)\right\}}{\{(\sqrt{(x^V{}_1)^2 + \cdots + (x^V{}_r)^2})^r (1 + O((|x^V|_r)^{r/2}))}.$$

Here $|x^V|_r$ means $\sqrt{(x^V{}_1)^2 + \cdots + (x^V{}_r)^2}$. Then we get

$$\lim_{\varepsilon\to 0} \int_{\substack{(|x^V|_r)^2 = \varepsilon \\ x^V{}_{r+i} = a_i}} \frac{(\sum_{i=1}^{r} x^V{}_i \check{d} x^V{}_i)}{(\sqrt{(x^V{}_1)^2 + \cdots + (x^V{}_r)^2})^r (1 + O((|x^V|_r)^{r/2}))}$$

$$= \sigma^{r-1},$$

$$\lim_{\varepsilon\to 0} \int_{\substack{(|x^V|_r)^2 = \varepsilon \\ x^V{}_{r+i} = a_i}} \frac{O((|x^V|_r)^2)}{(\sqrt{(x^V{}_1)^2 + \cdots + (x^V{}_r)^2})^r (1 + O((|x^V|_r)^{r/2}))}$$

$$= 0.$$

Hence we obtain the lemma.

Note 1. In general, we get

$$\omega_U = det.(g^1{}_{UV})\omega_V + d\psi.$$

Here $\{g^1{}_{UV}\}$ means the transition function of the normal bundle of Y.

Note 2. If $r=1$ and $\varphi \in \mathfrak{G}^0[Y]_x$, then

$$res.{}^U{}_x\varphi = det.(g^1{}_{UV}) res.{}^V{}_x\varphi, \quad x \in U \cap V.$$

3. By Lemma 2, Lemma 3 and above Note 2, we have

Theorem 1. *$\mathfrak{G}^p[Y]$ is equal to $d\mathfrak{D}^{p-1}[Y]$ if $q \neq 0$, $r-1$, $\mathfrak{G}^0[Y]$ is equal to $\mathfrak{G}^0 = \boldsymbol{C}$, the constant sheaf of complex numbers on X if $r \neq 1$ and for $p = r-1$, we have the following exact sequences.*

$$0 \longrightarrow d\mathfrak{D}^{r-2}[Y] \xrightarrow{i'} \mathfrak{G}^{r-1}[Y] \xrightarrow{res.'} \mathscr{C}_Y \longrightarrow 0, \quad r \geqq 2, \tag{4}$$

$$0 \longrightarrow \mathfrak{G}^0 \xrightarrow{i'} \mathfrak{G}^0[Y] \xrightarrow{res.'} \mathscr{C}_Y \longrightarrow 0, \quad r = 1,$$

where *$\mathscr{C}_Y$ is the trivial extension of the local constant sheaf of complex numbers on Y with operation of the determinant bundle of the normal bundle of Y, to X. This sheaf is written by $\mathscr{C}$ if it is considered on Y. If the normal bundle of Y in X is orientable, then $\mathscr{C}_Y = \boldsymbol{C}_Y$, the trivial extension of the constant sheaf of complex numbers on Y, to X. i' is the inclusion and $res.'$ is given by*

$$res.'(\tilde{\varphi}_x) = \frac{1}{\sigma^{r-1}} \int_{\substack{x_1^2 + \cdots + x_r^2 = \varepsilon \\ x_{r+i} = a_i}} \varphi_x,$$

if $r \geqq 2$, *and if* $r=1$, *then*

$$res.'(\varphi_x)=\varphi(\sqrt{\varepsilon}, a_1, \cdots, a_{n-1})-\varphi(-\sqrt{\varepsilon}, a_1, \cdots, a_{n-1}),$$

where φ_x *is a representation of* $\tilde{\varphi}_x$.

Note 1. By de Rham's theorem, a current T_x on $U-Y$ is written

$$T_x=\varphi_x+dS_x,\ \varphi_x \text{ is a differntial form on } U-Y.$$

Therefore we can define $res.'(T_x)$ by

$$res.'(\widetilde{T}_x)=res.'(\tilde{\varphi}_x),\ \widetilde{T}_x \text{ and } \tilde{\varphi}_x \text{ are the classes of } T_x \text{ and } \varphi_x.$$

Then we get

$$\mathfrak{G}'^p[Y]=d\mathfrak{D}'^{p-1}[Y],\ p\neq 0,\ r-1,\ \mathfrak{G}'^0[Y]=\mathfrak{G}'^0=\boldsymbol{C},\ (r\geqq 2),$$

and the exact sequence

$$0\longrightarrow d\mathfrak{D}'^{r-2}[Y]\xrightarrow{i'}\mathfrak{G}'^{r-1}[Y]\xrightarrow{res.'}\mathscr{C}_Y\longrightarrow 0,\ r\geqq 2.$$

Lemma 4. *If we consider* ω *to be a current* T *defined by* $T_\omega[\varphi]=\int\omega\wedge\varphi$, *then*

(5) $$dT_\omega[\varphi]=(-1)^{r-1}\int_Y\varphi=(-1)^{r-1}T_Y[\varphi].$$

Proof. By the definition of ω^{r-1}, we have

$$\int_Y\varphi=\int_{x_1^2+\cdots+x_r^2=\varepsilon}\omega\wedge\varphi(0, \cdots, 0, x_{r+1}, \cdots, x_n).$$

Then we get

$$(-1)^{r-1}\lim_{\varepsilon\to 0}\int_{x_1^2+\cdots+x_r^2\geqq\varepsilon}\omega\wedge d\varphi$$

$$=\lim_{\varepsilon\to 0}\int_{x_1^2+\cdots+x_r^2=\varepsilon}\omega<\varphi=\int_Y\varphi.$$

This shows the lemma for $r\geqq 2$. If $r=1$, then setting $\omega(x)=1$, $x\in U^+$, $\omega(x)=0$, $x\in U^-$, we obtain same result.

By this lemma, a current T on $U-Y$ is cohomologous to a current T' on U and

(5)′ $$res.'T_x=(-1)^{r-1}(dT'|Y)_x.$$

§ 2. Residue exact sequences, I.

4. **Theorem 2.** ([6], [12], [16]) *On X, we have the following exact sequence.*

$$\cdots\longrightarrow H^p(X, \boldsymbol{C})\xrightarrow{i} H^p(X-Y, \boldsymbol{C})\xrightarrow{res.} H^{p-r+1}(Y, \mathscr{C})\xrightarrow{\delta} H^{p+1}(X, \boldsymbol{C})\longrightarrow\cdots. \tag{6}$$

Here i is the map induced from the inclusion.

Proof. If $r=1$, (6) follows from the second exact sequence of (4), because we have

$$H^p(X, \mathfrak{G}^0[Y]) = H^p(X-Y, \boldsymbol{C}),$$

in this case, and since i' is the inclusion, i should be the map induced from the inclusion.

Next we assume $r \geqq 2$. Then by the first exact sequence of (4), we obtain the following exact sequence.

$$\begin{aligned}&\cdots\longrightarrow H^p(X, d\mathfrak{D}^{r-2}[Y])\xrightarrow{i'^*} H^p(X, \mathfrak{G}^{r-1}[Y])\xrightarrow{res.'^*} H^p(X, \mathscr{C}_Y)\\ &\xrightarrow{\delta'^*} H^{p+1}(X, d\mathfrak{D}^{r-2}[Y])\longrightarrow\cdots.\end{aligned}$$

By theorem 1, we get

$$H^p(X, d\mathfrak{D}^{r-2}[Y]) \simeq H^{p+r-1}(X, \boldsymbol{C}),$$

$$H^p(X, \mathfrak{G}^{r-1}[Y]) \simeq H^0(X, \mathfrak{G}^{p+r-1}[Y])/dH^0(X, \mathfrak{D}^{p+r-2}[Y]),$$

because $\mathfrak{D}^q[Y]$ is a fine sheaf for any q. In the second formula, the right hand side is isomorphic to $H^{p+r-1}(X-Y, \boldsymbol{C})$ by de Rham's theorem. Hence (6) is exact for $p \geqq r-1$.

By theorem 1, $d\mathfrak{D}^{p-1}[Y] = \mathfrak{G}^p[Y]$ and $\mathfrak{G}^0[Y] = \mathfrak{G}^0 = \boldsymbol{C}$, the constant sheaf of complex numbers, if $p \geqq r-2$. Therefore $H^p(X, \boldsymbol{C})$ is isomorphic to $H^p(X-Y, \boldsymbol{C})$ by the inclusion map. Hence (6) is exact for all p.

The assertion about i for $p \geqq r-1$ follows from the following commutative diagram.

$$\begin{tikzcd}[column sep=small]
H^{p+r-1}(X, C) \arrow[d, "\simeq"] & & H^{p+r-1}(X-Y, C) \arrow[d] \\
H^0(X, \mathfrak{S}^{p+r-1})/dH^0(X, \mathfrak{D}^{p+r-2}) \arrow[rr, "i"] \arrow[d, "\delta"', "\simeq"] & & H^0(X, \mathfrak{S}^{p+r-1}[Y])/dH^0)X, \mathfrak{D}^{p+r-2}[Y]) \arrow[d, "\delta"'] \\
H^1(X, \mathfrak{S}^{p+r-2}) \arrow[r, "i^*"] \arrow[d, "\delta"', "\simeq"] & H^1(X, d\mathfrak{D}^{p+r-3}[Y]) \arrow[r, "i'^*"] \arrow[ddr, dotted, "\delta", "\simeq"'] & H^1(X, \mathfrak{S}^{p+r-2}[Y]) \arrow[d] \\
\vdots \arrow[d, "\delta"', "\simeq"] & & \vdots \arrow[d, "\delta"'] \\
H^p(X, \mathfrak{S}^{r-1}) \arrow[r, "i^*"] \arrow[d, "\delta", "\simeq"'] & H^p(X, d\mathfrak{D}^{r-2}[Y]) \arrow[r, "i'^*"] \arrow[dr, "\simeq"'] & H^p(X, \mathfrak{S}^{r-1}[Y]) \arrow[d, "\delta"'] \\
H^{p+1}(X, \mathfrak{S}^{r-2}) \arrow[rr, "i^*"] \arrow[d, "\delta", "\simeq"'] & & H^{p+1}(X, \mathfrak{S}^{r-2}[Y]) \arrow[d, "\delta"', "\simeq"] \\
\vdots \arrow[d, "\delta", "\simeq"'] & & \vdots \arrow[d, "\delta"', "\simeq"] \\
H^{p+r-1}(X, \mathfrak{S}^0) \arrow[rr, "i^*"] \arrow[d, "="'] & & H^{p+r-1}(X, \mathfrak{S}^0[Y]) \arrow[d, "="'] \\
H^{p+r-1}(X, C) & & H^{p+r-1}(X, C).
\end{tikzcd}$$

Definition. *The exact sequence (6) is called the residue exact sequence.*

Note. If the normal bundle of Y in X is orientable, then (6) is written

$$(6)' \qquad \cdots \longrightarrow H^p(X, C) \xrightarrow{i} H^p(X-Y, C) \xrightarrow{res.} H^{p-r+1}(Y, C) \xrightarrow{\delta} H^{p+1}(X, C) \longrightarrow \cdots.$$

5. Theorem 2′. *If X and Y are both orientable, then the value of res. on a $(p+r-1)$-chain c is given by*

$$(7) \qquad (res.\ \alpha,\ c) = (\alpha,\ \partial\gamma),\quad \gamma \cdot Y = c,$$

and δ (β) is the class of current $(-1)^{(r-1)(p-r)}T_{Y,\varphi}$. Where $\beta \in H^{p-r+1}(Y)$ is the class of φ and $T_{Y,\varphi}$ is given by

$$(8) \qquad T_{Y,\varphi}[\psi] = \int_Y \varphi\wedge\psi.$$

Proof. First we prove (8).

By the definition of *res.*′, we get

$$res.'\,\{\omega_{i_0}{}^{r-1}\, c_{i_0,\dots,i_p}\} = \{c_{i_0,\dots,i_p}\},$$

$$\{c_{i_0,\dots,i_p}\} \in Z^p(\{U \cap Y,\ C).$$

Since $(\delta c)_{i_0,\dots,i_{p+1}} = 0$, we have

$$c_{i_1,\dots,i_{p+1}} = c_{i_0,i_2,\dots,i_{p+1}} - c_{i_0,i_1,i_3,\dots,i_{p+1}} + \cdots + (-1)^p c_{i_0,\dots,i_p}.$$

Hence

$$\delta\{\omega_{i_0}{}^{r-1}c_{i_0,\ldots,i_p}\}_{i_0,\ldots,i_{p+1}}$$

$$=\{\omega_{i_0}{}^{r-1}c_{i_1,\ldots,i_{p+1}}-\omega_{i_0}{}^{r-1}c_{i_0,i_2,\ldots,i_p}+\cdots+(-1)^p\omega_{i_0}{}^{r-1}c_{i_0,\ldots,i_p}\}$$

$$=\{(\omega_{i_1}{}^{r-1}-\omega_{i_0}{}^{r-1})c_{i_1,\ldots,i_{p+1}}\}.$$

On the other hand, we obtain by (5),

$$\{(-1)^{r-1}dT_{\omega_{i_0}}{}^{r-1}c_{i_0,\ldots,i_p}\}$$

$$=\{T_{Y\cap U_{i_0}\cap\cdots\cap U_{i_p}},\ c^{i_0,\ldots,i_p}\}.$$

Therefore, if we consider δ' to be the map from $H^p(Y, C)$ to $H^p(X, \mathfrak{G}'^r)$, we get

(9) $$\delta'(\{c_{i_0,\ldots,i_p}\})=\{(-1)^{r-1}T_{Y\cap U_{i_0}\cap\cdots\cap U_{i_p}},\ c_{i_0,\ldots,i_p}\}.$$

Then as we know

$$\delta(T_{Y,}\varphi_{i_0,\ldots,i_q})_{i_0,\ldots,i_{q+1}}=T_{Y,}(\delta(\varphi_{i_0,\ldots,i_q}))_{i_0,\ldots,i_{q+1}},$$

$$dT_{Y,}\varphi^{pp}=(-1)^{p+1}T_{Y,}d\varphi,$$

we have (8) by (9).

To prove (7)', it is sufficient to prove

(7)' $$\int_{\partial r}\varphi=\int_{r.Y}res.\varphi.$$ res. φ is a representation of $res.\langle\varphi\rangle$, where $\langle\varphi\rangle$ is the class of φ.

To prove (7)', we take $\{\varphi^{i_0,\ldots,i_p}\}\in{}^p(\{U_i\}, \mathfrak{G}^{r-1}[Y])$. Then

$$res.'^*(\{\varphi^{i_0,\ldots,i_p}\})$$

$$=\left\{\frac{1}{\sigma^{r-1}}\int_{\substack{x_1^2+\cdots+x_r^2=\varepsilon\\ x_{r+i}=a_i}}\varphi i_0,\ldots,i_p\right\}.$$

If $\varphi^{i_0,\ldots,i_p}=(\delta\psi)^{i_0,\ldots,i_p}$, $\{\psi^{i_0,\ldots,i_{p-1}}\}\in C^{p-1}(\{U_i\},\mathfrak{D}^{r-1}[Y])$, then

$$\delta\left\{\frac{1}{\sigma^{r-1}}\int_{x_1^2+\cdots+x_r^2=\varepsilon}\psi i_0,\ldots,i_{p-1}(x_{r+1},\ldots,x_n)\right\}_{i_0,\ldots,i_p}$$

$$=\delta\left\{\frac{1}{\sigma^{r-1}}\int_{\substack{x_1^2+\cdots+x_r^2=\varepsilon\\ y_{r+i}=x_{r+i}}}\psi i_0,\ldots,i_{p-1}\right\}_{i_0,\ldots,i_p}$$

$$=\frac{1}{\sigma^{r-1}}\int_{\substack{x_1^2+\cdots+x_r^2=\varepsilon\\ y_{r+i}=x_{r+i}}}\varphi i_0,\ldots,i_p.$$

Here $x_{r+1}, \cdots, x_n$ are considered to be parameters in the first formula and they are written as $y_{r+1}, \cdots, y_n$ in the second formula. Hence as the map from $H^{p-1}(X, \mathfrak{G}^r[Y])$ to $H^{p-1}(Y, \mathfrak{G}')$, $res.'^*$ is given by

$$res.'^*(\{d\phi^i{}_0, \cdots, {}^i{}_{p-1})$$
$$=\Big\{\frac{1}{\sigma^{r-1}}d\Big(\int_{x_1^2+\cdots+x_r^2=\varepsilon}\phi_{i_0,\cdots,i_{p-1}}(x_{r+1},\cdots,x_n)\Big)\Big\}.$$

We assume that $res.'^*: H^{q-p}(X, \mathfrak{G}^{r+q-1}[Y])\longrightarrow H^{p-q}(Y, \mathfrak{G}^q)$ is given by

$$res.'^*(\{d\phi^i{}_0, \cdots, {}^i{}_{p-q}\})$$
$$=\Big\{\frac{1}{\sigma^{r-1}}d\Big(\sum_{k_{q-1}>\cdots>k_1\geq r+1}\Big(\int_{x_1^2+\cdots+x_r^2=\varepsilon}\phi'^{k_1,\cdots,k_{r-1}}_{i_0,\cdots,i_{p-q}}(x_{q+1},\cdots,x_n)\Big)\wedge$$
$$\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_{q-1}}\Big)\Big\},$$

$$\phi_{i_0,\cdots,i_{p-1}}$$
$$=\sum_{k_q>\cdots>k_1\geq r+1}\phi'^{k_1,\cdots,k_{q-1}}_{i_0,\cdots,i_{p-1}}\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q}.$$

Then we obtain

$$\delta\Big\{\frac{1}{\sigma^{r-1}}\Big(\sum_{k_q>\cdots>k_1\geq r+1}\Big(\int_{x_1^2+\cdots+{}_r^2=\varepsilon}\phi'^{k_1,\cdots,k_q}_{i_0,\cdots,i_{p-q-1}}(x_{r+1},\cdots,x_n)\Big)\wedge$$
$$\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q}\Big)\Big\}_{i_0,\cdots,i_{p-1}}$$
$$=\frac{1}{\sigma^{r-1}}\int\sum_{k_q>\cdots>k_1\geq r+1}\Big(\int_{x_1^2+\cdots+x_r^2=\varepsilon}(\delta\phi')^{k_1,\cdots,k_q}_{i_0,\cdots,i_{p-q}}(x_{r+1},\cdots,x_n)\Big)\wedge$$
$$\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q}\Big)\Big\}_{i_0,\cdots,i_{p-1}},$$

if $d\phi_{i_0,\cdots,i_{p-1}}=(\delta\phi')_{i_0,\cdots,i_{p-1}}$, because

$$(\delta\phi')_{i_0,\cdots,i_{p-q}}$$
$$=\sum_{k_q>\cdots>k_1\geq r+1}(\delta\phi')^{k_1,\cdots,k_q}_{i_0,\cdots,i_{p-1}}\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q},$$
$$\phi_{i_0,\cdots,i_{p-q-1}}=\sum_{k_q>\cdots>k_1\geq r+1}\phi'^{k_1,\cdots,k_q}_{i_0,\cdots,i_{p-q-1}}\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q}.$$

Hence $res.'^*: H^{p-q-1}(X, \mathfrak{G}^{r+1}[Y])\longrightarrow H^{p-r-1}(Y, \mathfrak{G}^{q+1})$ is given by

$$res.'^*(\{d\phi_{i_0,\cdots,i_{p-q-1}}\})$$

$$=\left\{\frac{1}{\sigma^{r-1}}d(\sum_{k_q>\cdots>k_1\geqq r+1}(\int_{x_1^2+\cdots+x_r^2=\varepsilon}\phi_{i_0,\cdots,k_{p-q-1}}^{k_1,\cdots,k_q}(x_{r+1},\cdots,x_n))\wedge\right.$$

$$\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_q}).$$

Therefore $res.: H^0(X, \mathfrak{S}^{p+r-1}[Y])\longrightarrow H^0(Y, \mathfrak{S}^p)$ is given by

(10) $$res.(\{d\phi_{i_0}\})$$

$$=\left\{\frac{1}{\sigma^{r-1}}d(\sum_{k_{p-1}>\cdots>k_1\geqq r+1}(\int_{x_1^2+\cdots+x_r^2=\varepsilon}\phi_{i_0}^{k_1,\cdots,k_{p-1}}(x_{r+1},\cdots,x_n))\wedge\right.$$

$$\left.\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_{p-1}})\right\}.$$

To show (7)$'$, it is sufficient to assume γ is a chain of $\cup_i U_i$, $U_i \cap Y\neq\varphi$. Then we have

$$\int_{\partial\gamma}\varphi=\sum_i\int_{\partial\gamma\cap V_i}\varphi_i,\quad \varphi_i=\varphi\,|V_i,$$

where $\{V_i\}$ satisfy

$$V_i\subset U_i,\ \ U_i\cap Y\neq\varphi,\ \ V_i\cap V_j\subset\partial V_i\cap\partial V_j,$$

$$\partial\gamma\cap V_i=\partial(Q\times c_i),\ \ c_i=c\cap V_i,\ \ Q \text{ is the qube.}$$

Then we get

$$\sum_i\int_{\partial\gamma\cap V_i}\varphi_i=\sum_i\int_{S^{r-1}\times c_i}\varphi_i=$$

$$=\sum_i\int_{x_1^2+\cdots+x_r^2=\varepsilon}\left\{\sum_{k_1,\cdots,k_p}\int_{c_i}\varphi_i^{k_1,\cdots,k_p}(x_{r+1},\cdots,x_n))\wedge\right.$$

$$\wedge dx_{k_1}\wedge\cdots\wedge dx_{k_p}.$$

This shows (7)$'$.

Definition. (7)$'$ *is called the residue formula.* ([12]).

6. If φ is a differential form on X with singularities on Y, then we call the current T_φ and dT_φ are defined on X if the limits

$$\lim_{U(Y)\to Y}\int_{X-U(Y)}\varphi\wedge\psi,\quad \lim_{U(Y)\to Y}\int_{\partial U(Y)}\varphi\wedge\psi$$

exist for all compact carrier differential forms ψ on X. If dT_φ is defined on X, then we define a current $dT_\varphi|Y$ on Y by

$$(dT_\varphi|Y)[\psi] = \lim_{U(Y)\to Y} \int_{\partial U(Y)} \varphi\wedge\overline{\psi},$$

$\overline{\psi}$ is an extension of ψ to $\overline{U(Y)}$.

Then we get by (7),

Lemma 5. *If φ is closed on $X-Y$ and the currents T_φ and dT_φ are both defined on X, then*

$$res. \langle T_\varphi\rangle = \langle (dT_\varphi|Y)\rangle. \tag{11}$$

Lemma 6. *If the current $T_{Y,\varphi}$ is exact on X, then there exists a differential form ψ on X with singularities on Y such that*

$$T_{Y,\varphi} = dT_\psi.$$

Proof. We take a system of closed forms $\{\varphi_U\}$ such that

$$\varphi_U|Y = \varphi|Y \text{ on } U\cap Y, \ (\varphi_U = 0 \text{ if } U\cap Y = \varphi),$$

and set

$$\varphi_{U,V} = \omega_U\wedge\varphi_U - \omega_V\wedge\varphi_V, \text{ on } U\cap V.$$

Then $\{\varphi_{U,V}\}$ defines an element of $H^1(X, \mathfrak{G}^q)$ and it vanishes if and only if the current $T_{Y,\varphi}$ is exact, because $dT\omega_U\wedge\varphi_U = (-1)^{r-1}T_{Y\cap U}$ by lemma 4. Hence we have the lemma by Theorem 2.

We denote by $H_0^p(X-Y, C)$ the subgroup of $H^p(X-Y, C)$ defined by

$$H_0^p(X-Y, C)$$
$$= \{c \mid c = \langle\varphi\rangle, \ T_\varphi \text{ and } dT_\varphi \text{ are both defined on } X\}.$$

Then by Theorem 2′, Lemma 5 and Lemma 6, the following diagram is exact and their lines are exact.

