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A Note on the Uniform Distribution of Sequences of Integers

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1. Let $A=(a_n)$ be an infinite sequence of integers not necessarily different from each other. For any integers j and $m\geqq 2$ we denote by $A_N(j, m)$ the number of terms a_n $(1\leqq n\leqq N)$ satisfying the condition $a_n\equiv j \pmod m$. Following I. NIVEN [2], we say that the sequence A is uniformly distributed (mod m) if the limit

$$\lim_{N\to\infty}\frac{1}{N}A_N(j, m)=\frac{1}{m}$$

exists for all integers j, $1\leqq j\leqq m$. If A is uniformly distributed (mod m) for every integer $m\geqq 2$, A is said to be uniformly distributed.

NIVEN [2] has shown that the sequence $([n\theta])$ with irrational θ is uniformly distributed. Indeed, this fact is equivalent to the well-known BOHL-SIERPIŃSKI-WEYL theorem to the effect that the sequence $(n\theta)$ with irrational θ is uniformly distributed (mod 1).

We have proved in [3] the following theorem which is the analogue of the celebrated WEYL criterion for the uniform distribution (mod 1) of sequences of real numbers.

Theorem 1. *Let $A=(a_n)$ be a sequence of integers. A necessary and sufficient condition for the sequence A to be uniformly distributed* (mod m), $m\geqq 2$, *is that*

$$\lim_{N\to\infty}\frac{1}{N}S_N\left(A, \frac{h}{m}\right)=0$$

for all integers h, $1\leqq h\leqq m-1$, where

$$S_N(A, t)=\sum_{n=1}^{N}e(a_nt), \quad e(t)=e^{2\pi it}.$$

In fact, we have

$$S_N\left(A, \frac{h}{m}\right)=\sum_{j=1}^{m}A_N(j, m)e\left(\frac{jh}{m}\right),$$

so that

$$\sum_{h=1}^{m-1}\left|S_N\left(A,\ \frac{h}{m}\right)\right|^2 = m\sum_{j=1}^{m}\left