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & H^{p-r}(Y, C) & \xrightarrow{\delta} & H^p(X, C) & \xrightarrow{i^*} & H^p(X-Y, C) & \xrightarrow{res.} & H^{p-r+1}(Y, C) & \longrightarrow \\
& \downarrow = & & \downarrow = & & \downarrow & & \downarrow = & \\
\cdots\longrightarrow & H^{p-r}(Y, C) & \xrightarrow{\delta} & H^p(X, C) & \xrightarrow{i^*} & H_0^p(X-Y, C) & \xrightarrow{\widetilde{res.}} & H^{p-r+1}(Y, C) & \longrightarrow
\end{array}$$

$$\begin{array}{l} \xrightarrow{\delta} H^{p+1}(X, C) \longrightarrow \cdots \\ \qquad \downarrow = \\ \xrightarrow{\delta} H^{p+1}(X, C) \longrightarrow \cdots \end{array},$$

where $\widetilde{res.}\ \langle\varphi\rangle$ is defined by (11). Hence we have

Theorem 3. *If X and Y are both orientable, then $H_0{}^p(X-Y, C)$ is equal to $H^p(X-Y, C)$ unless $r=1$ and $p=0$.*

Since $\mathscr{D}^p(X-Y)$, the Schwartz space of p-forms on $X-Y$, is contained in $\mathscr{D}^p(X)$, any current on X can be considered to be a current on $X-Y$, and we call a current T on $X-Y$ is defined on X if it belongs in i^*-image, where i is the inclusion from $\mathscr{D}^p(X-Y)$ into $\mathscr{D}^p(X)$.

We note that if T is a closed current of $X-Y$ and defined on X, then

$$car.\, dT' \subset Y, \quad i^*(T') = T.$$

Hence dT' defines a current on Y. This current is written by $dT'\,|Y$. If $T=T_\varphi$, then these definitions coincide the previous definitions, and we have

Corollary of Theorem 3. *If T' is a current on X such that $i^*(T')$ is closed, then*

$$res.\ \langle i^*(T')\rangle = \langle dT'\,|Y\rangle. \tag{11$'$}$$

Note. By theorem 3, we can conclude the results of this § as follows: *If Y is a closed submanifold of X such that X and Y are both orientable, then a closed current on $X-Y$ is always cohomologous to a current on X and the following sequence is exact.*

$$\cdots \longrightarrow H^p(X, C) \xrightarrow{i^*} H^p(X-Y, C) \xrightarrow{res.} H^{p-r+1}(Y, C) \xrightarrow{\delta} H^{p+1}(X, C) \longrightarrow \cdots,$$

where i^* is the map induced from inclusion, res. is defined by (11)$'$ and δ is given by

$$\delta\langle T\rangle = \langle \delta_Y(T)\rangle, \quad \delta_Y(T)[\varphi] = T[\varphi\,|Y]. \tag{12}$$

§ 3. Residue exact sequences, II.

7. In this §, we assume X is an orientable real analytic manifold, Y is an its (closed) real analytic subvariety such that setting

$$Y = Y_1 \supset Y_2 \supset \cdots \supset Y_s \supset Y_{s+1} = \phi, \tag{13}$$

Y_{i+1} is the set of all multiple points of Y_i, each $Y_i - Y_{i+1}$ is an orientable submanifold of $X - Y_{i+1}$ with codimension r_i. We call Y_{i+1} the i-th multiple subvariety of Y.

We set

$$\begin{aligned} & H_{0,j}{}^p(X - Y_i,\ \boldsymbol{C}) \\ &= \{c \mid c \in H^p(X - Y_i,\ \boldsymbol{C}),\ c = \langle\varphi\rangle,\ T_\varphi \text{ and } dT_\varphi \text{ both defined on } X - Y_j\}. \\ & H_{0,j}{}^p(Y_i - Y_{i+1},\ \boldsymbol{C}) \\ &= \{c \mid c \in H^p(Y_i - Y_{i+1},\ \boldsymbol{C}),\ c = \langle\varphi\rangle,\ T_{Y_i - Y_{i+1},\varphi} \text{ and } dT_{Y_i-Y_{i+1},\varphi} \text{ both} \\ &\quad \text{defined on } X - Y_j\}. \end{aligned}$$

Here $T_\varphi[\psi]$, etc. mean $\lim._{U(Y_k)\to Y_k}\int_{X-U(Y_k)}\varphi\wedge\psi$, $car.\ \psi \subset X - Y_j$, $k = max. (i,\ j)$, etc.. We denote $H_0{}^p(X - Y_i,\ \boldsymbol{C})$ and $H_0{}^p(Y_i - Y_{i+1},\ \boldsymbol{C})$ instead of $H_{0,s+1}{}^p(X - Y_i,\ \boldsymbol{C})$ and $H_{0,s+1}{}^p(Y_i - Y_{i+1},\ \boldsymbol{C})$.

Theorem 4. *We have for all p and i,*

$$\begin{aligned} & H_0{}^p(X - Y,\ \boldsymbol{C}) = H^p(X - Y,\ \boldsymbol{C}), \\ & H_0{}^p(Y_i - Y_{i+1},\ \boldsymbol{C}) = H^p(Y_i - Y_{i+1},\ \boldsymbol{C}). \end{aligned} \tag{14}$$

Proof. Since the theorem is true if $dim.\ Y = 0$, we assume the theorem is true for any t-dimensional real analytic subvariety which satisfies the asuumptions of this §, of orientable real analytic manifold, $t \leqq s - 1$, and assume $dim.\,Y = s$.

First we assume that Y is simple in the sence of Atiyah-Hodge ([2], p. 77) with finite irreducible components, i.e. setting $Y = Y^1 \cup \cdots \cup Y^k$, each Y^i is irreducible, then each Y^i is non-singular and for all $(i_1, \cdots, i_j)$, $Y^{i_1} \cap \cdots \cap Y^{i_j}$ is a proper intersection.

Since the theorem is true if $k = 1$ by Theorem 3, we use the induction about k and assume the theorem is true for $Y^{(k-1)} = Y^1 \cup \cdots \cup Y^{k-1}$. Then we have the following commutative diagram with exact lines.

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_0{}^{p-r_1}(Y_k - Y_k \cap Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{\delta} & H_0{}^p(X - Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{i^*} & H_0{}^p(X - Y,\ \boldsymbol{C}) & \longrightarrow \\ & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota} & \\ \cdots \longrightarrow & H^{p-r_1}(Y_k - Y_k \cap Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{\delta} & H^p(X - Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{i^*} & H^p(X - Y,\ \boldsymbol{C}) & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{res.} & H_0{}^{p-r_1+1}(Y_k - Y_k \cap Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{\delta} & H_0{}^{p+1}(X - Y^{(k-1)},\ \boldsymbol{C}) & \longrightarrow \cdots. \\ & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & \\ \xrightarrow{res.} & H^{p-r_1+1}(Y_k - Y_k \cap Y^{(k-1)},\ \boldsymbol{C}) & \xrightarrow{\delta} & H^{p+1}(X - Y^{(k-1)}, \boldsymbol{C}) & \longrightarrow \cdots. \end{array}$$

Then ι is an isomorphism because ι_1 and ι_2 are isomorphisms by inductive assumption.

If Y is simple with infinite irreducible components, we set

$$X=\cup_n X_n,\ X_{n+1}\supseteq X_n,\ \text{each } X_n \text{ is a relative compact open set of } X.$$

Then for any $c=\langle\varphi\rangle\in H^p(X-Y,\ \boldsymbol{C})$, we find a closed form φ_n' on X_n such that

$$\langle\varphi\,|X_n\rangle=\langle\varphi_n'\rangle,\ T_{\varphi_n}'\ \text{and}\ dT_{\varphi_n}'\ \text{are both defined,}$$

because $Y\cap X_n$ containes only finite irreducible components. Since $\varphi_{n+1}'\,|X_n-\varphi_n'$ is exact on X_n, we can construct a system of closed forms $\{\varphi_n''\}$ on X_n such that

$$\begin{aligned}&\langle\varphi\,|X_n\rangle=\langle\varphi_n''\rangle,\ T_{\varphi_n}''\ \text{and}\ dT_{\varphi_n}''\ \text{are both defined.}\\ &\varphi_{n+1}''-\varphi_n''\ \text{is defined and}\ |\varphi_{n+1}''-\varphi_n''|<\frac{1}{2^n}\ \text{on}\ X_{n-1}.\end{aligned}\tag{15}$$

Therefore we can construct a closed form φ'' on X such that φ'' is cohomologous to φ and $T_{\varphi''}$ and $dT_{\varphi''}$ both define currents on X. Hence $H^p(X-Y,\ \boldsymbol{C})=H_0{}^p(X-Y,\ \boldsymbol{C})$ if Y is simple.

Next we consider the general case. By Hironaka's theorem ([9]), we can construct a real analytic manifold $\hat{X}$ and a map η from $\hat{X}$ to X such that η is biregular on $\hat{X}-\eta^{-1}(Y)$ and $\eta^{-1}(Y)$ is simple on $\hat{X}$ for arbitrary Y. Then by the above discussion

$$H_0{}^p(\hat{X}-\eta^{-1}(Y),\ \boldsymbol{C})=H^p(\hat{X}-\eta^{-1}(Y),\ \boldsymbol{C}),\tag{16}$$

if $\hat{X}$ is orientable.

By (16), for any closed form φ on $X-Y$, we obtain

$$\begin{aligned}&\eta^*(\varphi)=\varphi_1+d\alpha_1,\\ &\lim_{U(\eta^{-1}(Y))\to\eta^{-1}(Y)}\int_{U(\eta^{-1}(Y))}\varphi_1\wedge\eta^*(\psi)=0.\\ &\lim_{U(\eta^{-1}((Y)\to\eta^{-1}(Y)}\int_{\partial U(\eta^{-1}(Y))}\varphi_1\wedge\eta^*(\psi)\ \text{exists if the carrier of}\ \psi\ \text{is compact.}\end{aligned}\tag{17}$$

Since η is biregular on $\hat{X}-\eta^{-1}(Y)$, we have

$$\varphi=\eta^{-1*}(\varphi_1)+d\eta^{-1*}(\alpha_1),$$

$$\lim_{UY(\to Y)}\int_{U(Y)}\eta^{-1*}(\varphi_1)\wedge\psi=0,$$

$$\lim_{UY\to(Y)}\int_{\partial U(Y)}\eta^{-1*}(\varphi_1)\wedge\psi\ \text{exists if carrier}\ \psi\ \text{is compact.}$$

Therefore $H_0{}^p(X-Y,\ \boldsymbol{C})$ is equal to $H^p(X-Y,\ \boldsymbol{C})$ for any orientable Y. The second equality of (14) follows from the following commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & H_0{}^p(X-Y_2,\ C) \arrow[r, "i^*"] \arrow[d, "="] & H_0{}^p(X-Y,\ C) \arrow[r, "res."] \arrow[d, "="] & H_0{}^{p-r_1+1}(Y-Y_2,\ C) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & H^p(X-Y_2,\ C) \arrow[r, "i^*"] & H^p(X-Y,\ C) \arrow[r, "res."] & H^{p-r_1+1}(Y-Y_2,\ C) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H_0{}^{p+1}(X-Y_2,\ C) \arrow[r, "i^*"] \arrow[d, "="] & H_0{}^{p+1}(X-Y,\ C) \arrow[r] \arrow[d, "="] & \cdots \\
{} \arrow[r, "\delta"] & H^{p+1}(X-Y_2,\ C) \arrow[r, "i^*"] & H^{p+1}(X-Y,\ C) \arrow[r] & \cdots.
\end{tikzcd}$$

If $\hat{X}$ is not orientable, it is sufficient to consider the double covering of $\hat{X}$.

8. **Corollary**. *If $i^* : H^p(X, C) \longrightarrow H^p(X-Y, C)$ is the map induced from the inclusion, and c belongs in $ker. i^*$, then*

$$c = \langle \sum_i T_{Y_i - Y_{i+1},\ \varphi_i} \rangle. \tag{18}$$

Proof. We denote the inclusion from $X-Y_j$ to $X-Y_{j+1}$ by i_j and the inclusion frm $X-Y_j$ to X by i^j. Then we have the following commutative diagram.

$$\begin{tikzcd}
 & H^p(X-Y_s,\ C) \arrow[d, "i_s{}^*"] \\
 & H^p(X-Y_{s-1},\ C) \arrow[d, "i_{s-1}"] \\
H^p(X,\ C) \arrow[uur, "i^{s*}"] \arrow[ur, "i^{s-1*}"] \arrow[dr, "i^{2*}"] \arrow[ddr, "i^{1*}=i^*"'] & \vdots \arrow[d, "i_2{}^*"] \\
 & H^p(X-Y_2,\ C) \arrow[d, "i_1{}^*"] \\
 & H^p(X-Y_1,\ C) = H^p(X-Y,\ C).
\end{tikzcd}$$

On the other hand, we get the following residue exact sequence for all j.

$$\cdots \longrightarrow H^{p-r_j}(Y_j - Y_{j+1},\ C) \xrightarrow{\delta_j} H^p(X-Y_{j+1},\ C) \xrightarrow{i_j{}^*} H^p(X-Y_j,\ C) \longrightarrow \cdots,$$

$$\delta_j(c) = (-1)^{(r_j-1)(p-r_j)} \langle T_{Y_j - Y_{j+1},\varphi} \rangle, \text{ if } c = \langle \varphi \rangle.$$

Since $i_1{}^* i^{2*}(c)$ is equal to 0 if $i^*(c) = 0$, we get

$$i^{2*}(c) = \langle T_{Y_1 - Y_2, \varphi_1} \rangle.$$

Here we consider $T_{Y_1 - Y_2, \varphi_1}$ to be a current on $X-Y_3$ because $H_0{}^p(X-Y_1,\ C) = H^p(X-Y_1,\ C)$. Then

$$i_2{}^* \circ i^{3*}(c) = i^{2*}(c) = \langle T_{Y_1 - Y_2, \varphi_1} \rangle.$$

Therefore we get

$$i^{3*}(c) = \langle T_{Y_1 - Y_2, \varphi_1} + T_{Y_2 - Y_3, \varphi_2} \rangle.$$

Repeating this, we have the corollary.

Note. If $Y = Y^1 \cup \cdots \cup Y^r$, each Y^i is non-singular, then we can set

$$\sum_{i=1}^{s} T_{Y_i - Y_{i+1}, \varphi_i}$$
$$= \sum_{j=1}^{r} \sum_{i=1}^{s} T_{Y_i \cap Y^j - Y_{i+1} \cap Y^j, \varphi_i | (Y_i \cap Y^j - Y_{i+1} \cap Y^j)}.$$

Since $\sum_{i=1}^{s} T_{Y_i - Y_{i+1}, \varphi_i}$ is a closed current on X, we can take each $\sum_{i=1}^{s} T_{Y_i \cap Y^j - Y_{i+1} \cap Y^j, \varphi_i | (Y_i \cap Y^j - Y_{i+1} \cap Y^j)}$ to define a closed current on X. Then we can find a closed form φ^j on Y^j such that

$$\langle T_{Y^j, \varphi^j} \rangle$$
$$= \langle \sum_{i=1}^{s} T_{Y_i \cap Y^j - Y_{i+1} \cap Y^j, \varphi_i | (Y_i \cap Y^j - Y_{i+1} \cap Y^j)} \rangle,$$

because $\langle \sum_{i=1}^{s} T_{Y_i \cap Y^j - Y_{i+1} \cap Y^j, \varphi_i | (Y_i \cap Y^j - Y_{i+1} \cap Y^j)} \rangle$ vanishes in $H^p(X - Y^j, \boldsymbol{C})$ and Y^j is a submanifold. Therefore we obtain

$$(18)' \qquad c = \sum_{j=1}^{r} \langle T_{Y^j, \varphi^j} \rangle, \quad \text{if } i^*(c) = 0.$$

9. On $X - Y_i$ and $Y_j - Y_{j+1}$, we set

$$\Gamma_0(X - Y_i, \mathfrak{D}^p)$$
$$= \{\varphi \mid \varphi \in \Gamma(X - Y_i, \mathfrak{D}^p),\ T_\varphi \text{ and } dT_\varphi \text{ are defined on } X\}.$$

$$\Gamma_0(Y_j - Y_{j+1}, \mathfrak{D}^p)$$
$$= \{\varphi \mid \varphi \in \Gamma(Y_j - Y_{j+1}, \mathfrak{D}^p),\ T_{Y_j - Y_{j+1}, \varphi} \text{ and } dT_{Y_j - Y_{j+1}, \varphi} \text{ are defined on } X\}.$$

If φ belongs in $\Gamma_0(X - Y_i, \mathfrak{D}^p)$ or in $\Gamma_0(Y_j - Y_{j+1}, \mathfrak{D}^p)$, then

$$\lim_{U(Y_i) \to Y_i} \int_{\partial U(Y_i)} \varphi \wedge \psi$$
$$= \sum_{k \geqq 1} \lim_{\substack{U(Y_k) \to Y_k \\ V(Y_i) \to Y_i}} \int_{\partial U(Y_k - Y_{k+1}) - V(Y_i \cap \partial U(Y_k - Y_{k+1}))} \varphi \wedge \psi,$$

$$\lim_{U(Y_{j+1})\to Y_{j+1}} \int_{\partial U(Y_{j+1})\cap Y_j} \varphi\wedge\psi$$

$$=\sum_{k\geqq j+1} \lim_{\substack{U(Y_k)\to Y_k\\ V(Y_{j+1})\to Y_{j+1}}} \int_{(\partial U(Y_k-Y_{k+1})-V(Y_{j+1}\cap\partial U(Y_k-Y_{k+1})))\cap Y_j} \varphi\wedge\psi,$$

exist for all compact carrier forms ψ on X. Hence we get

$$dT_\varphi|(Y_k-Y_{k+1})[\psi]$$

$$=\lim_{\substack{U(Y_k)\to Y_k\\ V(Y_i)\to Y_i}} \int_{\partial U(Y_k-Y_{k+1})-V(Y_i\cap\partial U(Y_k-Y_{k+1}))} \varphi\wedge\bar{\psi}, \quad \varphi\in\Gamma_0(X-Y_i,\ \mathfrak{D}^p),$$

$$dT_\varphi|(Y_k-Y_{k+1})[\psi]$$

$$=\lim_{\substack{U(Y_k)\to Y_k\\ V(Y_{j+1})\to Y_{j+1}}} \int_{(\partial U(Y_k-Y_{k+1})-V(Y_{j+1}\cap\partial U(Y_k-Y_{k+1})))\cap Y_j} \varphi\wedge\bar{\psi},$$

$$\varphi\in\Gamma_0(Y_j-Y_{j+1},\ \mathfrak{D}^p).$$

Here $\bar{\psi}$ is an extension of ψ, a differential form on Y_k-Y_{k+1}.

By definition, $dT_\varphi|(Y_k-Y_{k+1})$ is a closed current on Y_k-Y_{k+1} for all k. Hence we can define the maps $\widehat{res}.:\Gamma_0(X-Y_i,\ \mathfrak{D}^p)\longrightarrow\sum_{k\geqq i}H^{p-r_k+1}(Y_k-Y_{k+1},\ \boldsymbol{C})$ and $\widehat{res}.:\Gamma_0(Y_j-Y_{j+1},\ \mathfrak{D}^p)\longrightarrow\sum_{k\geqq j+1}H^{p-r_k+r_{j+1}}(Y_k-Y_{k+1},\ \boldsymbol{C})$ by

(19) $$\widehat{res}.\ (\varphi)=\sum_k\langle\widehat{res}._k(\varphi)\rangle.\quad \widehat{res}._k(\varphi)=dT_\varphi|(Y_k-Y_{k+1}).$$

Definition. *$\widehat{res}.(\varphi)$ (or its class) is called the singular residue and $\widehat{res}._k(\varphi)$ is called the k-th residue.*

By the definition of $\widehat{res}._k(\varphi)$, we have

Lemma 7. *If $\varphi\in\Gamma_0(Y_i-Y_{i+1},\ \mathfrak{D}^p)$, then*

(20) $$dT_{Y_i-Y_{i+1},\varphi}=(-1)^{p+1}T_{Y_i-Y_{i+1},d\varphi}+(-1)^p\sum_{k\geqq i+1}\widehat{res}._k(\varphi).$$

Corollary. *$\sum_{k\geqq i+1}\widehat{res}._k(\varphi)$ is an exact current of $X-Y_{i+1}$.*

Note. $\widehat{res}._k(\varphi)$ is not exact in general even φ is exact.

We denote the subgroups of $\Gamma_0(X-Y_i,\ \mathfrak{D}^p)$ and $\Gamma_0(Y_j-Y_{j+1},\ \mathfrak{D}^p)$ consisted by the closed forms by $\Gamma_0(X-Y_i,\ \mathfrak{G}^p)$ and $\Gamma_0(Y_j-Y_{j+1},\ \mathfrak{G}^p)$. On the other hand, we set

$$(\sum_k H^{p-r_k}(Y_k-Y_{k+1},\ \boldsymbol{C}))_0$$

$=\{c_1+\cdots+c_s|\, c_k=\langle\varphi_k\rangle\in H^{p-r_k}(Y_k-Y_{k+1},\ \boldsymbol{C}),\ \sum_k T_{Y_k-Y_{k+1},\varphi_k}$ defines a closed current on $X\}$.

Definition. *We define the groups $R^p(Y)=R^p{}_X(Y)$ for all p by*

$$R^p(Y)$$
$$=(\sum_k H^{p-r_k+r_1}(Y_k-Y_{k+1},\ \boldsymbol{C}))_0/\sum_k \widehat{res}.(\Gamma_0(Y_k-Y_{k+1},\ \mathfrak{G}^{p-r_k+r_1-1})).$$

Example. If Y is defined by $z_1\cdot z_2=0$ in $\boldsymbol{C}^2$, then

$$R^0(Y)=H^0(\boldsymbol{C}^1-\{0\},\ \boldsymbol{C})\oplus H^0(\boldsymbol{C}^1-\{0\},\ \boldsymbol{C})=\boldsymbol{C}\oplus\boldsymbol{C},$$

$$R^1(Y)=(H^1(\boldsymbol{C}^1-\{0\},\ \boldsymbol{C})+H^1(\boldsymbol{C}^1-\{0\},\ \boldsymbol{C}))_0=\boldsymbol{C},$$
generator is the class of $dz_1/z_1-dz_2/z_2$,

$$R^2(Y)=H^0(\{0\},\ \boldsymbol{C})/res.(\Gamma_0(\boldsymbol{C}^1\cup\boldsymbol{C}^1-\{0\},\ \mathfrak{G}^1))=0.$$

$$R^p(Y)=0,\ \ p\geqq 3.$$

This example shows $R^p(Y)$ is not isomorphic to neither $H^p(Y,\ \boldsymbol{C})$ nor $H^p(\hat{Y},\ \boldsymbol{C})$ in general. Here $\hat{Y}$ is the non-singular model of Y.

10. Theorem 5. *We have the following exact sequence.*

$$\cdots\longrightarrow H^p(X,\ \boldsymbol{C})\xrightarrow{i}H^p(X-Y,\ \boldsymbol{C})\xrightarrow{res.}R^{p-r+1}(Y)\xrightarrow{\delta}H^{p+1}(X,\ \boldsymbol{C})\longrightarrow\cdots, \tag{21}$$

where i is the inclusion and $res.$ and δ are defined by

$$res.(c)=\langle\widehat{res}.(\varphi)\rangle,\ c=\langle\varphi\rangle,\ \langle\widehat{res}.\ (\varphi)\rangle \text{ is the class of } \widehat{res}.(\varphi) \text{ in } R^{p-r+1}(Y), \tag{22}$$

$$\delta(c)=\langle\sum_i T_{Y_i-Y_{i+1},\varphi_i}\rangle,\quad c=\langle\sum_i\langle\varphi_i\rangle\rangle. \tag{23}$$

Proof. Since the theorem is true if $s=1$, we use the induction about s and assume the theorem is true for Y_{j+1}.

First we prove that $res.$ is defined on $H^p(X-Y,\ \boldsymbol{C})$. To show this, we take representations $\varphi,\ \varphi'$ of $c\in H^p(X-Y,\ \boldsymbol{C})$. Then by residue exact sequence, $\langle T_{\varphi-\varphi'}\rangle$ belongs in $H^p(X-Y_{j+1},\ \boldsymbol{C})$ and it comes from $H^{p-r_i}(Y_i-Y_{i+1},\ \boldsymbol{C})$. Then since the diagram

$$
\begin{tikzcd}
H^{p-r_i}(Y_i - Y_{i+1}, C) \arrow[rr, "\delta' = (-1)^{(r_i-1)(p-r_i)}\delta"] & & H^p(X - Y_{i+1}, C) \arrow[d, "res."] \\
\Gamma_0(Y_i - Y_{i+1}, \mathfrak{G}^{p-r_i}) \arrow[u] \arrow[d, "\widehat{res.}"'] & & \\
\sum_{k \geq i+1} H^{p-r_{k+1}}(Y_k - Y_{k+1}, C) \arrow[rr] & & R^{p-r_{i+1}+1}(Y_{i+1}),
\end{tikzcd}
$$

is commutative, the class of $\widehat{res.}(T_{\varphi-\varphi'}) = \widehat{res.}\varphi - \widehat{res.}\varphi'$ is 0 in $R^{p-r_i+1}(Y_i)$. On the other hand, *res.* is defined on $H^p(X - Y_i, C)$ by theorem 4. This shows the assertion.

By the definition of *res.*, *Im. i* is contained in ker. *res.*. On the other hand, we get *Im. i* = *ker. res.* on Y_{j+1} by inductive assumption. Then considering following commutative diagram

$$
\begin{tikzcd}
 & H^{p-r_j}(Y_j - Y_{j+1}, C) \arrow[d, "\delta'"] \arrow[dr, "res.' = res. \cdot \delta'"] & \\
 & H^p(X - Y_{j+1}, C) \arrow[r, "res."] \arrow[dl, "i_{j+1}"'] \arrow[d, "i_j^{j+1}"] & R^{p-r_{j+1}+1}(Y_{j+1}) \arrow[d, "i_j^{j+1'}"] \\
H^p(X, C) \arrow[r, "i_j"] & H^p(X - Y_j, C) \arrow[r, "res."] & R^{p-r_j+1}(Y_j),
\end{tikzcd}
$$

if $c \in H^p(X - Y_j, C)$ belongs in *ker. res.*, then by residue exact sequence, $c = i_j^{j+1}(c')$ and we get

$$c' = i_{j+1}(c_1) + \delta'(c_2).$$

Hence $c = i_j(c_1)$ which means $Im.\ i_j \supset ker.\ res.$.

By the definitions of *res.* and δ, *Im. res.* is contained in $ker.\delta$ and on Y_{j+1}, we have $Im.\ res. = ker.\delta$.

If $c \in R^{p-r_j+1}(Y_j)$ and $\delta(c) = 0$, then there exists a c' such that

$$c - c' = i_j^{j+1'}(c_1), \quad c' = res.(c_0).$$

Then since the diagram

$$
\begin{tikzcd}
H^{p-r_{j+1}}(Y_j - Y_{j+1}, C) \arrow[d, "\delta'"] \arrow[dr, "res.' = res."] & & \\
H^p(X - Y_{j+1}, C) \arrow[r, "res."] \arrow[d, "i_j^{j+1}"] & R^{p-r_{j+1}+1}(Y_{j+1}) \arrow[d, "i_j^{j+1}"] \arrow[dr, "\delta_{j+1}"] & \\
H^p(X - Y_j, C) \arrow[r, "res."] & R^{p-r_j+1}(Y_j) \arrow[r, "\delta_j"] & H^{p+1}(X, C),
\end{tikzcd}
$$

is commutative, we obtain

$$c_1 = c' + c_1'', \quad \delta_{j+1}(c_1') = 0, \quad i_j^{j+1\prime}(c_1'') = 0.$$

Since $ker.\ i_j^{j+1\prime} = Im.\ res.'$ by inductive assumption, we get $Im.\ res. \supset ker.\ \delta$ by this equality.

$Im.\ \delta = ker.\ i$ follows from the corollary of Theorem 4.

§ 4. Properties of $R^p(X)$.

11. By (23), $\delta : R^{p-r_k+r_j+1}(Y_k) \longrightarrow R^{p+1}(Y_j)$ is also defined. On the other hand, $res. : R^p(Y_j - Y_k) \longrightarrow R^{p-r_k+r_j+1}(Y_k)$ is defined by using the diagram

$$\begin{tikzcd}
R^p(Y_j - Y_k) \arrow[rr, "res."] \arrow[dr, "\delta"'] & & R^{p-r_k+r_j+1}(Y_k) \\
& H^{p+r_j}(X - Y_k, C). \arrow[ur, "res."'] &
\end{tikzcd}$$

Then we get

Theorem 5′. *We have the following exact sequence.*

$$(21)' \qquad \cdots \longrightarrow R^p(Y_j) \xrightarrow{i} R^p(Y_j - Y_k) \xrightarrow{res.} R^{p-r_k+r_j+1}(Y_k) \longrightarrow$$
$$\xrightarrow{\delta} R^{p+1}(Y_j) \longrightarrow \cdots.$$

Here Y_k is a closed subvariety of Y_j and satisfies the assumptions of § 3, but need not be a multiple subvariety of Y_j.

Proof. By the definitions of i, $res.$, and δ, we have the following commutative diagram with exact raws and colums.

$$\begin{tikzcd}
& & & & \vdots \arrow[d] & \vdots \arrow[d] & \\
& \vdots \arrow[d, "i_1"] & \vdots \arrow[d, "i_2"] & \vdots \arrow[d] \arrow[r, phantom, "\cdots \longrightarrow"] & H^{p+r_j}(X, C) \arrow[r, "i_3"] \arrow[d, "i_1"] & H^{p+r_j}(X - Y_k, C) \arrow[r] \arrow[d, "i_2"] & \cdots \\
\cdots \arrow[r] & H^{p+r_j-1}(X - Y_j, C) \arrow[r, "="] \arrow[d, "res._1"] & H^{p+r_j-1}(X - Y_j, C) \arrow[r] \arrow[d, "res._2"] & 0 \arrow[r] \arrow[d] & H^{p+r_j}(X - Y_j, C) \arrow[r] \arrow[d, "res._1"] & H^{p+r_j}(X - Y_j, C) \arrow[r] \arrow[d, "res._2"] \arrow[dl, "res._3" near start, "i"' near end] & \cdots \\
\cdots \arrow[r] & R^p(Y_j) \arrow[r, "i"] \arrow[d, "\delta_1"] & R^p(Y_j - Y_k) \arrow[r, "res."] \arrow[d, "\delta_2"] & R^{p-r_k+r_j+1}(Y_k) \arrow[r, "\delta"] \arrow[d, "="] & R^{p+1}(Y_j) \arrow[r] \arrow[d, "\delta_1"] & R^{p+1}(Y_j - Y_k) \arrow[r] \arrow[d, "\delta_2"] & \cdots \\
\cdots \arrow[r] & H^{p+r_j}(X, C) \arrow[r, "i_3"] \arrow[d, "i_1"] \arrow[dr, "i_4"] & H^{p+r_j}(X - Y_k, C) \arrow[r, "res._3"] \arrow[d, "i_2"] & R^{p-r_k+r_j+1}(Y_k) \arrow[r, "\delta_3"] \arrow[d] & H^{p+r_j+1}(X; C) \arrow[r] \arrow[d] & H^{p+r_j+1}(X - Y_k, C) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{p+r_j}(X - Y_j, C) \arrow[r, "="'] \arrow[d] & H^{p+r_j}(X - Y_j, C) \arrow[r] \arrow[d] \arrow[ur, "\delta_4"] & \cdots & & & \\
& \vdots & \vdots & & & &
\end{tikzcd}$$

By the definitions of i, $res.$ and δ, we have $Im.\ i \subset ker.\ \delta$ and $Im.\ \delta \subset ker\ i$.

If $res.\ (c) = 0$, then $\delta_2\ (c) = i_3\ (c_1)$ and $i_1\ (c_1) = i_2\ (c) = 0$. Hence $c_1 = \delta_1\ (c_2)$ and $c - i\ (c_2) = res._2\ (c_3)$. Therefore

$$c = i(c_2 + res._1(c_3)).$$

This shows $ker.\ res. \subset Im.\ i$.

If $\delta\ (c) = 0$, then $c = res._3\ (c_1)$ and $i_2\ (c_1) = i_4\ (c_2)$ because $\delta_4\ i_2\ (c_1) = 0$. Then $c_1 - i_3\ (c_2) = \delta_2\ (c_3)$. Hence

$$c = res._3(c_1 - i_3(c_2)) = res.\ (c_3),$$

because $res._3 i_3\ (c_2) = 0$. This shows $ker.\ \delta \subset Im.\ res.$.

If $i\ (c) = 0$, then $\delta_1\ (c) = \delta_3\ (c_1)$ and $c - \delta\ (c_1) = res._1\ (c_2)$. Since $res._2\ (c_2) = 0$, $c_2 = i_2\ (c_3)$. Hence

$$c = \delta\ (c_1) + res._1\ (c_2) = \delta\ (c_1 + res._3(c_3)).$$

This shows $ker.\ i \subset Im.\ \delta$.

Note. Explicitly, $res.$ is given by

$$res.\ (\sum_i \varphi_i) = \langle \sum_i \sum_k \widehat{res.}_k(\varphi_i) \rangle. \tag{24}$$

12. Lemma 8. *For all k, j $(j > k)$, we have*

$$\sum_{i \geqq j > k} res._k \Gamma_0(Y_j - Y_{j+1},\ \mathfrak{G}^p)/d\Gamma(Y_k - Y_{k+1},\ \mathfrak{D}'^{p-r_k+r_j-1}) \tag{25}$$
$$= \sum_{i \geqq j > k} \widehat{res.}_k \Gamma_0(Y_j - Y_{j+1},\ \mathfrak{D}^p)/d\Gamma(Y_k - Y_{k+1},\ \mathfrak{D}'^{p-r_k+r_j-1}).$$

Proof. As this left hand side is contained in the right hand side, we need only to show the right hand side is contained in the left hand side. But since the right hand side is contained in the δ-kernel of the exact sequence

$$\cdots \longrightarrow R^{p+r_k-r_i}(Y_i - Y_k) \xrightarrow{res.} H^{p+1}(Y_k - Y_{k+1},\ C) \xrightarrow{\delta}$$
$$\longrightarrow R^{p+r_k-r_i+1}(Y_i - Y_{k+1}) \longrightarrow \cdots,$$

we can find for any $\psi_j \in \Gamma_0(Y_j - Y_{j+1},\ \mathfrak{D}^p)$, a system of closed forms $\{\varphi_j\}$, $\varphi_j \in \Gamma_0(Y_j - Y_{j+1},\ \mathfrak{G}^p)$ such that

$$\widehat{res.}_k(\psi_j) = \sum_{i \geqq j > k} \widehat{res.}_k(\varphi_j).$$

Hence the right hand side is contained in the left hand side.

Corollary. *We obtain*

$$R^p(Y)$$
$$=(\sum_k H^{p-r_k+r_1}(Y_k-Y_{k+1},\ C))_0/\sum_k res.\ (\Gamma_0(Y_k-Y_{k+1},\ \mathfrak{D}^{p-r_k+r_1-1})).$$

We set

$$\Gamma_X(Y,\ \mathfrak{D}^p)=\{(\varphi_1,\ \cdots,\ \varphi_s)\,|\varphi_i\in\Gamma_0(Y_i-Y_{i+1},\ \mathfrak{D}^{p+r_i-r_1}),$$
$$\Gamma_X(Y,\ \mathfrak{G}^p)$$
$$=\{(\varphi_1,\ \cdots,\ \varphi_s)\,|\varphi_i\in\Gamma_0(Y_i-Y_{i+1},\ \mathfrak{G}^{p+r_i-r_1}),\quad d(\sum_i T_{Y_i-Y_{i+1},\varphi_i})=0\}.$$

Then we have by lemma 7 and the above corollary,

$$R^p(Y)=\Gamma_X(Y,\ \mathfrak{G}^p)/\hat{d}\Gamma_X(Y,\ \mathfrak{D}^{p-1}), \tag{26}$$

where $\hat{d}(\varphi_1,\ \cdots,\ \varphi_s)$ means $d(\sum_i T_{Y_i-Y_{i+1},\varphi_i})$.

Next, we set

$$\Gamma_X(Y,\ \mathfrak{D}'^p)=\{T|\ T \text{ is a } (p+r)\text{-current on } X,\ car.\,T\subset Y\},$$
$$\Gamma_X(Y,\ \mathfrak{G}'^p)=\{T|\ T \text{ is a closed } (p+r)\text{-current on } X,\ car.\,T\subset Y\}.$$

Theorem 6. *$R^p(Y)$ is isomorphic to $\Gamma_X(Y,\ \mathfrak{G}'^p)/d\Gamma_X(Y,\ \mathfrak{D}'^{p-1})$.*

Proof. By (26), we can consider $R^p(Y)$ to be a subgroup of $\Gamma_X(Y,\ \mathfrak{G}'^p)/d\Gamma_X(Y,\ \mathfrak{D}'^{p-1})$, which we set $R'^p(Y)$.

If Y is non-singular, then $R^p(Y)=R'^p(Y)=H^p(Y,\ C)$. Hence we use the induction about s and assume the theorem is true for Y_2.

Let T be a current in $\Gamma_X(Y,\ \mathfrak{G}'^p)$, then by theorem 4, we can find a closed form $\varphi\in\Gamma_0(Y_1-Y_2,\ \mathfrak{G}^p)$ such that

$$T|\,(Y_1-Y_2)=T_{\varphi_1}+dS \quad \text{on} \quad Y_1-Y_2. \tag{27}$$

Since T belongs in $\Gamma_X(Y,\ \mathfrak{G}'^p)$, dS also belongs in $\Gamma_X(Y,\ \mathfrak{G}'^p)$. Then by (27),

$$car.\,(T-(T_{\varphi_1}+dS))\subset Y_2.$$

Since the theorem is true on Y_2, we have

$$T-(T_{\varphi_1}+dS)=\sum_{k\geq 2}T_{Y_k-Y_{k+1},\varphi_k}+dS',\ \text{ on } Y_2.$$

This proves the theorem.

Corollary 1. *$R_X(Y)$ does not depend on X.*

Proof. By theorem 6, we have

$$R_X(Y) = R_{U(Y)}(Y),$$

where $U(Y)$ is a neighborhood of Y in X. Then since $U(Y)$ is a real analytic manifold, it is imbedded in R^n ([8]). Denoting this imbedding by ι, we have

$$R_{U(Y)}(Y) = R_{\boldsymbol{R}^m}(\iota(Y)).$$

Since this right hand side does not depend on ι and m, we obtain the theorem.

Corollary 2. *If Y is a topological submanifold of X, then $R^p(Y) \simeq H^p(X, \boldsymbol{C})$ for all p.*

Proof. By assumption, each irreducible component of Y is disjoint each other. Hence we get

$$\text{(28)} \qquad \begin{aligned} &R^0(Y) = H^0(Y, \boldsymbol{C}), \\ &\Gamma_X(Y, \mathfrak{G}'^0) = \Gamma(Y, \boldsymbol{C}). \end{aligned}$$

We denote the sheaves of germs of $\Gamma_X(Y, \mathfrak{D}'^p)$ and $\Gamma_X(Y, \mathfrak{G}'^p)$ by $\mathfrak{D}'_Y{}^p$ and $\mathfrak{G}'_Y{}^p$. Then since the sequence

$$\cdots \longrightarrow H^p(U, \boldsymbol{C}) \xrightarrow{i} H^p(U - Y, \boldsymbol{C}) \xrightarrow{res.} R^{p-r+1}(Y) \xrightarrow{\delta} \\ \longrightarrow H^{p+1}(U, \boldsymbol{C}) \longrightarrow \cdots$$

is exact, $\mathfrak{G}'_Y{}^p/d\mathfrak{D}'_Y{}^{p-1}$ is the sheaf of the correction of local $(p+r-1)$-cohomology groups of $U-Y$. Hence it vanishes for $p \geqq 1$ if Y is a topological submanifold of X. Hence we get the exact sequence

$$\text{(29)} \qquad 0 \longrightarrow \mathfrak{G}'_Y{}^p \longrightarrow \mathfrak{D}'_Y{}^p \xrightarrow{d} \mathfrak{G}'_Y{}^{p+1} \longrightarrow 0.$$

Since $\mathfrak{D}'_Y{}^p$ is a fine sheaf, we obtain by (28), (29),

$$R^p(Y) \simeq H^p(Y, \mathfrak{G}'_Y{}^0) \simeq H^p(Y, \boldsymbol{C}), \quad p \geqq 1.$$

Corollary 3. *If Y is an analytic subvariety and a topological submanifold of X with codimension r, then the following sequence is exact.*

$$\text{(6)}'' \qquad \cdots \longrightarrow H^p(X, \boldsymbol{C}) \xrightarrow{i} H^p(X - Y, \boldsymbol{C}) \xrightarrow{res.} H^{p-r+1}(Y, \boldsymbol{C}) \xrightarrow{\delta} \\ \longrightarrow H^{p+1}(X, \boldsymbol{C}) \longrightarrow \cdots.$$

Corollary 4. *If $Y = Y^1 \cup \cdots \cup Y^r$, each Y^i is irreducible, then there is a homomorphism*

$$h : H^p(Y^1, \boldsymbol{C}) \oplus \cdots \oplus H^p(Y^r, \boldsymbol{C}) \longrightarrow R^p(Y),$$

for all p. This h is an isomorphism onto for $p=0$ if each $Y^i - Y^i \cap Y^1 \cup \cdots \cup Y^{i-1} \cup Y^{i+1} \cup \cdots \cup Y^r)$ is connected and isomorphism into for $p=1$ if each Y^i is locally irreducible.

Proof. Since $\mathfrak{D}'_Y{}^p$ is a fine sheaf, there is a homomorphism from $H^p(Y, \mathfrak{G}'_Y{}^0)$ to $R^p(Y)$. Then since $\boldsymbol{C}_{Y^1} \oplus \cdots \oplus \boldsymbol{C}_{Y^r}$ is a subsheaf of $\mathfrak{G}'_Y{}^0$, h is defined. The rest follows from the definition.

13. Lemma 9. *Let Y' be a closed submanifold of $Y_i - Y_{i+1}$ or $X - Y_{i+1}$ and $T_{Y'}$ defines a current in $\Gamma_X(Y, \mathfrak{G}'^p)$, then*

$$\begin{aligned}(30)\qquad & dT_{Y',\varphi}[\psi] \\ &= (-1)^{p-1} T_{Y',d\varphi}[\psi] + \\ &+ (-1)^p \sum_{k>i} \lim_{\substack{U(Y_k)\to Y_k \\ V(Y_{i+1})\to Y_{i+1}}} \int_{(\partial U(Y_k - Y_{k+1}) - V(Y_{i+1} \cap \partial U(Y_k - Y_{k+1})))\cap Y'} \varphi \wedge \psi.\end{aligned}$$

Lemma 10. *If Y', Y'' are closed submanifolds in $Y_i - Y_{i+1}$ and homologous each other, then $T_{Y',\varphi}$ and $T_{Y'',\varphi}$ are cohomologous each other if φ is closed.*

We denote the self intersection of $Y_i - Y_{i+1}$ in $X - Y_{i+1}$ by $(Y_i - Y_{i+1}) \cdot (Y_i - Y_{i+1})$. Then by Lemma 9 and 10, we define the cup product of $R^*(Y) = \sum_p R^p(Y)$ by

$$\begin{aligned}(31)\qquad & \langle \sum_i T_{Y_i - Y_{i+1}, \varphi_i} \rangle \cup \langle \sum_i T_{Y_i - Y_{i+1}, \varphi_i'} \rangle \\ &= \langle (\sum_i T_{Y_i - Y_{i+1}, \varphi_i}) \wedge (\sum_i T_{Y_i - Y_{i+1}, \varphi_i'}) \rangle.\end{aligned}$$

Since we know

$$\langle \sum_i T_{Y_i - Y_{i+1}, \varphi_i} \rangle \cup \langle \psi \rangle = \langle \sum_i T_{Y_i - Y_{i+1}, \varphi_i \wedge \psi} \rangle,$$

we obtain by (31) and Lemma 9,

Theorem 7. *$\delta(R^*(Y_k))$ and res. $(H^*(X - Y_k, \boldsymbol{C})$ or res. $(R^*(Y_j - Y_k))$ are ideals of $H^*(X)$ or $R^*(Y_j - Y_k)$ and $R^*(Y_k)$ $(j<k)$, and*

$$(32)\qquad R^*(Y_j)/\delta(R^*(Y_k)) \simeq i(R^*(Y_j)),$$

$$(32)'\qquad H^*(X, \boldsymbol{C})/\delta(R^*(Y_k)) \simeq i(H^*(X, \boldsymbol{C})),$$

$$(33)\qquad R^*(Y_k)/res.\,(R^*(Y_j - Y_k)) \simeq \delta(R^*(Y_k)) \ (\subset R^*(Y_j)),$$

$$(33)'\qquad R^*(Y_k)/res.\,(H^*(X - Y_k, \boldsymbol{C})) \simeq \delta(R^*(Y_k)) \ (\subset H^*(X, \boldsymbol{C})).$$

Note. As (11)′, we obtain

$$res. \langle i^*(T')\rangle = \langle dT' \mid Y\rangle, \tag{11}''$$

in this case, too. Then by Theorem 4, we can conclude the results of § 3 and § 4 as follows : *If Y is a closed orientable subvariety of an orientable real analytic manifold X, then a closed current on $X-Y$ is always cohomologous to a current on X and the following sequence is exact.*

$$\cdots\longrightarrow H^p(X,\ C)\xrightarrow{i^*} H^p(X-Y,\ C)\xrightarrow{res.} R^{p-r+1}(Y)\xrightarrow{\delta}$$
$$\longrightarrow H^{p+1}(X,\ C)\longrightarrow\cdots,$$

where $R^p(Y)$ is given by theorem 6, res. is defined by (11)″.

§ 5. Applications of residue exact sequences.

14. In this n^0, we assume X is a complex manifold and Y is a complex analytic submanifold of X with complex codimension 1. We set

$\Omega^p\{kY\}$: *the sheaf of germs of meromorphic p-forms whose poles are in Y with degree at msot k on X.*

$\Psi^p\{kY\}$: *the sheaf of germs of closed meromorphic p-forms whose poles are in Y with degree at most k on X.*

Lemma 11. *On X, we have*

$$\Psi^0\{kY\} = C,\ \ \Psi^p\{kY\} = d\Omega^{p-1}\{(k-1)Y\},\ \ p\geqq 2 \tag{34}$$

and the following sequence is exact.

$$0\longrightarrow d\Omega^0\{(k-1)Y\}\longrightarrow \Psi^1\{kY\}\longrightarrow C_Y\longrightarrow 0,\ \ k\geqq 1. \tag{35}$$

Proof. If $\varphi\in\Gamma(U,\ \Omega^p\{kY\})$, then by Laurent expansion, we may set

$$\varphi = \sum_{j\leqq k} z^{-j}(\varphi_{j,1} + dz\wedge\varphi_{j,2}) + \varphi_0,$$

where φ_0 is holomorphic, Y is defined by $z=0$ on U, $\varphi_{j,1}$ and $\varphi_{j,2}$ both independent to z and $\varphi_{j,1}$ does not involve dz. Then since

$$d\varphi = \sum_{j\leqq k}(-j)z^{-j-1}dz\wedge\varphi_{j,1} + \sum_{j\leqq k} z^{-j}(d\varphi_{j,1} - dz\wedge d\varphi_{j,2}) + d\varphi_0,$$

we get

$$d\varphi_0 = 0,\ \ \varphi_{k,1} = 0,\ \ (-j)\varphi_{j,1} = d\varphi_{j+1,2},\ \ j=1,\cdots,k-1,\ \ d\varphi_{1,2} = 0.$$

Hence we have the lemma.

Theorem 8. *Let X be a Stein manifold, Y its non-singular divisor, then*

$$
\begin{aligned}
(36)\qquad & H^p(X-Y,\ \boldsymbol{C}) \\
& \simeq H^0(X,\ \Psi^p\{(k+p)Y\})/dH^0(X,\ \Omega^{p-1}\{(k+p-1)Y\}),\ k\geqq 0.
\end{aligned}
$$

Proof. By (35), we have the exact sequence

$$
\begin{aligned}
(37)\qquad & \cdots \longrightarrow H^p(X,\ d\Omega^0\{(k-1)Y\}) \longrightarrow H^p(X,\ \Psi^1\{kY\}) \longrightarrow H^p(Y,\ \boldsymbol{C}) \longrightarrow \\
& \longrightarrow H^{p+1}(X,\ d\Omega^0\{(k-1)Y\}) \longrightarrow \cdots,\quad k\geqq 1.
\end{aligned}
$$

Since X is a Stein manifold and the sequences

$$
\begin{gathered}
0 \longrightarrow \boldsymbol{C} \longrightarrow \Omega^0\{(k-1)Y\} \longrightarrow d\Omega^0\{(k-1)Y\} \longrightarrow 0, \\
0 \longrightarrow \Psi^q\{jY\} \longrightarrow \Omega^q\{jY\} \longrightarrow \Psi^{q+1}\{(j+1)Y\} \longrightarrow 0,
\end{gathered}
$$

sre exact, we have by (37), the following exact sequence.

$$
\begin{aligned}
(37)'\qquad & \cdots \longrightarrow H^{p+1}(X,\ \boldsymbol{C}) \longrightarrow H^0(X,\ \Psi^{p+1}\{(k+p)Y\})/dH^0(X,\ \Omega^{p-1}\{(k+p-1)Y\}) \longrightarrow \\
& \longrightarrow H^p(Y,\ \boldsymbol{C}) \longrightarrow H^{p+1}(X,\ \boldsymbol{C}) \longrightarrow \cdots.
\end{aligned}
$$

Then since the diagram

$$
\begin{tikzcd}
0 \arrow[r] & d\mathfrak{D}^0[Y] \arrow[r] & \mathfrak{S}^1[Y] \arrow[r] & \boldsymbol{C}_Y \arrow[r] & 0 \\
0 \arrow[r] & d\Omega^0\{(k-1)Y\} \arrow[r] \arrow[u] & \Psi^1\{kY\} \arrow[r] \arrow[u] & \boldsymbol{C}_Y \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

is commutative, we have the lemma by 5-lemma.

15. Lemma 12. *If Y is a boundary, then $\delta : R^{p-r}(Y) \rightarrow R^p(Y)$ is a 0-map.*

Proof. First we assume that Y and X are both non-singular.

Let T be a closed current with carrier in Y and $Y=\partial Z$. We take sequences of forms $\{\Psi_n\}$ and $\{\delta_Z{}^m\}$ who converge to T and T_Z. We may assume each Ψ_n is a closed form. Then

$$
\delta T[\Psi] = \lim_{n,m\to\infty} \int_X \psi_n \wedge d\delta_Z{}^m \wedge \varphi = \pm \lim_{n,m\to\infty} (\int_X \Psi_n \wedge \delta_Z{}^m \wedge d\varphi).
$$

Hence $\delta T[\varphi]=0$ if $d\varphi|X=0$ in this case.

The general case follows from the definitions of $R^p(X)$ and δ (cf. $n^0 9$ and $n^0 10$).

Lemma 13. *If Y is an irreducible complex analytic subvariety of $\boldsymbol{C}^n$ with complex dimension r, then there exists an analytic Zariski open set Y' of Y such that*

$$R^p(Y') = 0, \quad p \geqq r+1.$$

Proof. By assumption, there exists a principal divisor with carrier D of C^n such that $D\cdot Y$ is defined and contains the multiple points of Y. Then $Y' = Y - D\cdot Y$ is a Stein manifold and we have the lemma.

Corollary. *If Y and Y' are same as above, then the following sequence is exact.*

$$0 \longrightarrow R^p(Y) \xrightarrow{i} H^p(Y', C) \xrightarrow{res.} R^{p-1}(Y - Y') \longrightarrow 0. \tag{38}$$

Theorem 9. (cf. [1]). *Let X be an n-dimensional Stein manifold, Y its complex subvariety with complex dimension r, then*

$$R^p(Y) = 0, \quad p \geqq r+1, \tag{39}$$

$$H^p(X - Y, C) = 0, \quad p \geqq 2n - r. \tag{39'}$$

Proof. Since $H^p(X, C) = 0$, $p \geqq n+1$ ([20]), (39) and (39)′ are equivalent by residue exact sequence.

To prove the theorem, we use the induction about r because the theorem is true for $r = 0$.

If Y is irreducible, then the theorem is true by (38) because X is imbedded in C^n ([15], [19]).

If Y is reducible, first we assume the number of irreducible components of Y is finite. Then we set $Y = Y^1 \cup \cdots \cup Y^k$ and use the induction about k. Therefore we assume the theorem is true for $Y_{k-1} = Y^1 \cup \cdots \cup Y^{k-1}$. Then since the sequence

$$\cdots \longrightarrow R^p(Y^k) \xrightarrow{i} R^p(Y^k - Y^k \cap Y_{k-1}) \xrightarrow{res.} R^{p-1}(Y^k \cap Y_{k-1}) \longrightarrow \cdots$$

is exact, $R^p(Y^k - Y^k \cap Y_{k-1}) = 0$, $p \geqq r+1$. Then we have $H^p(X - Y, C) = 0$, $p \geqq 2n - r$ because the sequence

$$\cdots \longrightarrow H^p(X - Y_{k-1}, C) \xrightarrow{i} H^p(X - Y, C) \xrightarrow{res.}$$
$$\longrightarrow R^{p-2n+2r+1}(Y^k - Y^k Y_{k-1}) \longrightarrow \cdots$$

is exact and we get the theorem for Y.

If Y contains infinitely many irreducible components, then we set

$$X = \cup X_m, \quad X_{m+1} \Supset X_m, \text{ each } X_m \text{ is a Stein manifold and relative compact in } X.$$

Since $X_m \cap Y$ contains only finite irreducible components for each m, we obtain

(40) $$R^p(X_m \cap Y) = 0, \quad p \geqq r+1.$$

Then we have the theorem because a compact subset of X always containd in some X_m.

16. Theorem 10. *Let X be an orientable n-dimensional real analytic manifold, Y its orientable $(n-r)$-dimensional real analytic subvariety and satisfies*

$$R^0(Y) = \boldsymbol{C}\langle Y^1\rangle \oplus \cdots \oplus \boldsymbol{C}\langle Y^k\rangle,$$
$$Y = Y^1 \cup \cdots \cup Y^k, \text{ each } Y^i \text{ is irreducible},$$

then a closed $(n-r-1)$-form φ on X with singularities on Y is written

(41) $$\varphi = \varphi_1 + \varphi_2 + \varphi_3,$$
φ_1 is closed and differentiable on X, φ_2 is exact,
$\varphi_3 | U = c_i \omega_{Y_i} + o(r(p, Y_i)^{1-r})$, $U \cap Y_i \neq \phi$, $U \cap Y_i$ is non-singular,
$\sum c_i \langle Y_i\rangle = 0$ in $H^{n-r}(X, \boldsymbol{C})$.

Proof. By assumption, we have the exact sequence

$$0 \longrightarrow H^{n-r-1}(X, \boldsymbol{C}) \xrightarrow{i} H^{n-r-1}(X-Y, \boldsymbol{C}) \xrightarrow{res.} R^0(Y) \longrightarrow$$
$$\xrightarrow{\delta} H^{n-r}(X, \boldsymbol{C}) \longrightarrow \cdots.$$

Hence we obtain the theorem by theorem 4.

Theorem 10′. *Let X be an orientable n-dimensional manifold, φ a closed $(n-1)$-form on X with singularities on discrete set of poonts $\{p_i\}$, then*

(41)′ $$\varphi = \varphi_1 + \varphi_2 + \varphi_3,$$
φ_1 is closed and differentiable on X, φ_2 is exact,
$\varphi_3 | U(p_i) = c_i \omega_{p_i} + o(r(x, p_i)^{1-n})$,
$\{c_i\}$ is arbitrary if X is open and $\sum c_i = 0$ if X is compact.

Moreover, setting $\varphi_3 = \varphi_3(\{p_i\}, \{c_i\})$, we can take φ_3 to depend differentiably on $\{p_i\}$ and $\{c_i\}$ if X is open and depend real analytically on $\{p_i\}$ and $\{c_i\}$ if X is compact and real analytic.

Proof. We need only to prove the second assertion. For this, it is sufficient to show $\varphi_3(p, c)$ depends differentiably on p, c if X is open and $\varphi_3(p_1, p_2, c, -c)$ depends real analyticaly on p_1, p_2, c if X is compact. To show this, we first note

(42) $$\langle T_{\Delta, \varphi n}\rangle = 0 \text{ in } H^{2n}(X \times X, \boldsymbol{C}) \text{ if } X \text{ is open},$$

$$\langle T_{\Delta\times X,\varphi}{}^n - T_{X\times\Delta,\varphi}{}^n\rangle = 0 \text{ in } H^{2n}(X\times X\times X,\ \boldsymbol{C}) \text{ if } X \text{ is compact,}$$

where Δ is the diagonal of $X\times X$. Then we get the theorem by the following commutative diagrams with exact lines.

$$\begin{tikzcd}
 & H^{2n-1}(X\times X,\ C) \arrow[r, "i"] & H^{2n-1}(X\times X-\Delta,\ C) \arrow[r, "res."] & H^n(\Delta,\ C) \arrow[r] & {} \\
0 \arrow[r] & H^{n-1}(X\times\{p\},\ C) \arrow[r, "i"] \arrow[u, "\delta"] & H^{n-1}(X\times\{p\}-\{p,\ p\},\ C) \arrow[r, "res."] \arrow[u, "\delta"] & H^0(\{p,\ p\},\ C) \arrow[r] \arrow[u, "\delta"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta"] & H^{2n}(X\times X,\ C) \\
{} \arrow[r, "\delta"] & H^n(X\times\{p\},\ C) = 0. \arrow[u, "\delta"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^{2n-1}(X\times X\times X,\ C) \arrow[r, "i"] & H^{2n-1}(X\times X\times X-\Delta\times X\cup X\times\Delta,\ C) \arrow[r] & {} \\
0 \arrow[r] & H^{n-1}(\{p_1\}\times X\times\{p_2\},\ C) \arrow[r, "i"] \arrow[u, "\delta"] & H^{n-1}(\{p_1\}\times X\times\{p_2\}-(\{p_1,\ p_1,p_2),\ (p_1,\ p_2,\ p_2)\}\ C), \arrow[r] \arrow[u, "\delta"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "res."] & R^n(\Delta\times X\cup X\times\Delta) \arrow[r, "\delta"] & H^{2n}(X\times X\times X,\ C) \\
{} \arrow[r, "res."] & H^0(\{(p_1,\ p_1,\ p_2),\ (p_1,\ p_2,\ p_2\}),\ C) \arrow[r, "\delta"] \arrow[u, "\delta"] & H^n(\{p_1\}\times X\times\{p_2\},\ C). \arrow[u, "\delta"]
\end{tikzcd}$$

§ 6. Residue formula and composed residue.

17. Lemma 14. *Let γ be a chain of X, an orientable manifold, Y an orientable subvariety of X such that Y and γ intersect properly. Then there exists a series of forms $\{\delta_\gamma{}^n\}$ such that*

$$\int_{\gamma\cdot Y}\varphi = \lim_{n\to\infty}\int_Y \varphi\wedge\delta_\gamma{}^n. \tag{43}$$

Proof. We assume γ is defined by $x_1=\cdots=x_p=0$ on U, a coördinate neighborhood of X (x_i need not be smooth). Then denoting the Heaviside function for x_i by l_{xi}, we get

$$\int_{\gamma\cap U}\varphi = \lim.\int_U \varphi\wedge dl_{x1}{}^n\wedge\cdots\wedge dl_{xp}{}^n,\quad \lim.\, l_{xi}{}^n = l^{x}{}_{i}.$$

Since $\mathfrak{D}^p(\gamma)$, the sheaf of germs of p-forms on X which vanishes on γ, is fine, $H^p(X,\ \mathfrak{D}^p(\gamma))=0$ and we can define $\delta_\gamma{}^n$ by

$$\delta_\gamma{}^n\,|\,U = (dl_{x1}{}^n\wedge\cdots\wedge dl_{xp}{}^n)U + \psi_U{}^n,$$

$$\psi_U{}^n - \psi_V{}^n = (dl_{x1}{}^n\wedge\cdots\wedge dl_{xp}{}^n)_V - (dl_{x1}{}^n\wedge\cdots\wedge dl_{xp}{}^n)_U,$$

$$\lim_{n\to\infty}\int_U\varphi\wedge\psi_U{}^n = 0.$$

By the definition of $\delta_r{}^n$, we get

$$(44)\qquad \left|\int_{U(Y)}\varphi\wedge\delta_r{}^n - \int_{V(Y)}\varphi\wedge\delta_r{}^m\right| \leqq \|\varphi\|_{r,U(Y)}\varepsilon(n, m) + \|\varphi\|_{r,V(Y)}\varepsilon(n, m)$$

if $\varphi \in \Gamma_0(X-Y, \mathfrak{G}^{n-p})$, where $\|\varphi\|_{r,U(Y)}$ is given by

$$\|\varphi\|_{r,U(Y)} = \max_{x\in U(Y)} \|\rho(x, Y)^{r-1}\varphi(x)\|,$$

and $\lim._{n,m\to\infty}\varepsilon(n, m)=0$, $\lim._{U(Y),V(Y)\to Y}\varepsilon(U(Y), V(Y))=0$.

Theorem 11. *Let γ be a compact chain on X such that γ and $Y_i - Y_{i+1}$ intersect properly for all i, $res._i'(\varphi)$ is a closed form on $Y_i - Y_{i+1}$ such that cohomologous to $\widehat{res}._i(\varphi)$, then*

$$(45)\qquad \int_{\partial\gamma}\varphi = \sum_i \int_{\gamma(Y_i - Y_{i+1})} res._i'(\varphi).$$

Proof. We may assume $\varphi \in \Gamma_0(X-Y, \mathfrak{G}^{n-p})$. Then

$$\int_{\partial\gamma}\varphi = \lim_{U(Y)\to Y}\int_{\gamma\partial}\varphi|(X-U(Y)) = \lim_{U(Y)\to Y}\int_{\gamma\cdot\partial U(Y)}\varphi$$

$$= \lim_{U(Y)\to Y}\left(\lim_{n\to\infty}\int_{\gamma\cdot\partial U(Y)}\varphi\wedge\delta_r{}^n\right).$$

By (44), this last formula is equal to $\lim._{n\to\infty}(\lim._{U(Y)\to Y}\int_{\gamma\cdot\partial U(Y)}\varphi\wedge\delta_r{}^n)$. Then by (43) and the definition of $\widehat{res}._i(\varphi)$, we obtain (45).

Definition. (45) *is called residue formula.*

18. Lemma 15. *Let γ be a cycle on $X - Y_{i+1}$ and in general position with $Y_i - Y_{i+1}$, φa closed form on $X - Y_{i+1}$ and cohomologous to $T_{Y_i - Y_{i+1}}, \varphi$, then*

$$(46)\qquad \int_\gamma \phi = \int_{\gamma\cdot(Y_i - Y_{i+1})}\varphi.$$

Proof. By assumption, we have

$$\int\phi\wedge\delta_r{}^n = \int_{Y_i - Y_{i+1}}\varphi\wedge\delta_r{}^n.$$

Hence we get the lemma.

Theorem 11′. *Let φ be a closed p-form on $X-Y$ and γ an $(p+1)$-chain on X such that γ and each $Y_i - Y_{i+1}$ is in general position and*

$$\gamma\cdot(Y_i - Y_{i+1}) = \gamma_i\cdot(Y_i - Y_{i+1}),\ \gamma_i \text{ is a cycle of } X - Y_{i+1},$$

then

$$\int_{\partial\gamma}\varphi = 0.$$

Proof. If γ is a chain of $X - Y_2$, then the theorem follows from residue exact sequence and residue formula. Hence we assume the theorem is true for the chains of $X - Y_j$ and assume γ is a chain of $X - Y_{j+1}$.

By assumption, we set

$$\gamma = \gamma' + \gamma'',\ \gamma' \text{ is a chain of } X - Y_j,$$
$$\gamma'' \cap (Y_k - Y_{k+1}) = \varphi,\ k < j.$$

Then we get

$$\int_{\partial\gamma}\varphi = \int_{\partial\gamma''}\varphi = \int_{\gamma\cdot(Y_j - Y_{j+1})} res._j'(\varphi).$$

On the other hand, since the diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \\
H^p(X-Y, \boldsymbol{C}) \arrow[d, "res."] \arrow[dr, "res._j^*"] & \\
R^{p-r_1+1}(Y - Y_{i+1}) \arrow[r, "\delta_2"] \arrow[d, "\delta_1"] & H^{p-r_j+1}(Y_j - Y_{j+1}, \boldsymbol{C}) \\
H^{p+1}(X - Y_{j+1}, \boldsymbol{C}) \arrow[ur, "\delta_3"] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

is commutative, $\delta_3(\langle res._j'\varphi\rangle) = 0$. Therefore

$$\int_\alpha res._j'(\varphi) = 0,\ \ \alpha = \beta.(Y_j - Y_{j+1}),\ \ \partial\beta = 0$$

by Lemma 15. This proves the theorem.

19. In this n^0, we assume X is a smooth orientable manifold. (cf. [2], § 1).

Let $Y_1, Y_2, \cdots, Y_s$ be orientable submanifolds with codimensions $r_1, r_2, \cdots, r_s$ and assume they satisfy following condition.

Y_1 and Y_2, $(Y_2 \cup \cdots \cup Y_s)$ and $(Y_3 \cup \cdots \cup Y_s)$, $Y_1 \cap Y_2$ and Y_3, $(Y_3 \cup \cdots \cup Y_s)$ and $(Y_4 \cup \cdots \cup Y_s)$, $\cdots$, $Y_1 \cap \cdots \cap Y_{s-1}$ and Y_s are in general positions.

Then we obtain residue exact sequences for the pairs

$$
\begin{gathered}
(X-(Y_2\cup\cdots\cup Y_s),\ X-(Y_1\cup Y_2\cup\cdots\cup Y_s),\ Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s)),\\
(Y_1-Y_1\cap(Y_3\cup\cdots\cup Y_s),\ Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s),\ Y_1\cap Y_2-Y_1\cap Y_2\cap(Y_3\cup\cdots\cup Y_s)),\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(Y_1\cap\cdots\cap Y_{s-1},\ Y_1\cap\cdots\cap Y_{s-1}-Y_1\cap\cdots\cap Y_s,\ Y_1\cap\cdots\cap Y_s).
\end{gathered}
$$

We denote the residue maps in these pairs by $res._{Y_1}$, $res._{Y_1\cap Y_2}, \cdots$, $res._{Y_1\cap\cdots\cap Y_s}$ and set

$$res._{Y_1,\cdots,Y_s} = res._{Y_1\cap\cdots\cap Y_s}\cdots res._{Y_1\cap Y_2}\cdots res._{Y_1}. \tag{47}$$

Definition. ([12], cf. [2]). *$res._{Y_1,\cdots,Y_s}$ is called the composed residue.*

By definition, $res._{Y_1,\cdots,Y_s}$ is a homomorphism from $H^p(X-(Y_1\cup\cdots\cup Y_s), C)$ into $H^{p-(r_1+\cdots+r_s)+s}(Y_1\cap\cdots\cap Y_s, C)$ and the following diagram is commutative.

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^p(X-(Y_2\cup\cdots\cup Y_s), C) \arrow[r, "i"] & {} & & \\
{} \arrow[r] & H^p(X-(Y_1\cup\cdots\cup Y_s), C) \arrow[rrr, "res._{Y_1}"] \arrow[ddd, "res._{Y_1,\cdots,Y_s}"'] & & & H^{p-r_1+1}(Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s), C) \arrow[r] \arrow[ddlll, "="'] & \cdots \\
\cdots \arrow[r] & H^{p-r_1+1}(Y_1-Y_1\cap(Y_3\cup\cdots\cup Y_s), C) \arrow[r, "i"] & {} & & & \\
{} \arrow[r] & H^{p-r_1+1}(Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s), C) \arrow[rrr, "res._{Y_1\cap Y_2}"] & & & H^{p-(r_1+r_2)+2}(Y_1\cap Y_2-Y_1\cap Y_2\cap(Y_3\cup\cdots\cup Y_s), C) \arrow[r] \arrow[dlll, "="'] & \cdots \\
 & \cdots\cdots\cdots\cdots\cdots\cdots \arrow[ddrrr, dotted] \arrow[drr, dotted, "="] & & & & \\
\cdots \arrow[r] & H^{p-(r_1+\cdots+r_{s-1})+s-1}(Y_1\cap\cdots\cap Y_{s-1}, C) \arrow[r, "i"] & {} \arrow[d] & & & \\
{} \arrow[r] & H^{p-(r_1+\cdots+r_{s-1})+s-1}(Y_1\cap\cdots\cap Y_{s-1}-Y_1\cap\cdots\cap Y_s, C) \arrow[rrr, "res._{Y_1\cap\cdots\cap Y_s}"] & {} & & H^{p-(r_1+\cdots+r_s)+s}(Y_1\cap\cdots\cap Y_s, C) \arrow[r] & \cdots
\end{tikzcd}
$$

By the residue formula (7)$'$, we obtain

Theorem 12. *Let γ be a chain of X such that there exists a series of chains γ_1 of Y_1, γ_2 of $Y_1\cap Y_2, \cdots, \gamma_s$ of $Y_1\cap\cdots\cap Y_s$ such that*

(i) γ and Y_1, γ_1 and $Y_1\cap Y_2, \cdots, \gamma_{s-1}$ and $Y_1\cap\cdots\cap Y_s$ are in general positions.

(ii) $\gamma\cdot Y_1=\partial\gamma_1,\ \gamma_1\cdot(Y_1\cap Y_2)=\partial\gamma_2, \cdots,\ \gamma_{s-2}\cdot(Y_1\cap\cdots\cap Y_{s-1})=\partial\gamma_{s-1},\ \gamma_{s-1}\cdot(Y_1\cap\cdots\cap Y_s)=\gamma_s.$

Then we have

$$\int_{\partial\gamma}\varphi = \int_{\gamma_s} res._{Y_1,\cdots,Y_s}\varphi. \tag{48}$$

Note. If Y_i is given by $f_{i,1} = \cdots = f_{i,r_i} = 0$, $i = 1, \cdots, s$ then for sufficiently small $\varepsilon_1, \cdots, \varepsilon_s$, the series of chains $\gamma, \gamma_1, \cdots, \gamma_s$ given by

$$\gamma = \{x \mid |f_{i,j}(x)| \leqq \varepsilon_i\} \cap \bigcap_{i=1}^{s} \bigcup_{j=1}^{r_i} \{x \mid |f_{i,j}(x)| = \varepsilon_i\},$$

$$\gamma_1 = \{x \mid |f_{i,j}(x)| \leqq \varepsilon_i\} \cap \bigcup_{i=2}^{s} \bigcup_{j=1}^{r_i} \{x \mid |f_{i,j}(x)| = \varepsilon_i\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\gamma_s = \{x \mid f_{i,j}(x) = 0,\ i = 1, \cdots, s,\ j = 1, \cdots, r_i\}$$
$$= Y_1 \cap \cdots \cap Y_s$$

satisfies the assumption of the theorem. The simplest example is $X = C^n$, each Y_i is given by $z_i = 0$, $i = 1, \cdots, n$ and $\gamma = \{(z_1, \cdots, z_n) \mid |z_i| = a_i\}$.

20. We can define the composed residue for singular residues. We denote the j-th multiple subvarieties of Y_i or $Y_1 \cap \cdots \cap Y_s$ by $Y_{i,j+1}$ or $Y_1 \cap \cdots \cap Y_{s,j+1}$. Then we define the $(j_1, \cdots, j_s)$-composed residue by

$$res.^{j_1, \cdots, j_s}_{Y_1, \cdots, Y_s} = res._{Y_1 \cap \cdots \cap Y_s, j_s} \cdots res._{Y_1 \cap Y_2, j_2}\, res._{Y_1, j_1}, \tag{47)$'$}$$

where $res._{Y_1 \cap \cdots \cap Y_i, j_i}$ means the j_1-residue of the pair

$$(Y_1 \cap \cdots \cap Y_{i-1} - Y_1 \cap \cdots \cap Y_{i-1} \cap (Y_{i+1} \cup \cdots \cup Y_s),$$
$$Y_1 \cap \cdots \cap Y_{i-1} - Y_1 \cap \cdots \cap Y_{i-1} \cap (Y_i \cup \cdots \cup Y_s),$$
$$Y_1 \cap \cdots \cap Y_i - Y_1 \cap \cdots Y_i \cap (Y_{i+1} \cup \cdots \cup Y_s)).$$

By Theorem 11, if a series of chains γ of X, γ_1 of $Y_1, \cdots, \gamma_s$ of $Y_1 \cap \cdots \cap Y_s$ satisfies

(i)′ γ and $Y_{1,j} - Y_{1,j+1}$, γ_1 and $Y_1 \cap Y_{2,j+1}, \cdots, \gamma_{s-1}$ and $Y_1 \cap \cdots \cap Y_{s,j} - Y_1 \cap \cdots \cap Y_{s,j+1}$ are in general positions.

(ii)′
$$(Y_{1,j_1} - Y_{1,j_1+1}) = \partial\gamma_{1,j_1},$$
$$\gamma_{1,j_1} \cdot (Y_1 \cap Y_{2,j_2} - Y_1 \cap Y_{2,j_2+1}) = \partial\gamma_{2,j_1,j_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\gamma_{s-2,j_1,\cdots,j_{s-2}}(Y_1 \cap \cdots \cap Y_{s-1,j_{s-1}} - Y_1 \cap \cdots \cap Y_{s-1,j_{s-1}+1})$$
$$= \partial\gamma_{s-1,j_1,\cdots,j_{s-1}},$$
$$\gamma_{s-1,j_1,\cdots,j_{s-1}} \cdot (Y_1 \cap \cdots \cap Y_{s,j_s} - Y_1 \cap \cdots \cap Y_{s,j_s+1})$$
$$= \gamma_{s,j_1,\cdots,j_s},$$

then we have

(48)′ $$\int_{\partial\gamma}\varphi = \sum_{j_1,\cdots,j_s}\int_{\gamma_{s,j_1,\cdots,j_s}} res.{}^{j_1,\cdots,j_s}_{Y_1,\cdots,Y_s}\varphi.$$

§ 7. Integral expressions.

21. Lemma 16. *Let $a_{ij}(X)$, $i, j=1, \cdots, n$ be a system of differentiable functions such that*

(49) $$\frac{\partial a_{ij}(x)}{\partial x_i}=\frac{\partial a_{ij}(x)}{\partial x_j}=0,\ \ a_{ii}(x)=1,\ \ a_{ij}(x)+a_{ji}(x)=0,\ \ i\neq j,$$

then the differential form $\sum_i\sum_{j\neq i} a_{ij}(x)(x_j-\xi_j)\check{dx}_i/r(x, \xi)^n$ is an exact form. Here $\check{dx}_i$ and $r(x, \xi)$ are

$$\check{dx}_i=(-1)^{i-1}dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_n,$$

$$r(x, \xi)=\sqrt{\sum_i(x_i-\xi_i)^2}.$$

Proof. We set

$$\widetilde{dx_i\wedge dx_j}=(-1)^{i+j}dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_n.$$

Then we get

$$d\left(\frac{\sum_{i<j}a_{ij}(x)\widetilde{dx_i\wedge dx_j}}{r(x, \xi)^{n-2}}\right)$$

$$=\frac{\sum_{i<i}\left(\frac{\partial a_{ij}(x)}{\partial x_i}\check{dx}_j-\frac{\partial a_{ij}(x)}{\partial x_j}\check{dx}_i\right)}{r(x, \xi)^{n-2}}-$$

$$-\frac{\frac{n-2}{r(x, \xi)}\sum_{i<j}(a_{ij}(x)\,(x_i-\xi_i)\,\check{dx}_j-a_{ij}(x)\,\,(x_j-\xi_j)\,\check{dx}_i)}{r(x, \xi)^{n-1}}$$

$$=\frac{(2-n)\sum_{k<i}(a_{ki}(x)\,(x_k-\xi_k)\,-\sum_{i<m}a_{im}(x)\,(x_m-\xi_m))\,\check{dx}_i}{r(x, \xi)^n}$$

$$=\frac{(2-n)\sum_{i\neq j}a_{ij}(x)\,(x_j-\xi_j)\check{dx}_i}{r(x, \xi)^n}.$$

Hence we obtain the lemma if $n\geqq 3$. If $n=2$, $a_{ij}(x)$ must be constants for all (i, j). Therefore we need only to show the exactness of $\{(x-\xi)dx+(y-\eta)dy\}/(x-\xi)^2+(y-\eta)^2\}$, which is equal to $1/2(d\log(1/\{(x-\xi)^2+(y-\eta)^2\}))$. Hence we have the lemma.

22. Theorem 13. *Let $f(x)$ be a differentiable function on a domain D of $\boldsymbol{R}^n$ such that $D-\{x\}$ has same homotopy type of S^{n-1} for any $x \in D$, and satisfies the equation*

$$\sum_{i=1}^{n} a_{ij}(x)\frac{\partial f(x)}{\partial x_i} = 0, \quad j=1, \cdots, n. \tag{50}$$

Then we have on D

$$f(\xi) = \frac{1}{\sigma^{n-1}}\int_{\partial D} f(x)\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)d\check{x}_i}{r(x, \xi)^n}. \tag{51}$$

Proof. Denoting $dV = dx_1\wedge\cdots\wedge dx_n$, we get

$$\begin{aligned}
&d_x\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)f(x)d\check{x}_i}{r(x, \xi)^n}\right)\\
&=\frac{\sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)\frac{\partial f(x)}{\partial x_i}+f(x))}{r(x, \xi)^n}-\\
&\quad-\frac{\frac{n}{r(x, \xi)}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)(x_i-\xi_i)f(x)}{r(x, \xi)^{n+1}}dV\\
&=r(x, \xi)^{-(n+2)}\{nr(x, \xi)^2 f(x)+(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)\frac{\partial f(x)}{\partial x_i})r(x, \xi)^2-\\
&\quad-n\sum_{i=1}^{n}(x_i-\xi_i)^2 f(x)+\sum_{i\neq j} a_{ji}(x)(x_i-\xi_i)(x_j-\xi_j)f(x)\}dV\\
&=r(x, \xi)^{-(n+2)}\{r(x, \xi)^2\sum_{j=1}^{n}(\sum_{i=1}^{n} a_{ij}(x)\frac{\partial f(x)}{\partial x_i}((x_j-\xi_j)+\\
&\quad+\sum_{i<j}(a_{ij}(x)+a_{ji}(x))(x_i-\xi_i)(x_j-\xi_j)f(x)\}dV\\
&=0.
\end{aligned}$$

Hence $(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)(x_j-\xi_j)f(x)d\check{x}_i)/r(x, \xi)^n$ is a closed form for all ξ.

We set

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)f(x)d\check{x}_i}{r(x, \xi)^n}$$

$$= f(\xi)\frac{\sum_{i=1}^{n}(x_i-\xi_i)d\check{x}_i}{r(x,\ \xi)^n} + f(\xi)\frac{\sum_{i\neq j} a_{ji}(x)(x_j-\xi_j)d\check{x}_i}{r(x,\ \xi)^n} +$$

$$+ (f(x)-f(\xi))\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)d\check{x}_i}{r(x,\ \xi)^n}.$$

Then we get

$$\int_{\partial D}(f(x)-f(\xi))\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}(x)(x_j-\xi_j)d\check{x}_i}{r(x,\ \xi)^n} = 0,$$

$$\int_{\partial D} f(\xi)\frac{\sum_{i\neq j} a_{ji}(x)(x_j-\xi_j)d\check{x}_i}{r(x,\ \xi)^n} = 0$$

because $|f(x)-f(\xi)| = O(r(x,\ \xi))$ *and* $f(\xi)(\sum_{i\neq j}a_{ji}(x)(x_j-\xi)/r(x,\ \xi)^n)$ is exact. Therefore we obtain the theorem.

Corollary 1. *If a function f satisfies (50), then f is real analytic.*

Note. In this corollary, we need not the real analyticity of $a_{ij}(x)$.

Corollary 2. If f is holomorphic on $\bar{D} \subset C^m$, then

(52)
$$f(\zeta,\ \cdots,\ \zeta_m)$$
$$= \frac{1}{\sigma^{2m-1}}\int_{\partial D} f(z_1,\ \cdots,\ z_m)\left(\frac{\sum_{k=1}^{m}\{(x_{2k-1}-\xi_{2k-1})-\sqrt{-1}(x_{2k}-\xi_{2k})\}d\check{x}_{2k-1}}{\{(x_1-\xi_1)^2+\cdots+(x_{2m}-\xi_{2m})^2\}^m}\right.$$
$$\left. - \frac{\sum_{k=1}^{m}\{\sqrt{-1}(x_{2k-1}-\xi_{2k-1})+(x_{2k}-\xi_{2k})\}d\check{x}_{2k}}{\{(x_1-\xi_1)^2+\cdots+(x_{2m}-\xi_{2m})^2\}^m}\right),$$

$$\zeta_k = \xi_{2k-1}+\sqrt{-1}\,\xi_{2k},\quad z_k = x_{2k-1}+\sqrt{-1}\,x_{2k},\quad k=1,\ \cdots,\ \mathrm{m}.$$

Proof. We take $(a_{ij}(x))$ to be

$$(a_{ij}(x)) = \begin{pmatrix} 1, & -\sqrt{-1}, & & & \\ \sqrt{-1}, & 1, & & & \\ & & \ddots & 0 & \\ & 0 & & \ddots & \\ & & & 1, & -\sqrt{-1}, \\ & & & \sqrt{-1}, & 1, \end{pmatrix}$$

then $(a_{ij}(x))$ satisfies (49) and the equation (50) reduces to

$$\frac{\partial f(x)}{\partial x_{2k-1}} + \sqrt{-1}\,\frac{\partial f(x)}{\partial x_{2k}} = \frac{\partial f(x)}{\partial \bar{z}_k} = 0, \quad k = 1, \cdots, m.$$

Hence f is holomorphic and we have (52) by (51).

Note. If we use the complex coordinate, (52) is the Bochner-Martinelli, formula ([4])

$$f(\zeta) = \left(\frac{\sqrt{-1}}{2}\right)^{n-1} \frac{1}{\sigma^{2m-1}} \int_{\partial D} f(z) \frac{\sum_{k=1}^{m} \overline{(z_k - \zeta_k)} d\check{\bar{z}}_k}{\left(\sum_{k=1}^{m} (z_k - \zeta_k)\overline{(z_k - \zeta_k)}\right)^m}, \tag{52$'$}$$

$$\check{d}_k = dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1} \wedge dz_k \wedge dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

We also note that

$$\frac{\sum_{i=1}^{2m} (x_i - \xi_i) \check{d}x_i}{\left(\sqrt{\sum_{i=1}^{2m} (x_i - \xi_i)^2}\right)^{2m}}$$

$$= -\left(\frac{\sqrt{-1}}{2}\right)^m \left(\frac{\sum_{k=1}^{m} \overline{(z_k - \zeta_k)} d\check{\bar{z}}_k}{\left(\sum_{k=1}^{m} (z_k - \zeta_k)\overline{(z_k - \zeta_k)}\right)^m} + \frac{\sum_{k=1}^{m} (z_k - \zeta_k) d\check{z}_k}{\left(\sum_{k=1}^{m} (z_k - \zeta_k)\overline{(z_k - \zeta_k)}\right)^m} \right),$$

$$d\check{z}_k = -d\bar{z}_1 \wedge dz_1 \wedge \cdots \wedge d\bar{z}_{k-1} \wedge d\bar{z}_{k-1} \wedge dz_k \wedge dz_{k+1} \wedge d\bar{z}_{k+1} \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

The first term of this right hand side is type $(n, n-1)$ and the second term is type $(n-1, n)$. They are both closed and non-exact.

23. We take integers $r_1, \cdots, r_s$ such that

$$r_k \geqq 2, \quad k = 1, \cdots, s, \quad r_1 + \cdots + r_k = n_k, \quad n_s = n.$$

We set Y_{ξ, r_k} the linear space in $\boldsymbol{R}^n$ defined by $x_{r_1+\cdots+r_{k-1}} = \xi_{r_1+\cdots+r_{k-1}+1}, \cdots, x_{r_1+\cdots+r_k} = \xi_{r_1+\cdots+r_k}$. Then

$$Y_{\xi, r_1} \cap \cdots \cap Y_{\xi, r_s} = \xi = (\xi_1, \cdots, \xi_n).$$

We take a domain D of $\boldsymbol{R}^n$ such that for any $\xi \in D$, we can take a path $\partial_{r_1, \cdots, r_s, \xi} D$ which is containd in ∂D and satisfies (i), (ii) of $n^0 19$ for $(Y_{\xi, r_1}, \cdots, Y_{\xi, r_s})$.

Example. If $D = D_{r_1} \times \cdots \times D_{r_s}$, where D_{r_k} is a domain in $(x_{r_1+\cdots+r_{k-1}+1}, \cdots, x_{r_1+\cdots+r_k})$-space, $k = 1, \cdots, s$, then we can take $\partial_{r_1, \cdots, r_s, \xi} D = \partial D_{r_1} \times \cdots \times \partial D_{r_s}$ for all $\xi \in D$.

We also assume that the matrix $(a_{ij}(x))$ is given by

$$(a_{i,j}(x)) = \begin{pmatrix} a_{1,1}(x), \cdots, a_{r_1,1}(x), & & & \\ \cdots\cdots\cdots & & & \\ a_{1,r_1}(x), \cdots, a_{r_1,r_1}(x), & \ddots & 0 & \\ 0 & & a_{n_{s-1}+1,n_{s-1}+1}(x), \cdots, a_{n,n_{s-1}+1}(x), & \\ & & \cdots\cdots\cdots & \\ & & a_{n_{s-1}+1,n}(x), \cdots\cdots, a_{n,n}(x), & \end{pmatrix}$$

We also set

$$\rho_k(x, \xi) = \sqrt{\sum_{i=n_{k-1}+1}^{i=n_k} (x_i - \xi_i)^2},$$

$$(\check{dx_i})_k = (-1)^{n_{k-1}+i+1} dx_{n_{k-1}+1} \wedge \cdots \wedge dx_{n_k+i-1} \wedge dx_{n_k+i+1} \wedge \cdots \wedge dx_{n_k}.$$

Then we obtain by theorem 12 and Theorem 13,

Theorem 14. *Under the above assumptions and notations, if $f(x)$ satisfies (50) on $\bar{D}$, then*

(53) $$f(\xi_1, \cdots, \xi_n)$$

$$= \frac{1}{\sigma^{r_1-1} \cdots \sigma^{r_s-1}} \int_{\partial_{r_1} \cdots r_s, \xi D} f(x_1, \cdots, x_n).$$

$$\frac{\sum_{i=1}^{r_1} \sum_{j=1}^{r_1} a_{ji}(x)(x_j - \xi_j)(\check{dx_i})_1}{\rho_1(x, \xi)^{r_1}} \wedge \cdots \wedge \frac{\sum_{i=n_{s-1}+1}^{n} \sum_{j=n_{s-1}+1}^{n} a_{ji}(x)(x_j - \xi_j)(\check{dx_i})_s}{\rho_s(x, \xi)^{r_s}}.$$

Note. In this theorem, the linearity of Y_{ξ,r_i} is not essential. For example, we set

$$D = \{(x) \| g_i(x)| < 1, \ i = 1, \cdots, m\},$$

$$\partial_{j_1, \cdots, j_s} D = \{(x) \| g_{j_1}(x)| = \cdots = |g_{j_s}(x)| = 1,$$

$$Y_{\xi, r_k} = \{(x) | g_{n_{k-1}+1}(x) = g_{n_{k-1}+1}(\xi), \cdots, g_{n_k}(x) g_{n_k}(\xi)\},$$

and assume $\partial_{j_1, \cdots, j_s} D$ satisfies (i), (ii) of $n^0 19$ for Y_{ξ, r_k}. Then we get similar integral expression as (53) replacing x_i by g_i and $\partial_{r_1, \cdots, r_s, \xi} D$ by $\partial_{j_1, \cdots, j_s} D$ with suitable assumptions about $(a_{ij}(x))$. If D is an analytic polyheder, $m = 2n$, $s = n$ and f is holomorphic on $\bar{D}$, then the formula is

$$(53)' \qquad f(\zeta)=\frac{1}{(2\pi\sqrt{-1})^n}\sum_{j_1,\cdots,j_n}\int_{\partial j_1,\cdots,j_nD} f(z).$$

$$\cdot\frac{dg_{j1}\wedge\cdots\wedge dg_{jn}}{\sum_{k=1}^{n}(g_{jk}(z)-g_{jk}(\zeta))}.$$

Since we know

$$(54) \qquad \frac{dg_1\wedge\cdots\wedge dg_n}{(g_1(z)-g_1(\zeta))\cdots(g_n(z)-g_n(\zeta))}$$

$$=\frac{det.(g_{ij}(z,\ \zeta))dz_1\wedge\cdots\wedge dz_n+O(z-\zeta)}{(g_1(z)-g_1(\zeta))\cdots(g_n(z)-g_n(\zeta))},$$

$$g_i(z)-g_i(\zeta)=\sum_j(z_j-\zeta_j)g_{ij}(z,\ \zeta),$$

(53)′ is the Weil's formula ([22]). We note that (53)′ is true on any complex manifold.

Chapter II. Residue exact sequence of Banach manifolds.

§ 1. Currents on C^∞-smooth Banach spaces.

24. We denote by $\boldsymbol{E}$ a separable C^∞-smooth Banach space, that is, on which the partition of unity by C^∞-class functions is always possible (cf. [7]). We set
$C^{\infty,p}(\boldsymbol{E})$: *the vector space consisted by C^∞-class p-forms on* $\boldsymbol{E}$ (cf. [11]).
$C_0^{\infty,p}(\boldsymbol{E})$: *the vector space consisted by C^∞-class p-forms with bounded carrier on* $\boldsymbol{E}$.

We define the topologies of $C_0^{\infty,p}(\boldsymbol{E})$ by the semi-norms

$$\rho_{K,m}(\varphi)=\sup_{s\leq m,\,x\in K}\|D^s(\varphi x)\|,\ K \text{ is a compact set, } m<\infty.$$

Then $C^{\infty,p}(\boldsymbol{E})$ and $C_0^{\infty,p}(\boldsymbol{E})$ become locally convex topological vector spaces. These spaces are denoted by $\mathscr{E}^p(\boldsymbol{E})$ and $\mathscr{D}^p(\boldsymbol{E})$.

By definitions, if $\boldsymbol{E}^r$ is an r-dimensional subspace of $\boldsymbol{E}$, then

$$(1)' \qquad \pi_r(\mathscr{E}^p(\boldsymbol{E}))=\mathscr{E}^p(\boldsymbol{E}^r),\ \ \pi_r(\varphi)=\varphi\,|\,\boldsymbol{E}^r,$$

$$(1) \qquad \pi_r(\mathscr{D}^p(\boldsymbol{E}))=\mathscr{D}^p(\boldsymbol{E}^r),$$

where $\mathscr{E}^p(\boldsymbol{E}^r)$ and $\mathscr{D}^p(\boldsymbol{E}^r)$ are the usual Schwartz spaces of p-forms on $\boldsymbol{E}^r$.

Similarly, we can define the onto homomorphisms $\pi_r{}^s$: $\mathscr{E}^p(\boldsymbol{E}^r)\to\mathscr{E}^p(\boldsymbol{E}^s)$,

$\pi_r^s : \mathscr{D}^p(E^r) \to \mathscr{D}(E^s)$ and we get

$$\pi_s^r \pi_t^s = \pi_t^r, \quad \pi_r \pi_s^r = \pi_s.$$

Hence we can define the projective limits $\varprojlim [\mathscr{E}^p(E^r) : \pi_s^r]$ and $\varprojlim [\mathscr{D}^p(E^r) : \pi_s^r]$.

Lemma 17. *If* $\cup_r E^r$ *is dense in* E, *then*

$$\mathscr{E}^p(E) = \varprojlim [\mathscr{E}^p(E^r) : \pi_s^r], \tag{2$'$}$$

$$\mathscr{D}^p(E) = \varprojlim [\mathscr{D}^p(E^r) : \pi_s^r]. \tag{2}$$

Proof. We define the into homomorphisms $\rho : \mathscr{E}^p(E) \to \varprojlim [\mathscr{E}^p(E^r) : \pi_s^r]$ and $\rho : \mathscr{D}^p(E) \to \varprojlim [\mathscr{D}^p(E) : \pi_s^r]$ by

$$\rho(\varphi) = (\pi_1(\varphi), \ \pi_2(\varphi), \ \cdots, \ \pi_r(\varphi), \ \cdots\cdots).$$

Then by assumption, $\rho(\varphi) = 0$ if and only if $\varphi = 0$. Hence ρ is an into isomorphism. Therefore we obtain the lemma by (1)$'$ and (1).

Corollary. *If* $\varphi_n \in \mathscr{D}^p(E)$ *and* $\lim.\varphi_n = \varphi$ *in* $\mathscr{E}^p(E)$, *then* φ *belongs in* $\mathscr{D}^p(E)$.

Definition. *The subspaces of* $\mathscr{E}^p(E)$ *and* $\mathscr{D}^p(E)$ *(or* $\mathscr{E}^p(E^r)$ *and* $\mathscr{D}^p(E^r)$*) consisted by closed forms are denoted by* $\mathscr{B}^p(E)$ *and* $\mathscr{A}^p(E)$ *(or* $\mathscr{B}^p(E^r)$ *and* $\mathscr{A}^p(E)$*).*

Since we know

$$d\pi_r = \pi_r d, \quad d\pi_s^r = \pi_s^r d,$$

we obtain

$$\mathscr{B}^p(E) = \varprojlim [\mathscr{B}^p(E^r) : \pi_s^r], \tag{3$'$}$$

$$\mathscr{A}^p(E) = \varprojlim [\mathscr{A}^p(E^r) : \pi_s^r]. \tag{3}$$

Lemma 18. *If* $p \neq 0$, *then*

$$\mathscr{E}^p(E) = d\,\mathscr{B}^{p-1}(E), \tag{4$'$}$$

$$\mathscr{A}^p(E) = d\,\mathscr{D}^{p-1}(E). \tag{4}$$

Proof. We need not prove the first equality (cf. [11]). To get the second equality, we note

$$d\mathscr{D}^{p-1}(\boldsymbol{E}) = \underset{\longleftarrow}{lim.}\,[d\mathscr{D}^{p-1}(\boldsymbol{E}^r) : \pi_r^s].$$

Then since we know

$$d\mathscr{D}^{p-1}(\boldsymbol{E}^r) = \mathscr{A}^p(\boldsymbol{E}^r),\ \ 0 < p < r,$$

we obtain the lemma by (3).

25. We denote the dual spaces of $\mathscr{E}^p(\boldsymbol{E})$, etc. by $\mathscr{E}^p(\boldsymbol{E})'$, etc..

Definition. *An element of $\mathscr{D}^p(\boldsymbol{E})'$ is called an $(\infty - p)$-current of $\boldsymbol{E}$.*

Definition. *For an element T of $\mathscr{D}^p(\boldsymbol{E})'$ (or $\mathscr{E}^p(\boldsymbol{E})'$), we define its exterior differential dT by*

$$dT[\varphi] = T[d\varphi]. \tag{5}$$

Definition. *If a current T satisfies $dT = 0$, then T is called a closed current. A current of the form dS is called an exact current.*

Lemma 19. *On $\boldsymbol{E}$, a closed $(\infty - p)$-current T is always exact.*

Proof. Since T is closed, we can define an element S' of $(d\mathscr{D}^p(\boldsymbol{E}))'$ by

$$S'[d\varphi] = T[\varphi],$$

because by (4), if $d\varphi_1 = d\varphi_2$, then $\varphi_2 = \varphi_1 + d\psi$, $\psi \in \mathscr{D}^{p-1}(\boldsymbol{E})$.

Then since $\mathscr{D}^{p+1}(\boldsymbol{E})$ is locally convex, there exists a current S of $\boldsymbol{E}$ such that

$$S\,|\,d\,\mathscr{D}^p(\boldsymbol{E}) = S',$$

by Hahn-Banach' theorem ([10]). Hence we have the lemma for $p>0$. For $p=0$, we note that

$$\mathscr{A}^0(\boldsymbol{E}) = \{0\}.$$

Therefore we obtain the lemma in this case.

Note. If $T \in (\mathscr{E}^p(\boldsymbol{E}))'$, then this lemma is also true for $p > 0$. For $p = 0$, T is exact if and only if

$$dT = 0,\ \ T[1]=0.\ \ 1 \text{ is the constant function on } \boldsymbol{E} \text{ with value } 1.$$

Since $\sum_p \mathscr{D}^p(\boldsymbol{E})$ is an (algebraic) ideal of $\sum_p \mathscr{E}^p(\boldsymbol{E})$, we can define the product of a current T and a form φ by

$$T\wedge\varphi[\psi] = T[\varphi\wedge\psi].$$

If f is a function, then we can define $f\cdot T$ by $T\cdot f$. Then since $\boldsymbol{E}$ is paracompact and C^∞-smooth, we get

Lemma 20. *Let T and S be two $(\infty - p)$-currents of $\boldsymbol{E}$ such that for any $x\in\boldsymbol{E}$, there exists a neighborhood $U(x)$ of x and*

$$T[\varphi] = S[\varphi], \qquad car.\,\varphi \subset U(x).$$

Then $T=S$. This is also true for the elements of $\mathscr{E}^p(\boldsymbol{E})'$.

By this lemma, we can define the carrier $car.(T)$ of a current T.

Lemma 21. *We set*

$$U(E^r, \varepsilon) = \{y \mid \min_{x\in E} \| y - x \| < \varepsilon\},$$

and assume there exists C^∞-class projection $\rho^r : U(\boldsymbol{E}^r, \varepsilon) \to \boldsymbol{E}^r$ such that

$$\rho_r{}^s\rho_s{}^t = \rho_r{}^t, \quad \rho_r{}^s = \rho^s \,|\, U(\boldsymbol{E}^s, \varepsilon) \cap \boldsymbol{E}^r,$$

for some ε. Then $\underrightarrow{lim.}[\mathscr{E}^p(\boldsymbol{E}^r)' \,|\, \pi_r{}^{s*}]$ and $\underrightarrow{lim.}[\mathscr{D}^p(\boldsymbol{E}^r)' \,|\, \pi_r{}^{s*}]$ are dense in $\mathscr{E}^p(\boldsymbol{E})'$ and $\mathscr{D}^p(\boldsymbol{E})'$.

Corollary. *If $\boldsymbol{E}$ satisfies the assumptions of Lemma 21, then*

(6)' $$\mathscr{E}^p(\boldsymbol{E})'' = \mathscr{E}^p(\boldsymbol{E}),$$

(6) $$\mathscr{D}^p(\boldsymbol{E})'' = \mathscr{D}^p(\boldsymbol{E}).$$

Note. If $\boldsymbol{E}$ is a Hilbert space, then the assumption of Lemma 21 is fulfilled.

26. Let Y be a closed p-dimensional orientable submanifold of $\boldsymbol{E}$ such that

(7) $Y \cup B$ is compact if B is a bounded closed set of $\boldsymbol{E}$,

then the currents T_Y and $T_{Y,\varphi}$ given by

$$T_Y[\varphi] = \int_Y \varphi,$$

$$T_{Y,\varphi}[\psi] = \int_Y \varphi\wedge\psi,$$

are defined, if we fix the orientation of Y.

Note. A closed p-dimensional submanifold Y of $\boldsymbol{E}$ satisfies (7) if and only if there exists a series of integers $\{s(n)\}$ and subspaces $\{\boldsymbol{E}^{s(n)}\}$ such that

$$Y \cap B(n) \subset \boldsymbol{E}^{s(n)}, \quad \mathrm{B(n)} = \{\mathrm{x} \mid \|\mathrm{x}\| \leq \mathrm{n}\},$$

$$\boldsymbol{E}^{s(n)} \subset \boldsymbol{E}^{s(n+1)}, \quad dim.\,\boldsymbol{E}^{s(n)} = s(n).$$

Theorem 15. *If Y is a closed p-dimensional oriented submanifold of $\boldsymbol{E}$ and satisfies (7), φ a closed q-form on Y, then there exists an $(\infty-(p+q+1))$-current S on $\boldsymbol{E}$ such that*

(8) $$dS = T_{Y,\varphi}.$$

Proof. We can define an element S' of $(d\mathscr{D}^{p+q}(\boldsymbol{E}))'$ by

$$S['d\psi] = T_{Y,\varphi}[\psi],$$

because $d\psi = d\psi'$ implies $\psi' = \psi + d\alpha$, $\alpha \in \mathscr{D}^{p+q-1}(\boldsymbol{E})$ by Lemma 18 and

$$T_{Y,\varphi}[d\alpha] = 0.$$

Then since $\mathscr{D}^{p+q+1}(\boldsymbol{E})$ is locally convex, there is an $(\infty-(p+q+1))$-current S on $\boldsymbol{E}$ such that

$$S \,|\, d\mathscr{D}^{p+q}(\boldsymbol{E}) = S',$$

by Hahn-Banach' theorem ([10]). Hence we have the theorem.

Corollary. *There is an $(\infty-1)$-current T such that*

(9) $$T[df] = f(0).$$

Note. If Y is compact, then Y satisfies (7) and the current (8) is taken to be an element of $(\mathscr{E}^{p+q+1}(\boldsymbol{E}))'$. Especially, we can take T of (9) to be an element of $(\mathscr{E}^{1}(\boldsymbol{E}))'$.

Similarly, if Y is a closed real analytic subvariety of $\boldsymbol{E}$ such that Y satisfies (7) and each $Y_i - Y_{i+1}$ is oriented, then we can define the groups $\Gamma_0(Y_i - Y_{i+1}, \mathfrak{G}')$ and we obtain

Theorem 15′. *Under the above assumptions, if $\varphi_i \in \Gamma_0(Y_i - Y_{i+1}, \mathfrak{G}^{p-r_i})$ and*

$$\sum_{i=1}^{s-1}\sum_{k>i} res._k(\varphi_i) = 0,$$

then there exists a current S of $\boldsymbol{E}$ such that

(8)′ $$dS = \sum_{i=1}^{s} T_{Y_i - Y_{i+1},\varphi_i}.$$

§ 2. $(\infty - p)$-de Rham groups.

27. For any (∞-dimensional) Banach manifold X, we can define $C^{\infty,p}(X)$ and

$\mathscr{E}^p(X)$ similarly as $C^{\infty,p}(\boldsymbol{E})$ and $\mathscr{E}^p(\boldsymbol{E})$. To define $C_0^{\infty,p}(X)$ and $\mathscr{D}^p(X)$, we assume X is a (not necessarily closed) submanifold of $\boldsymbol{E}$, a separable C^∞-smooth Banach space. In the rest, we fix this inbedding $i: X \to \boldsymbol{E}$. Similarly, if X is a Banach analytic space (cf. [5]), then we consider X to be a (not necessarily closed) subanalytic space of $\boldsymbol{E}$.

Definition. *$car.(\varphi)$ of a differential form φ of X is called absolutely closed, if it is a closed set of $\boldsymbol{E}$.*

Definition. *$car.(\varphi)$ of a differential form φ of X is called bounded if it is a bounded set of $\boldsymbol{E}$.*

Then we define

$C_0^{\infty,p}(X)$: *the vector space consisted by C^∞-class p-forms with absolutely closed and bounded carrier of X.*

$\mathscr{D}^p(X)$: *the topological vector space regarded $C_0^{\infty,p}(X)$ to be a subspace of $\mathscr{E}^p(X)$.*

Since $\boldsymbol{E}$ is C^∞-smooth, we have

Lemma 22. *If X is a closed submanifold of $\boldsymbol{E}$, then*

$$\mathscr{E}^p(X) = \mathscr{E}^p(\boldsymbol{E})|X, \tag{10)'}$$

$$\mathscr{D}^p(X) = \mathscr{D}^p(\boldsymbol{E})|X. \tag{10}$$

Lemma 23. *If there is a sequence of closed submanifolds $\{X^r\}$ of X such that*

$$dim.\ X^r = n(r),\quad \lim_{r\to\infty} n(r) = \infty,\quad X^{r+1} \supset X^r,\quad \overline{\bigcup_r X^r} = X, \tag{i}$$

$$\text{each } X^r \text{ satisfies (7)}, \tag{ii}$$

then denoting $\pi^r{}_s : \mathscr{D}^p(X^r) \to \mathscr{D}^p(X^s)$, $r \geqq s$ the map defined by $\pi^r{}_s\varphi = \varphi|X^s$, we have

$$\mathscr{D}^p(X) = \varprojlim\ [\mathscr{D}^p(X^r) : \pi^r{}_s]. \tag{11}$$

Note. For $\mathscr{E}^p(X)$, we obtain

$$\mathscr{E}^p(X) = \varprojlim\ [\mathscr{E}^p(X^r) : \pi^r{}_s], \tag{11)'}$$

if $\{X^r\}$ satisfies only (i) (cf. [14]).

We denote the subspaces of $\mathscr{E}^p(X)$ and $\mathscr{D}^p(X)$ consisted by closed forms by $\mathscr{B}^p(X)$ and $\mathscr{A}^p(X)$. Then we know ([11]),

$$H^p(X,\ \boldsymbol{C}) = \mathscr{B}^p(X) / d\,\mathscr{E}^{p-1}(X), \tag{12)'}$$

where the left hand side is the Čech cohomology group. We also set

$$H^p{}_b(X, \boldsymbol{C}) = \mathscr{D}^p(X)/d\,\mathscr{A}^{p-1}(X). \tag{12}$$

Note. By definition, we get

$$\mathscr{E}^p(X) = \mathscr{D}^p(X), \quad H^p{}_b(X, \boldsymbol{C}) = H^p(X, \boldsymbol{C}) \text{ for all } p, \tag{13}$$
$$\text{if } X \text{ is a bounded closed set of } \boldsymbol{E}.$$

But in general, $H^p{}_b(X, \boldsymbol{C}) \neq H^p(X, \boldsymbol{C})$ and $H^p{}_b(X, \boldsymbol{C})$ is not a differential structure invariant.

28. Definition. *An element T of $\mathscr{D}^p(X)'$ is called an $(\infty - p)$-current of X.*

As in $n^0 25$, we define exterior differential d, etc. for the currents of X.

By (10), if T is an $(\infty - p)$-currents of X, then we can define a current $\pi^*(T) = \pi_X{}^*(T)$ by

$$\pi^*(T)[\varphi] = T[\varphi | X].$$

Theorem 15″. *If T is a closed $(\infty - p)$-current of X, then there is an $(\infty - p - 1)$-current S on $\boldsymbol{E}$ such that*

$$dS = \pi^*(T). \tag{8″}$$

Proof. Since $d\varphi | X = d(\varphi | X)$, $T[d\varphi | X] = 0$ for any $\varphi \in \mathscr{D}^{p-1}(\boldsymbol{E})$. Hence by (4), we can define $S' \in (d\mathscr{D}^{p-1}(\boldsymbol{E}))'$ by

$$S'[d\varphi] = T[\varphi | X].$$

Therefore we have the theorem by Hahn -Banach' theorem.

Note. In general, if Y is a closed submanifold of X, then

$$\pi(\mathscr{D}^p(X)) = \mathscr{D}^p(Y), \quad \pi(\varphi) = \varphi | Y.$$

Hence $\pi^* : \mathscr{D}^p(Y)' \to \mathscr{D}^p(X)'$ is defined. On the other hand, if Y is an open set of X, then there is an inclusion $\iota : \mathscr{D}^p(Y) \to \mathscr{D}^p(X)$. Hence $\iota^* : \mathscr{D}^p(X)' \to \mathscr{D}^p(Y)'$ is defined in this case. Since π is an onto homomorphism and ι is an into isomorphism, π^* is an into isomorphism and ι^* is an onto homomorphism.

Definition. *We set*

$$\mathscr{H}^{\infty - p}(X) = (d\mathscr{D}^{p-1}(X))^{\perp} / d\mathscr{D}^{p-1}(X))', \tag{15}$$

and call the $(\infty - p)$-de Rham group of X. Here $(d\mathscr{D}^{p-1}(X))^{\perp}$ is the space of closed $(\infty - p)$-currents of X.

Similarly, we set

(15)′ $$\mathscr{H}_c^{\infty-p}(X)=(d\mathscr{E}^{p-1}(X))^{\perp}/d(\mathscr{E}^{p-1}(X))'.$$

Similar as $H^p(X, \boldsymbol{C})$ and $H^p{}_b(X, \boldsymbol{C})$, we have

(13)′ $$\mathscr{H}^{\infty-p}(X)=\mathscr{H}_c^{\infty-p}(X),\ \text{if } X \text{ is a bounded closed set of } \boldsymbol{E}.$$

Lemma 24. *If Y is a closed submanifold of X, then there are homomorphisms*

(16)′ $$\pi' : H^p{}_b(X, \boldsymbol{C})\to H^p{}_b(Y, \boldsymbol{C}),$$

(16) $$\pi^{*\prime} : \mathscr{H}^{\infty-p}(Y)\to \mathscr{H}^{\infty-p}(X)$$

Lemma 24′. *If Y is an open set of X, then there are homomorphisms*

(17)′ $$\iota' : H^p{}_b(Y, \boldsymbol{C})\to H^p{}_b(X, \boldsymbol{C}),$$

(17) $$\iota^{*\prime} : \mathscr{H}^{\infty-p}(X)\to \mathscr{H}^{\infty-p}(Y).$$

Definition. *If Y^r is an r-dimensional oriented submanifold of X and satisfies* (7), *then the class $\langle T_Y\rangle$ of T_Y in $\mathscr{H}^{\infty-r}(X)$ is called the dual class of Y and denoted by $\langle T\rangle$.*

If Y is an r-dimnesional real analytic subvariety of X, a real analytic Banach manifold in $\boldsymbol{E}$, such that Y satisfies (7) and each $Y_i - Y_{i+1}$ is oriented and has only finite irreducible components $Y^1, \cdots, Y^s$, then by Theorem 6, corollary 4, there is an into isomorphism $h : H^0(Y^1, \boldsymbol{C})\oplus\cdots\oplus H^0(Y^s, \boldsymbol{C})\to R^0(Y)$. Then denoting $\langle Y^i\rangle$ the class of Y^i in $H^0(Y^i, \boldsymbol{C})$, we set

(18) $$\langle T_Y\rangle=\sum_{i=1}^{s}h(\langle Y^i\rangle).$$

Then the class $\langle T_Y\rangle$ induces an element of $\mathscr{H}^{\infty-r}(X)$. It is also called the dual class of Y.

29. Since $d\mathscr{E}^{p-1}(X)$, $d\mathscr{D}^{p-1}(X)$, $d(\mathscr{E}^{p-1}(X))'$ and $d\mathscr{D}^{p-1}(X))'$ are closed subspaces of $\mathscr{B}^p(X)$, $\mathscr{A}^p(X)$, $(d\mathscr{E}^{p-1}(X))^{\perp}$ and $(d\mathscr{D}^{p-1}(X))^{\perp}$, $H^p(X, \boldsymbol{C})$, $H_p{}^b(X, \boldsymbol{C})$, $\mathscr{H}^{\infty-p}(X)$ and $\mathscr{H}_c^{\infty-p}(X)$ are (locally convex) topological vector spaces.

Theorem 16. *For any p, we have*

(19) $$(H^p{}_b(X, \boldsymbol{C}))'=\mathscr{H}^{\infty-p}(X),$$

(19)′ $$(H^p(X, \boldsymbol{C}))'=\mathscr{H}_c^{\infty-p}(X).$$

Proof. For $\langle\varphi\rangle\in H_p{}^b(X, \boldsymbol{C})$ $(H^p(X, \boldsymbol{C}))$ and $\langle T\rangle\in\mathscr{H}^{\infty-p}(X)$ $(\mathscr{H}_c^{\infty-p}(X))$, we define

$$\langle\langle\varphi\rangle, \langle T\rangle\rangle = T[\varphi].$$

Then $\langle\langle\varphi\rangle, \langle T\rangle\rangle$ is well defined. Hence we get a homomorphism ϑ from $\mathscr{H}^{\infty-p}(X)$ ($\mathscr{H}_c^{\infty-p}(X)$) into $(H_p{}^b(X, \boldsymbol{C}))'$ $((H^p(X, \boldsymbol{C}))')$. If $\vartheta(\langle T\rangle) = 0$, then $T[\varphi] = 0$ if φ is closed. Therefore we define $S \in (d\mathscr{D}^{p-1}(X))'$ $((d\mathscr{E}^{p-1}(X))')$ by

$$S[d\varphi] = T[\varphi].$$

Then we get $T = d\tilde{S}$, where $\tilde{S}$ is an extension of S. This shows $\boldsymbol{ker}.\,\vartheta = 0$.

If τ belongs in $(H^p{}_b(X, \boldsymbol{C}))'$ $((H^p(X, \boldsymbol{C}))')$, then we define $T \in (\mathscr{A}^p(X))'$ $((\mathscr{B}(X))')$ by

$$T[\varphi] = \tau(\langle\varphi\rangle).$$

Then we get $\tau = \vartheta(\langle\widetilde{T}\rangle)$, where $\widetilde{T}$ is an extension of T. Hence we have the theorem.

Corollary. *If X is connected, closed and bounded, then*

$$\mathscr{H}^{\infty}(X) = \boldsymbol{C}. \tag{20}$$

On the other hand, if X is closed, connected and unbounded, then

$$\mathscr{H}^{\infty}(X) = \{0\}. \tag{20$'$}$$

Note. 1. Since $H^0{}_b(\boldsymbol{E}, \boldsymbol{C}) = \{0\}$, $\mathscr{H}^{\infty}(\boldsymbol{E})$ vanishes. On the other hand, we know that the unit sphere S^{∞} of $\boldsymbol{H}$, the Hilbert space, is diffeomorphic to $\boldsymbol{H}$ ([7]). Therefore $\mathscr{H}^{\infty-p}(X)$ is not a differential structure invariant.

Note. 2. If $\boldsymbol{E}$ satisfies the assumption of Lemma 21, then $\mathscr{D}^p(X)')' = \mathscr{D}^p(X)$. Therefore $(\mathscr{H}^{\infty-p}(X))'$ is isomorphic to $H_p{}^b(X, \boldsymbol{C})$ for all p.

Note. 3. If X is an (infinite dimensional) Banach analytic space in $\boldsymbol{E}$, then we define

$$R^{\infty-p}(X) = \frac{\{\text{the space of closed } (\infty-p)\text{-currents of } \boldsymbol{E} \text{ with carrier in } X\}}{d\{\text{the space of } (\infty-p-1)\text{-currents of } \boldsymbol{E} \text{ with carrier in } X\}}.$$

§ 3. Residue exact sequences, III.

30. In this §, we assume that the pair (X, Y) satisfies either of following (i) or (ii).

(i). *X is a Banach manifold, Y is an (orientable) closed submanifold of X such that there exists a series of orientable submanifolds* $\{X^r\}$ *which satisfies the assumption of Lemma 23, and* $X^r \cap Y$ *is an orientable submanifold of* X^r *for each r.*

(ii). *X is a real analytic Banach manifold, Y is a real analytic subvariety of X such that there exists a series of orientable real analytic submanifolds* $\{X^r\}$ *of X which satisfies the assumption of Lemma 23,* $X^r \cap Y$ *is a real analytic subvariety of* X^r *for each r and each* $(X^r \cap Y)_i - (X^r \cap Y)_{i+1}$ *is orientable.*

Since $X - Y$ is open in X in each case, there are following inclusion maps.

$$\iota : \mathscr{D}^p(X-Y) \to \mathscr{D}^p(X),$$

$$\iota^* : \mathscr{D}^p(X)' \to \mathscr{D}^p(X-Y)'.$$

By definition, $\iota \cdot d = d \cdot \iota$. Hence $\iota^* \cdot d = d \cdot \iota^*$. Therefore we get

$$\iota(\mathscr{A}^p(X-Y)) \subset \mathscr{A}^p(X), \quad \iota(d\mathscr{D}^{p-1}(X-Y)) \subset d\mathscr{D}^{p-1}(X), \tag{21}$$

$$\iota^*(d\mathscr{D}^{p-1}(X)^{\perp}) \subset d\mathscr{D}^p(X-Y), \tag{21'}$$

$$\iota^*(d\mathscr{D}^{p-1}(X))') \subset d(\mathscr{D}^{p-1}(X-Y))'.$$

Moreover, if X^r is an r-dimensional closed submanifold of X and satisfies (7), then denoting π_r and $\pi^r{}_s$ the homomorphisms from $\mathscr{D}^p(X)$ into $\mathscr{D}^p(X^r)$, etc. defined by $\pi_r(\varphi) = \varphi | X^r$, etc., we have the following commutative diagrams.

$$\begin{array}{ccc} \mathscr{D}^p(X-Y) & \xrightarrow{\iota} & \mathscr{D}^p(X) \\ \downarrow \pi_r & & \downarrow \pi_r \\ \mathscr{D}^p((X-Y)\cap X^r) & \xrightarrow{\iota} & \mathscr{D}^p(X^r), \end{array} \qquad \begin{array}{ccc} \mathscr{D}^p((X-Y)\cap X^r) & \xrightarrow{\iota} & \mathscr{D}^p(X^r) \\ \downarrow \pi^r{}_s & & \downarrow \pi^r{}_s \\ \mathscr{D}^p((X-Y)X^s) & \xrightarrow{\iota} & \mathscr{D}^p(X^s). \end{array} \tag{22}$$

Theorem 17. *If the pair (X, Y) satisfies (i) or (ii), then a closed current of* $X - Y$ *is always cohomologous to a current in* ι^**-image.*

Proof. We take a non-exact $(\infty - p)$-current T of $X - Y$. Then there exists a closed form $\varphi \in \mathscr{D}^p(X-Y)$ such that

$$T[\varphi] = 1. \tag{23}$$

On the other hand, since T is closed, *ker.* T contains $d\mathscr{D}^{p-1}(X-Y)$. Hence we get

$$\pi_r(ker.\, T) \cap \mathscr{A}^p((X-Y)\cap X^r) \supset d\mathscr{D}^{p-1}((X-Y)\cap X^r), \quad r > p, \tag{24}$$

$$\pi_r(\varphi) \notin \pi_r(ker.\, T) \cap \mathscr{A}^p((X-Y)\cap X^r), \quad r > p. \tag{24'}$$

Hence there is a closed $(r-p)$-current T_r of $X^r\cap(X-Y)$ such that

$$T_r(\pi_r(\varphi))=1, \tag{25}$$
$$ker.\ T_r\cap\mathscr{A}^p((X-Y)\cap X^r)\supset\pi_r(ker.\ T)\cap\mathscr{A}^p((X-Y)\cap X^r).$$

Since $ker.\ T_r\cap\mathscr{A}^p((X-Y)\cap X^r)$ is determined by the cohomology class of T_r, we have by Theorem 3 or Theorem 4,

$$\overline{\iota(ker.\ T_r\cap\mathscr{A}^p((X-Y)\cap X^r)}\not\ni\iota(\pi_r(\varphi)). \tag{26}$$

Hence by (25), we get

$$\overline{\iota(\pi_r(ker.\ T)\cap\mathscr{A}^p((X-Y)\cap X^r)}\not\ni(\pi_r(\varphi)). \tag{27}$$

Then since we have by the commutativity of the diagrams (22),

$$\overline{\iota(ker.\ T\cap\mathscr{A}^p(X-Y))}$$
$$=\underset{\longleftarrow}{lim.}\ [\overline{\iota(\pi_r(ker.\ T)\cap\mathscr{A}^p((X-Y)\cap X^r)}:\pi^r{}_s],$$

we obtain by (27),

$$\varphi\notin\overline{\iota(ker.\ T\cap\mathscr{A}^p(X-Y))}. \tag{28}$$

Therefore by Hahn-Banach' theorem, there is an $(\infty-p)$-current $\widetilde{T}$ of X such that

$$\widetilde{T}(\varphi)=1,\quad ker.\ \widetilde{T}\supset ker.\ T\cap\mathscr{A}^p(X-Y). \tag{29}$$

By (29), we get

$$\langle\iota^*(\widetilde{T})\rangle=\langle T\rangle.$$

Hence we have the theorem.

31. Theorem 18. *Under the same assumption of Theorem 17, we have the following exact sequences.*

(i). *(X, Y) is the case (i) of n^o30.*

$$(30)_f\qquad \cdots\longrightarrow\mathscr{H}^{\infty-p}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-p}(X-Y)\xrightarrow{res.}H^{r-p+1}(Y,\ C)\longrightarrow$$
$$\xrightarrow{\pi^*}\mathscr{H}^{\infty-p+1}(X)\longrightarrow\cdots,\ dim.\ Y=r.$$

$$(30)_i\qquad \cdots\longrightarrow\mathscr{H}^{\infty-p}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-p}(X-Y)\xrightarrow{res.}\mathscr{H}^{\infty-p+}(Y)\longrightarrow$$

$$\xrightarrow{\pi^*} \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots, \qquad dim.\, Y = \infty.$$

(ii). *(X, Y) is the case (ii) of $n^0 30$.*

$$(30)_f' \qquad \cdots\longrightarrow \mathscr{H}^{\infty-p}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-p}(X-Y) \xrightarrow{res.} R^{r-p+1}(Y) \longrightarrow$$

$$\xrightarrow{\pi^*} \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots, \qquad dim.\, Y = r.$$

$$(30)_i' \qquad \cdots\longrightarrow \mathscr{H}^{\infty-p}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-p}(X-Y) \xrightarrow{res.} R^{\infty-p+1}(Y)$$

$$\xrightarrow{\pi^*} \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots, \qquad dim.\, Y = \infty.$$

Here res. is defined by

$$(31) \qquad res.\,(\langle\iota^*(T)\rangle) = \langle dT\rangle.$$

Proof. By Theorem 17, *res.* is defined and $Im.\, \iota^* = ker.\, res.$.

By the definition of π^*, $Im.\, res.$ is contained in $ker.\, \pi^*$. On the other hand, if $\pi^*(\langle T\rangle) = 0$, then we can define $S \in (d\mathscr{D}^p(X))'$ by

$$S[d\varphi] = T[\varphi].$$

Then the extension $\tilde{S}$ satisfies

$$res.\,(\langle\tilde{S}\rangle) = \langle T\rangle.$$

Hence $Im.\, res. = ker.\, \pi^*$.

By the definition of ι^*, $Im.\, \pi^*$ is contained in $ker.\, \iota^*$. If $\iota^*(\langle S\rangle) = 0$, then we have

$$\iota^*(S) = dT.$$

Therefore

$$(32) \qquad S[\varphi] = 0, \quad \varphi \in \iota(\mathscr{A}^{p+1}(X-Y)).$$

Since $\mathscr{D}^{p-1}{}_Y(X) = \{\varphi \mid \varphi \in \mathscr{D}^{p-1}(X),\ \varphi(x) = 0,\ x \in Y\}$ is a closed subspace of $\mathscr{D}^{p-1}(X)$, the quotient space $\mathscr{D}^{p-1}(X)\mid Y = \mathscr{D}^{p-1}(X)/\mathscr{D}^{p-1}{}_Y(X)$ is a locally convex topological vector space and $\mathscr{A}^{p-1}(X)\mid Y = \mathscr{A}^{p-1}(X)/\mathscr{A}^{p-1}(X)\mathscr{D}^{p-1}{}_Y(X)$ is an its closed subspace. Since S can be considered to be an element $S \mid Y$ of $(\mathscr{A}^{p-1}(X) \mid Y)'$ by (32), $S \mid Y$ is extended to an element $\tilde{S} \mid Y$ of $\mathscr{D}(^{p-1}(X) \mid Y)'$. Then by the definition of $\mathscr{D}^{p-1}(X) \mid Y$, $\tilde{S} \mid Y$ defines an $(\infty-p+1)$-current S' of X which satisfies

(33) $$S[\varphi]=S'[\varphi], \quad \varphi\in\mathscr{A}^{p-1}(X),$$

(33)′ $$S'[\phi]=0, \quad car.\,\phi\subset X-Y.$$

Hence $\langle S\rangle=\langle S'\rangle$ and $\langle S'\rangle$ belongs in π^*-image. Therefore we have the theorem.

Note. In the cases $(30)_i$ and $(30)_i'$, we may set

$$\mathscr{H}^{\infty+r-p}(X) \text{ instead of } \mathscr{H}^{\infty-p}(X) \text{ if codim. } Y=r,$$

$$\mathscr{H}^{\infty+\infty-p}(X) \text{ instead of } \mathscr{H}^{\infty-p}(X) \text{ if codim. } Y=\infty.$$

Then $(30)_i$ and $(30)_i'$ are rewritten as the following exact sequences.

$(34)_r$ $$\cdots\longrightarrow\mathscr{H}^{\infty+r-p}(X)\longrightarrow\mathscr{H}^{\infty+r-p}(X-Y)\longrightarrow\mathscr{H}^{\infty-p+1}(Y)\longrightarrow$$
$$\longrightarrow\mathscr{H}^{\infty+r-p+1}(X)\longrightarrow\cdots,\qquad \text{codim. } Y=r,$$

$(34)_\infty$ $$\cdots\longrightarrow\mathscr{H}^{\infty+\infty-p}(X)\longrightarrow\mathscr{H}^{\infty+\infty-p}(X-Y)\mathscr{H}^{-p+1}(Y)\longrightarrow$$
$$\longrightarrow\mathscr{H}^{\infty+\infty-p+1}(X)\longrightarrow\cdots,\qquad \text{codim. } Y=\infty,$$

$(34)_r'$ $$\cdots\longrightarrow\mathscr{H}^{\infty+r-p}(X)\longrightarrow\mathscr{H}^{\infty+r-p}(X-Y)\longrightarrow R^{\infty-p+1}(Y)\longrightarrow$$
$$\longrightarrow\mathscr{H}^{\infty+r-p+1}(X),\qquad \text{codim. } Y=r,$$

$(34)_\infty'$ $$\cdots\longrightarrow\mathscr{H}^{\infty+\infty-p}(X)\longrightarrow\mathscr{H}^{\infty+\infty-p}(X-Y)\longrightarrow R^{\infty-p+1}(Y)\longrightarrow$$
$$\longrightarrow\mathscr{H}^{\infty+\infty-p+1}(X),\qquad \text{codim. } Y=\infty.$$

Definition. *The exact sequences* $(30)_f$, $(30)_i$, $(30)_f'$, $(30)_i'$, $(34)_r$, $(34)_\infty$, $(34)_r'$ *and* $(34)_\infty'$ *are called residue exact sequences for* $(\infty-p)$*-de Rham groups.*

32. Similarly as Theorem 5′, if X is a real Banach analytic space, Y its closed real analytic subvariety such that there exists a series of closed orientable real analytic subvarieties $\{X^r\}$ which satisfies

(i) $$dim.\, X^r=n(r),\ \lim_{r\to\infty} n(r)=\infty,\ X^{r+1}\supset X^r,\ \bigcup_r X^r=X,$$

(ii) each X^r satisfies (7),

(iii) each $(X^r\cap Y)_i-(X^r\cap Y)_{i+1}$ is orientable,

then we have

Theorem 18′. *The following sequences are exact.*

$(35)_f$ $$\cdots\longrightarrow R^{\infty-p}(X)\xrightarrow{\iota^*}R^{\infty-p}(X-Y)\xrightarrow{res.}R^{r-p+1}(Y)\longrightarrow$$
$$\xrightarrow{\pi^*}R^{\infty-p+1}(X)\cdots,\qquad dim.\, Y=r \text{ and } Y \text{ satisfies (7),}$$

$$(35)_i \qquad \cdots \longrightarrow R^{\infty-p}(X) \xrightarrow{\iota^*} R^{\infty-p}(X-Y) \xrightarrow{res.} R^{\infty-p+1}(Y) \longrightarrow$$

$$\xrightarrow{\pi^*} R^{\infty+\infty-p+1}(X) \longrightarrow \cdots, \qquad dim.\ Y=\infty.$$

Note. In the case $(35)_i$, we may set

$R^{\infty+r-p}(X)$ instead of $R^{\infty-p}(X)$ if codim. $Y=r$,

$R^{\infty+\infty-p}(X)$ instead of $R^{\infty-p}(X)$ if codim. $Y=\infty$.

Then $(35)_i$ is rewritten as the following exact sequences.

$$(36)_r \qquad \cdots \longrightarrow R^{\infty+r-p}(X) \xrightarrow{\iota^*} R^{\infty+r-p}(X-Y) \xrightarrow{res.} R^{\infty-p+1}(Y) \longrightarrow$$

$$\xrightarrow{\pi^*} R^{\infty+r-p+1}(X) \longrightarrow \cdots, \qquad \text{codim. } Y=r,$$

$$(36)_\infty \qquad \cdots \longrightarrow R^{\infty+\infty-p}(X) \xrightarrow{\iota^*} R^{\infty+\infty-p}(X-Y) \xrightarrow{res.} R^{\infty-p+1}(Y) \longrightarrow$$

$$\xrightarrow{\pi^*} R^{\infty+\infty-p+1}(X) \longrightarrow \cdots, \qquad \text{codim. } Y=\infty.$$

We also obtain

Theorem 19. *We define the homomorphism* $\delta' : H^{r_s-p+1}(Y \cap X^s, \boldsymbol{C}) \longrightarrow H^{n(s)-p+1}(X, \boldsymbol{C})$ *by* $(-1)^{(n(s)-r^s)(p-1)}\delta$, *where* $r_s = dim.\ Y \cap X^s$ *and* δ *is the map defined in* (6). *Then the following diagrams are commutative.*

case. (i)

$$\begin{array}{ccccccc}
\cdots \longrightarrow \mathscr{H}^{\infty-p}(X) & \xrightarrow{\iota^*} & \mathscr{H}^{\infty-p}(X-Y) & \longrightarrow & & & \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & & & & \\
\cdots \longrightarrow H^{n(s)-p}(X^s, \boldsymbol{C}) & \xrightarrow{i^*} & H^{n(s)-p}((X-Y)\cap X^s, \boldsymbol{C}) & \longrightarrow & & &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{res.} H^{r-p+1}(Y, \boldsymbol{C}) & \xrightarrow{\pi^*} & \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots & & \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & & \\
\xrightarrow{res.} H^{r_s-p+1}(Y \cap X^s, \boldsymbol{C}) & \xrightarrow{\delta'} & H^{n(s)-p+1}(X^s, \boldsymbol{C}) \longrightarrow \cdots, & & dim.\ Y=r,
\end{array}$$

$$\begin{array}{ccccc}
\cdots \longrightarrow \mathscr{H}^{\infty-p}(X) & \xrightarrow{\iota^*} & \mathscr{H}^{\infty-p}(X-Y) & \longrightarrow & \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & & \\
\cdots \longrightarrow H^{n(s)-p}(X^s, \boldsymbol{C}) & \xrightarrow{i^*} & H^{n(s)-p}((X-Y)\cap X^s, \boldsymbol{C}) & \longrightarrow &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{res.} \mathscr{H}^{\infty-p+1}(Y) & \xrightarrow{\pi^*} & \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots & & \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & & \\
\xrightarrow{res.} H^{r_s-p+1}(Y \cap X^s, \boldsymbol{C}) & \xrightarrow{\delta'} & H^{n(s)-p+1}(X^s, \boldsymbol{C}) \longrightarrow \cdots, & & dim.\ Y=\infty,
\end{array}$$

case. (ii).

$$\begin{array}{ccccccc}
\cdots\longrightarrow & \mathscr{H}^{\infty-p}(X) & \xrightarrow{\iota^*} & \mathscr{H}^{\infty-p}(X-Y) & \longrightarrow \\
& \downarrow \pi_s^* & & \downarrow \pi_s^* & \\
\cdots\longrightarrow & H^{n(s)-p}(X^s,\ C) & \xrightarrow{i^*} & H^{n(s)-p}((X-Y)\cap X^s,\ C) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{res.} & R^{r-p+1}(X) & \xrightarrow{\pi^*} & \mathscr{H}^{\infty-p+1}(X) & \longrightarrow\cdots \\
& \downarrow \pi_s^* & & \downarrow \pi_s^* & \\
\xrightarrow{res.} & R^{r_s-p+1}(Y\cap X^s) & \xrightarrow{\delta'} & H^{n(s)-p+1}(X^s,\ C) & \longrightarrow\cdots, & dim.\, Y=r,
\end{array}$$

$$\begin{array}{ccccccc}
\cdots\longrightarrow & \mathscr{H}^{\infty-p}(X) & \xrightarrow{\iota^*} & \mathscr{H}^{\infty-p}(X-Y) & \longrightarrow \\
& \downarrow \pi_s^* & & \downarrow \pi_s^* & \\
\cdots\longrightarrow & H^{n(s)-p}(X^s, C) & \xrightarrow{i^*} & H^{n(s)-p}((X-Y)\cap X^s,\ C) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\xrightarrow{res.} & R^{\infty-p+1}(Y) & \xrightarrow{\pi^*} & \mathscr{H}^{\infty-p+1}(X) & \longrightarrow\cdots \\
& \downarrow \pi_s^* & & \downarrow \pi_s^* & \\
\xrightarrow{res.} & R^{r_s-p+1}(Y\cap X^s) & \xrightarrow{\delta'} & H^{n(s)-p+1}(X_s,\ C) & \longrightarrow\cdots, & dim.\, Y=\infty.
\end{array}$$

§ 4. Residue exact sequences, IV.

33. Lemma 25. *If $U=\boldsymbol{E}$ or an open ball in $\boldsymbol{E}$, $\boldsymbol{E}^r$ is an r-dimensional subspace of $\boldsymbol{E}$ such that $U\cap\boldsymbol{E}^r\neq\varphi$, then*

(37) $$\mathscr{H}^{\infty-p}(U-\boldsymbol{E}^r\cap U)=\{0\},\quad p\neq r+1,$$
$$\mathscr{H}^{\infty-r-1}(U-\boldsymbol{E}^r\cap U)=\boldsymbol{C}.$$

Proof. Since $\mathscr{H}^{\infty-p}(U)=\{0\}$ for all p by Lemma 18 and Theorem 16, we get

$$res.\ :\ \mathscr{H}^{\infty-p}(U-\boldsymbol{E}^r\cap U)\simeq H^{r-p+1}(\boldsymbol{E}^r\cap U,\ \boldsymbol{C}),$$

for all p. Hence we have the lemma.

Similarly as in $n^0 1$, we denote $\mathfrak{D}'^{\infty-p}$, $\mathfrak{G}'^{\infty-p}$, $\mathfrak{D}'^{\infty-p}[Y]$ and $\mathfrak{G}'^{\infty-p}[Y]$ the sheaves on X, a Banach manifold in $\boldsymbol{E}$, consisted by germs of $(\infty-p)$-currents etc.. Here Y means a closed submanifold of X. We also denote the stalks of $\mathfrak{D}'^{\infty-p}$ etc. at x by $\mathfrak{D}'^{\infty-p}_x$ etc.. Then by Lemma 25, we have

Lemma 26. *If Y is an r-dimensional closed submanifold of X, then*

(38) $$d\mathfrak{D}'^{\infty-p}[Y]_x=\mathfrak{G}'^{\infty-p+1}[Y]_x,\quad p\neq r+2,$$
$$\mathfrak{G}'^{\infty-r-1}[Y]_x/d\mathfrak{D}'^{\infty-r-2}[Y]_x\simeq\boldsymbol{C}.$$

Corollary. *Under the same assumptions about X, Y, we have*

$$d\mathfrak{D}'^{\infty-p}[Y] = \mathfrak{G}'^{\infty-p+1}[Y], \quad p \neq r+2, \tag{39}$$

and the exact sequence

$$0 \longrightarrow d\mathfrak{D}'^{\infty-r-2}[Y] \xrightarrow{i'} \mathfrak{G}'^{\infty-r-1}[Y]' \xrightarrow{res.} \mathscr{C}_Y \longrightarrow 0, \tag{40}$$

where the stalk $\mathscr{C}_{Y,x}$ of $\mathscr{C}_Y$ at x is equal to $\{0\}$ if $x \in Y$ and $\mathscr{C}_{Y,x} = \boldsymbol{C}$ if $x \in Y$.

Lemma 27. *The sheaves $\mathscr{C}_Y$ and $d\mathfrak{D}'^{\infty-p}[Y]/d\mathfrak{D}'^{\infty-p}$ are determined by $U(Y)$, an (arbitrary) neighborhood of Y in X.*

Corollary. *If Y satisfies* (7) *and has the tublar neighborhood in X and h is a diffeomorphism of Y, then*

$$\begin{aligned} &\mathscr{C}_Y \simeq \mathscr{C}_{h(Y)}, \\ &d\mathfrak{D}'^{\infty-p}[Y]/d\mathfrak{D}'^{\infty-p} \simeq d\mathfrak{D}'^{\infty-p}[h(Y)]/d\mathfrak{D}'^{\infty-p}. \end{aligned} \tag{41}$$

Here the sheaves in the right hand sides are considered in $\boldsymbol{E}$.

Proof. By assumption, h is extended to a diffeomorphism $\bar{h} : U(Y) \longrightarrow V(h(Y))$, where $U(Y)$ and $V(h(Y))$ are suitable neighborhoods of Y and $h(Y)$ in X and in $\boldsymbol{E}$. Hence for any $x \in Y$, there exists a neighborhood $W(x)$ of x in Y such that

$$\begin{aligned} &\mathscr{C}_Y \mid W(x) \simeq \mathscr{C}_{h(Y)} \mid h(W(x)), \\ &d\mathfrak{D}'^{\infty-p}Y/d\mathfrak{D}'^{\infty-p} \mid W(x) \simeq d\mathfrak{D}'^{\infty-p}[h(Y)]/d\mathfrak{D}'^{\infty-p} \mid h(W(x)). \end{aligned}$$

Then since Y satisfies (7), we have the lemma by the definitions of the sheaves $\mathscr{C}_Y$ and $d\mathfrak{D}'^{\infty-p}[Y]/d\mathfrak{D}'^{\infty-p}$.

34. Since *dim.* Y is finite, we can imbed Y in a finite dimensional subspace of $\boldsymbol{E}$. We take a sequence of subspaces $\{\boldsymbol{E}^s\}$ of $\boldsymbol{E}$ such that

(i) $\quad dim.\, \boldsymbol{E}^s = s, \ \boldsymbol{E}^{s+1} \supset \boldsymbol{E}^s$, each $\boldsymbol{E}^s$ contains Y.

(ii) $\quad \overline{\cup \boldsymbol{E}^s} = \boldsymbol{E}$.

Then, if Y is orientable, we have the following commutative diagrams

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathscr{H}^{\infty-p}(\boldsymbol{E}) \arrow[r, "\iota^*"] \arrow[d, "\pi_s^*"] & \mathscr{H}^{\infty-p}(\boldsymbol{E}-Y) \arrow[r] \arrow[d, "\pi_s^*"] & {} \\
\cdots \arrow[r] & H^{s-p}(\boldsymbol{E}^s, \boldsymbol{C}) \arrow[r, "i^*"] \arrow[d, "\pi_s^{t*}"] & H^{s-p}(\boldsymbol{E}^s - Y, \boldsymbol{C}) \arrow[r] \arrow[d, "\pi_s^{t*}"] & {} \\
\cdots \arrow[r] & H^{t-p}(\boldsymbol{E}^t, \boldsymbol{C}) \arrow[r, "i^*"] \arrow[uu, bend left, "\pi_t^*"] & H^{t-p}(\boldsymbol{E}^t - Y, \boldsymbol{C}) \arrow[r] \arrow[uu, bend left, "\pi_t^*"] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r, "res."] & H^{r-p+1}(Y, \boldsymbol{C}) \arrow[r, "\pi^*"] \arrow[d, "="] & \mathscr{H}^{\infty-p+1}(\boldsymbol{E}) \arrow[r] \arrow[d, "\pi_s^*"] & \cdots \\
{} \arrow[r, "res."] & H^{r-p+1}(Y, \boldsymbol{C}) \arrow[r, "\delta"] \arrow[d, "="] & H^{s-p+1}(\boldsymbol{E}^s, \boldsymbol{C}) \arrow[r] \arrow[d, "\pi_s^{t*}"] & \cdots \\
{} \arrow[r, "res."] & H^{r-p+1}(Y, \boldsymbol{C}) \arrow[r, "\delta"] \arrow[uu, bend left, "="] & H^{t-p+1}(\boldsymbol{E}^t, \boldsymbol{C}), \arrow[r] \arrow[uu, bend left, "\pi_t^*"] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & d\mathfrak{D}'^{\infty-r-2}[Y] \arrow[r, "i'"] \arrow[d, "\pi_s^*"] & \mathfrak{G}'^{\infty-r-1}[Y] \arrow[r, "res.'"] \arrow[d, "\pi_s^*"] & \mathscr{C}_Y \arrow[r] \arrow[d, "\pi_s^*"] & 0 \\
0 \arrow[r] & d\mathfrak{D}'^{s-r-2}[Y] \arrow[r, "i'"] \arrow[d, "\pi^s_t{}^*"] & \mathfrak{G}'^{s-r-1}[Y] \arrow[r, "res."] \arrow[d, "\pi^s_t{}^*"] & \mathscr{C}_Y \arrow[r] \arrow[d, "\pi^s_t{}^*"] & 0 \\
0 \arrow[r] & d\mathfrak{D}'^{t-r-2}[Y] \arrow[r, "i'"] \arrow[uu, bend left, "\pi_t^*"] & \mathfrak{G}'^{t-r-1}[Y] \arrow[r, "res.'"] \arrow[uu, bend left, "\pi_t^*"] & \mathscr{C}_Y \arrow[r] \arrow[uu, bend left, "\pi_t^*"] & 0.
\end{tikzcd}
$$

By the commutativity of the first diagram, we get

$$\pi_s^* : \mathscr{H}^{\infty-p}(\boldsymbol{E}-Y) \simeq H^{s-p}(\boldsymbol{E}^s - Y, \boldsymbol{C}).$$

Then by the commutativity of the second diagram, we obtain the following commutative diagram.

$$
\begin{tikzcd}
\cdots \arrow[r] & H^{r-p+1}(\boldsymbol{E}, d\mathfrak{D}'^{\infty-r-2}[Y]) \arrow[r] \arrow[d, "\pi^*"] & H^{r-p+1}(\boldsymbol{E}, \mathfrak{G}'^{\infty-r-1}[Y]) \arrow[r] \arrow[d, "\pi^*"] & {} \\
\cdots \arrow[r] & \mathscr{H}^{\infty-p}(\boldsymbol{E}) \arrow[r] & \mathscr{H}^{\infty-p}(\boldsymbol{E}-Y) \arrow[r] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[r] & H^{r-p+1}(\boldsymbol{E}, \mathscr{C}_Y) \arrow[r] \arrow[d, "\pi^*"] & H^{r-p+2}(\boldsymbol{E}, d\mathfrak{D}'^{\infty-r-2}[Y]) \arrow[r] \arrow[d, "\pi^*"] & \cdots \\
{} \arrow[r] & H^{r-p+1}(Y, \boldsymbol{C}) \arrow[r] & \mathscr{H}^{\infty-p}(\boldsymbol{E}) \arrow[r] & \cdots.
\end{tikzcd}
$$

On the other hand, we get $\mathscr{C}_Y = \boldsymbol{C}_Y$ if we consider Y is contained in $\boldsymbol{E}^s$ and by the corollary of Lemma 26, we have

$$\pi^* : H^{r-p+1}(\boldsymbol{E}, \mathfrak{G}'^{\infty-r-1}[Y] \simeq \mathscr{H}^{\infty-p}(\boldsymbol{E}-Y).$$

Therefore we have

$$\pi^* : H^{r-p+1}(\boldsymbol{E}, d\mathfrak{D}'^{\infty-r-2}[Y]) \simeq \mathscr{H}^{\infty-p}(\boldsymbol{E}). \tag{42}$$

Since the sequence

$$0 \longrightarrow d\mathfrak{D}'^{\infty-k} \longrightarrow d\mathfrak{D}'^{\infty-k}[Y] \longrightarrow d\mathfrak{D}'^{\infty-k}Y/d\mathfrak{D}'^{\infty-k} \longrightarrow 0$$

is exact, we get by (42) and the corollary of Theorem 26,

Lemma 28. *If Y is an r-dimensional orientable manifold and satisfies* (7), *then*

$$H^q(\boldsymbol{E},\ d\mathfrak{D}'^{\infty-k}[Y]/d\mathfrak{D}'^{\infty-k}) = \{0\},\ q \geqq 1,\ k \geqq r+2. \tag{43)'}$$

Corollary. *If Y is an r-dimensional closed orientable submanifold of a Banach manifold X such that*

(i) *Y has a tublar neighborhood in X,*

(ii) *Y satisfies* (7),

then

$$H^q(X,\ d\mathfrak{D}'^{\infty-k}[Y]/d\mathfrak{D}'^{\infty-k}) = 0,\ q \geqq 1,\ k \geqq r+2. \tag{43}$$

35. Theorem 20. *Let X be a Banach manifold, Y an r-dimensional closed orientable submanifold of X and satisfies above* (*i*), (*ii*), *then the following sequence is exact.*

$$\cdots \longrightarrow \mathscr{H}^{\infty-p}(X) \xrightarrow{i^*} \mathscr{H}^{\infty-p}(X-Y) \xrightarrow{res.} H^{r-p+1}(Y,\ \boldsymbol{C}) \longrightarrow \tag{44}$$
$$\xrightarrow{\delta} \mathscr{H}^{\infty-p+1}(X) \longrightarrow \cdots.$$

Proof. By the exactnees of the sequence (40), we have the following exact sequence.

$$\cdots \longrightarrow H^{r-p+1}(X,\ d\mathfrak{D}'^{\infty-r-2}[Y]) \xrightarrow{i'^*} H^{r-p+1}(X,\ \mathfrak{G}'^{\infty r r-1}[Y]) \longrightarrow$$
$$\xrightarrow{res.'^*} H^{r-p+1}(X,\ \mathscr{C}_Y) \xrightarrow{\delta} H^{r-p+2}(X,\ d\mathfrak{D}'^{\infty-r-2}[Y]) \longrightarrow \cdots.$$

In this sequence, we have

$$H^{r-p+1}(X,\ \mathscr{C}_Y) = H^{r-p+1}(Y,\ \boldsymbol{C}),$$
$$H^{r-p+1}(X,\ \mathfrak{G}'^{\infty-r-1}[Y] \simeq \mathfrak{G}^{\infty-p}(X-Y),$$

by $n^0 34$ and the corollary of Theorem 26.

On the other hand, since the sequence

$$0 \longrightarrow d\mathfrak{D}'^{\infty-k} \longrightarrow d\mathfrak{D}'^{\infty-k}[Y] \longrightarrow d\mathfrak{D}'^{\infty-k}[Y]/d\mathfrak{D}'^{\infty-k} \longrightarrow 0$$

is exact on X, we have by (43)

$$H^q(X,\ d\mathfrak{D}'^{\infty-k}) \simeq H^q(X,\ d\mathfrak{D}'^{\infty-k}[Y]),\ q \geqq 2,\ k \geqq r+2. \tag{45}$$

By (45), we obtain

(46) $$H^{r-p+1}(X,\ d\mathfrak{D}'^{\infty-r-2}[Y]) \simeq H^{r-p+t}(X,\ d\mathfrak{D}'^{\infty-r-t+1}),\ t \geqq 1,\quad r-p+t \geqq 2,$$

because we know

$$H^{r-p+1}(X,\ d\mathfrak{D}'^{\infty-r-2}[Y]) \simeq H^{r-p+t}(X,\ d\mathfrak{D}'^{\infty-r-t+1}[Y]).$$

Then we have the theorem because we know

$$H^{r-p+t}(X,\ d\mathfrak{D}'^{\infty-r-t+1}) \simeq \mathscr{H}^{\infty-p}(X).$$

Similarly as Theorem 19, we have

Theorem 19′. *If X^s is an s-dimensional closed orientable submanifold of X such that $Y \cap X^s$ is a closed orientable submanifold of X^s, then setting dim. $Y \cap X^s = r_s$, we have the following commutative diagram.*

$$\begin{array}{ccccc}
\cdots\longrightarrow \mathscr{H}^{\infty-p}(X) & \xrightarrow{i^*} & \mathscr{H}^{\infty-p}(X-Y) & \longrightarrow & \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & & \\
\cdots\longrightarrow H^{s-p}(X^s,\ C) & \xrightarrow{i^*} & H^{s-p}(X^s - Y\cap X^s,\ C) & \longrightarrow &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{res.} H^{r-p+1}(Y,\ C) & \xrightarrow{\delta} & \mathscr{H}^{\infty-p+1}(X) & \longrightarrow\cdots \\
\downarrow \pi_s^* & & \downarrow \pi_s^* & \\
\xrightarrow{res.} H^{r_s-p+1}(Y\cap X^s,\ C) & \xrightarrow{\delta} & H^{s-p+1}(X^s,\ C). & \longrightarrow\cdots
\end{array}$$

Note. In this case, we need not change the sign of $\delta : H^{r_s-p+1}(Y \cap X^s,\ C) \longrightarrow H^{s-p+1}(X^s,\ C)$.

§ 5. Applications of residue exact sequences, II.

36. We assume the pair $(X,\ Y)$ satisfies either (i) or (ii) of $n^0 30$, and *dim.* $Y = r < \infty$. Moreover, if $(X,\ Y)$ satisfies (ii), then the homomorphism $h : R^0(Y) \longrightarrow H^0(Y,\ C)$ is an isomorphism. Then since the sequences

$$0 \longrightarrow \mathscr{H}^{\infty-r-1}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-r-1}(X-Y) \xrightarrow{res.} H^0(Y,\ C) \xrightarrow{\pi^*} \mathscr{H}^{\infty-r}(X) \longrightarrow\cdots$$

$$0 \longrightarrow \mathscr{H}^{\infty-r-1}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-r-1}(X-Y) \xrightarrow{res.} R^0(Y) \xrightarrow{\pi^*} \mathscr{H}^{\infty-r}(X) \longrightarrow\cdots$$

are exact, we get

Theorem 21. *Under the above assumptions, a closed $(\infty - r - 1)$-current T of $X - Y$ is written*

(47) $$T = T_1 + T_2 + T_3,$$

$$T_1 \in \iota^*((d\mathscr{D}^{\infty-r-2}(X))^{\perp}), \quad T_2 \in d(\mathscr{D}^{\infty-r-2}(X-Y))',$$
$$res.\, T_3 = \sum c_i\langle T_i\rangle, \quad c_i\langle Y_i\rangle = 0 \quad \text{in } \mathscr{H}^{\infty-r}(X),$$

where $H^0(Y, \boldsymbol{C}) = \sum_i \boldsymbol{C}\langle Y_i\rangle$.

Theorem 21$'$. *Let X be a Banach manifold and closed in $\boldsymbol{E}$, $Y=\{p_i\}$ is a closed 0-dimensional submanifold of X, then a closed $(\infty-1)$-current T on $X-Y$ is written*

$$(47)' \qquad T = T_1 + T_2 + T_3,$$
$$T_1 \in \iota^*((\mathscr{D}\, d^{\infty-2}(X))^{\perp}), \quad T_2 \in d(\mathscr{D}^{\infty-2}(X-Y))', \quad res.\, T_3 = \sum c_i\langle p_i\rangle,$$
$$\{c_i\} \text{ is arbitrary if } X \text{ is unbounded and } \sum c_i = 0 \text{ if } X \text{ is bounded.}$$

Moreover, setting $T_3 = T_3(\{p_i\}, \{c_i\})$, if $\{p_i\}$ move in finite dimensional domain, then we can take T_3 to depend differentiably on $\{p_i\}$ and $\{c_i\}$ if X is unbounded and depend real analytically on $\{p_i\}$ and $\{c_i\}$ if X is real analytic and $\{p_i\}$ and $\{c_i\}$ move real analytically.

Proof. We need only to prove the second assertion. For this, it is sufficient to show $T_3(p, c)$ depends differentiablly on p, c if X is unbounded, and $T_3(p_1, p_2, c, -c)$ depends real analytically on p_1, p_2, c if X is bounded.

By assumption, we assume p or p_1, p_2 move in $\boldsymbol{R}^n$ and denote their graphs in $X \times \boldsymbol{R}^n$ by Γ. We may assume Γ is an n-dimensional closed submanifold of $X \times \boldsymbol{R}^n$ if X is unbounded and an n-dimensional closed real analytic subvariety of $X \times \boldsymbol{R}^n$ if X is bounded.

Then since $\boldsymbol{E} \times \boldsymbol{R}^n$ is C^∞-smooth, we have the following commutative diagrams with exact raws,

$$\begin{tikzcd}
\mathscr{H}^{\infty-1}(X \times \boldsymbol{R}^n) \arrow[r] & \mathscr{D}^{\infty-1}(X \times \boldsymbol{R}^n - \Gamma) \arrow[r] & H^n(\Gamma, \boldsymbol{C}) \arrow[r] & \mathscr{H}^{\infty}(X \times \boldsymbol{R}^n) \\
\mathscr{H}^{\infty-1}(X \times \{p\}) \arrow[r] \arrow[u] & \mathscr{D}^{\infty-1}(X \times \{p\} - \{p, p\}) \arrow[r] \arrow[u] & H^0(\{p, p,\}\, \boldsymbol{C}) \arrow[r] \arrow[u] & \mathscr{H}^{\infty}(X \times \{p\}), \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{H}^{\infty-1}(X \times \boldsymbol{R}^n) \arrow[r] & \mathscr{H}^{\infty-1}(X \times \boldsymbol{R}^n - \Gamma) \arrow[r] & \boldsymbol{R}^n(\Gamma) \arrow[r] & \mathscr{H}^{\infty}(X \times \boldsymbol{R}^n) \\
\mathscr{H}^{\infty-1}(X \times \{p_1, p_2\}) \arrow[r] \arrow[u] & \mathscr{H}^{\infty-1}(X \times \{p_1, p_2\} - \{(p_1, p_1, p_2), (p_2, p_1, p_2)\}) \arrow[r] \arrow[u] & H^0(\{(p_1, p_1, p_2), (p_2, p_1, p_2)\}) \arrow[r] \arrow[u] & \mathscr{H}^{\infty}(X \times \{p_1, p_2\}), \arrow[u]
\end{tikzcd}$$

which show the theorem.

Corollary. *If X is unbounded, $D\subset X$ is an n-dimensional subset such that D is diffeomorphic to $\boldsymbol{R}^n$, f a smooth function on X, then there is a current $T_{D,\xi}$ of X such that $T_{D,\xi}$ depends differentiably on ξ, $\xi\in D$, and*

$$(48)\qquad T_{D,\xi}[df]=f(\xi),\ \ \xi\in D.$$

Note. Let $\boldsymbol{E}^r$ be an r-dimensional subspace of $\boldsymbol{E}$ with coordinate $(x_1,\cdots,x_r)$, then we denote the diagonal of $\boldsymbol{E}^r\times\boldsymbol{E}^r$ in $\boldsymbol{E}\times\boldsymbol{E}^r$ by Δ_r and set

$$S_{\xi,\varepsilon}=\{x_1,\cdots,x_r,\ \xi_1,\ \cdots,\ \xi_r)\mid\sqrt{\sum(x_i-\xi_i)^2}=\varepsilon\}$$
$$\subset\boldsymbol{E}^r\times\boldsymbol{E}^r\subset\boldsymbol{E}\times\boldsymbol{E}^r,$$

then there is an $(\infty-r)$-current $T_{r,\xi,\varepsilon}$ of $\boldsymbol{E}\times\boldsymbol{E}^r-\Delta_r$ such that $T_{r,\xi,\varepsilon}$ depends differentiably on ξ, $\xi\in\boldsymbol{E}^r$, and

$$T_{r,\xi,\varepsilon}[d\varphi]=\int_{S_{\xi,\varepsilon}}\varphi.$$

Moreover, Since $S_{\xi,\varepsilon}$ is compact, we can take $T_{r,\xi,\varepsilon}$ to belong in $(\mathscr{D}^p(\boldsymbol{E}\times\boldsymbol{E}^r-\Delta_r))'$. Then we get

$$(48)'\qquad T_{r,\xi,\varepsilon}[d(f\omega^{r-1}(x,\ \xi))]=\int_{S_{\xi,\varepsilon}}f\omega^{r-1}(x,\ \xi)=\int_{S_\varepsilon}f(x-\xi)\omega^{r-1},$$

$$S_\varepsilon=S_{0,\varepsilon},\ \ \omega^{r-1}(x,\ \xi)=\frac{1}{\sigma^{r-1}}\ \frac{\sum x_i\check{d}x_i}{(\sqrt{\sum(x_i-\xi_i)^2})^{r-1}}.$$

37. We assume X is connected, $p\in X$, then since $\mathscr{H}^\infty(X-\{p\})\simeq(H^0{}_b(X-\{p\}))'=0$, we have the exact sequence

$$0\longrightarrow\mathscr{H}^{\infty-1}(X)\xrightarrow{\iota^*}\mathscr{H}^{\infty-1}(X-\{p\})\xrightarrow{res.}H^0(\{p\},\ \boldsymbol{C})\longrightarrow$$
$$\xrightarrow{\pi^*}\mathscr{H}^\infty(X)\longrightarrow0.$$

Since we know $\mathscr{H}^\infty(X)=\boldsymbol{C}$ if X is bounded and closed and $\mathscr{H}^\infty(X)=\{0\}$ if X is unbounded, we get

$$\iota^*:\ \mathscr{H}^{\infty-1}(X)\simeq\mathscr{H}^{\infty-1}(X-\{p\}),$$
$$(49)\qquad \pi^*:H^0(\{p\},\ \boldsymbol{C})\simeq\mathscr{H}^\infty(X).$$

By (49), if T is an ∞-current of a closed bounded Banach manifold X, then

(50)′ $$T[f] = cf(p) + S_p[df].$$

We denote δ_p, the ∞-current of X defined by $\delta_p[f] = f(p)$, then since $c = T[1]$ in (50)′, we have

(50) $$T = T[1]\delta_p + dS_p.$$

By (50), we have

Theorem 22. *If X is a closed bounded Banach manifold, Y a closed submanifold of X and satisfies (7) if dim. $Y < \infty$, then*

$(51)_f$ $$\pi^* : H^r(Y, \boldsymbol{C}) \simeq \mathscr{H}^{\infty}(X), \ dim.\, Y = r,$$

$(51)_i$ $$\pi^* : \mathscr{H}^{\infty}(Y) \simeq \mathscr{H}^{\infty}(X), \ dim.\, Y = \infty.$$

Corollary. *Under the same assumptions about X, Y, the homomorphism $\iota^* : \mathscr{H}^{\infty-1}(X) \longrightarrow \mathscr{H}^{\infty-1}(X - Y)$ is onto.*

On the other hand, if X is unbounded or open, the sequences

$$\mathscr{H}^{\infty-1}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-1}(X - Y) \xrightarrow{res.} H^r(Y, \boldsymbol{C}) \longrightarrow 0,$$

$$\mathscr{H}^{\infty-1}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-1}(X - Y) \xrightarrow{res.} \mathscr{H}^{\infty}(Y) \longrightarrow 0$$

are exact. Hence we get

Theorem 22′. *If X is unbounded and Y is also unbounded, then $\iota^* : \mathscr{H}^{\infty-1}(X) \longrightarrow \mathscr{H}^{\infty-1}(X - Y)$ is onto.*

Similarly, we get, for example, denoting S^n the closed submanifold of X which is diffeomorphic to S^n, then

$\iota^* : \mathscr{H}^{\infty-p}(X) \simeq \mathscr{H}^{\infty-p}(X - S^n)$, $p \neq n + 1$,
if X is unbounded and S^n is homologous to 0 in X,

$\iota^* : \mathscr{H}^{\infty-p}(X) \simeq \mathscr{H}^{\infty-p}(X - S^n)$, $p \neq n$,
if X is bounded and S^n is not homologous to 0 in X,

$0 \longrightarrow \mathscr{H}^{\infty-n-1}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-n-1}(X - S^n) \xrightarrow{res.} H^0(S^n, \boldsymbol{C}) \longrightarrow 0$, is exact,
if S^n is homologous to 0 in X,

$0 \longrightarrow H^0(S^n, \boldsymbol{C}) \xrightarrow{\pi^*} \mathscr{H}^{\infty-n}(X) \xrightarrow{\iota^*} \mathscr{H}^{\infty-n}(X - S^n) \longrightarrow 0$, is exact,
if S^n is not homologous to 0 in X.

38. As in $n^0 19$, we can define composed residues for the pair $(X, Y_1, \cdots, Y_n)$, where X is a Banach manifold and $Y_1, \cdots, Y_n$ are closed (orientable) submanifolds of X such that

$$Y_1 \text{ and } Y_2,\ (Y_2\cup\cdots\cup Y_s) \text{ and } (Y_3\cup\cdots\cup Y_s),$$
$$Y_1\cap Y_2 \text{ and } Y_3,\ (Y_3\cup\cdots\cup Y_s) \text{ and } (Y_4\cup\cdots\cup Y_s),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Y_1\cap\cdots\cap Y_{s-1} \text{ and } Y_s \text{ are in general posisions.}$$

Moreover, we assume that Y_i satisfies (7) if $dim.\, Y_i < \infty$. Then denoting the residue maps in the pairs

$$(X-(Y_2\cup\cdots\cup Y_s),\ X-(Y_1\cup Y_2\cup\cdots\cup Y_s),\ Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s)),$$
$$(Y_1-Y_1\cap(Y_3\cup\cdots\cup Y_s),\ Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s),$$
$$Y_1\cap Y_2-Y_1\cap Y_2\cap(Y_3\cup\cdots\cup Y_s)),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(Y_1\cap\cdots\cap Y_{s-1},\ Y_1\cap\cdots\cap Y_{s-1}-Y_1\cap\cdots\cap Y_s,\ Y_1\cap\cdots\cap Y_s),$$

by $res._{Y_1}$, $res._{Y_1\cap Y_2}$, $\cdots$, $res._{Y_1\cap\cdots\cap Y_s}$, we define the composed residue map $res._{Y_1,\cdots,Y_s}$ by

$$res._{Y_1,\cdots,Y_s}=res._{Y_1\cap\cdots\cap Y_s}\cdots res._{Y_1\cap Y_2}res._{Y_1}. \tag{52}$$

By the definition of $res._{Y_1,\cdots,Y_s}$, if $dim.\, Y_1\cap\cdots\cap Y_i < \infty$ for some i, then $res._{Y_1,\cdots,Y_s}$ reduces essentially to the finite dimensional case which has been treated in $n^0 19$. On the other hand, if $dim.\, Y_1\cap\cdots\cap Y_s=\infty$, then $res._{Y_1,\cdots,Y_s}$ is a homomorphism from $\mathscr{H}^{\infty-p}(X-(Y\cup\cdots\cup Y_s))$ into $\mathscr{H}^{\infty-p+s}(Y_1\cap\cdots\cap Y_s)$. But if we set

$$\text{codim.}\, Y_1\cap\cdots\cap Y_i \quad \text{in } Y_1\cap\cdots\cap Y_{i-1}=r_i,$$
$$r_i \text{ is a positive integer or } \infty,\ Y_0=X,$$

then denoting

$$\mathscr{H}^{\infty+r_1+\cdots+r_s-p}(X) \text{ instead of } \mathscr{H}^{\infty-p}(X),$$
$$\mathscr{H}^{\infty+r_2+\cdots+r_s-p}(Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s)) \text{ instead of}$$
$$\mathscr{H}^{\infty-p}(Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s)),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$res._{Y_1,\cdots,Y_s}$ is the homomorphism

$$(53)\qquad res._{Y_1,\ \cdots,\ Y_s}: \mathscr{H}^{\infty+r_1+\cdots+r_s-p}(X-(Y_1\cup\cdots\cup Y_s))\longrightarrow$$
$$\longrightarrow\mathscr{H}^{\infty-p+s}(Y_1\cap\cdots\cap Y_s).$$

By the definition of res. $Y_1, \cdots, Y_s$, if the sequence of differential forms $\{\varphi, \varphi_1, \cdots, \varphi_s\}$, $\varphi\in\mathscr{D}({}^pX)$, $\varphi_1\in\mathscr{D}^{p-1}(Y_1)$, $\cdots$, $\varphi_s\in\mathscr{D}^{p-s}(Y_1\cap\cdots\cap Y_s)$ satisfies

$$\varphi\mid Y_1-Y_1\cap(Y_2\cup\cdots\cup Y_s)=d\varphi_1,$$
$$\varphi_1\mid Y_1\cap Y_2-Y_1\cap Y_2\cap(Y_3\cup\cdots\cup Y_s)=d\varphi_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\varphi_{s-2}\mid Y_1\cap\cdots\cap Y_{s-1}-Y_1\cap\cdots\cap Y_{s-1}\cap Y_s=d\varphi_{s-1},$$
$$\varphi_{s-1}\mid Y_1\cap\cdots\cap Y_s=d\varphi_s,$$

we get

$$(54)\qquad T[d\varphi]=S[\varphi_s],\ \langle S\rangle=res._{Y_1,\ \cdots,\ Y_s}\langle T\rangle.$$

Similarly, if X is a Banach analytic space, $Y_1, \cdots, Y_s$ are its closed analytic subvarieties which satisfy similar conditions as before, we can define the composed residue homomorphism, for example,

$$(53)'\qquad res._{Y_1,\ \cdots,\ Y_s}: \mathscr{H}^{\infty+r_1+\cdots+r_s-p}(X-(Y_1\cup\cdots\cup Y_s))\longrightarrow$$
$$\longrightarrow R^{\infty-p+s}(Y_1\cap\cdots\cap Y_s),$$

if *dim.* $(Y_1\cap\cdots\cap Y_s)=\infty$, *codim.* $(Y_1\cap\cdots\cap Y_i)$ in $Y_1\cap\cdots\cap Y_{i-1}=r_i$, r_i is a positive integer or ∞ and denote $\mathscr{H}^{\infty+r_{i-1}+\cdots+r_s-p}(Y_1\cap\cdots\cap Y_i-Y_1\cap\cdots\cap(Y_{i+1}\cap\cdots\cap Y_s))$ instead of $\mathscr{H}^{\infty-p}(Y_1\cap\cdots\cap Y_i-Y_1\cap\cdots\cap Y_i\cap(Y_{i+1}\cup\cdots\cup Y_s))$.

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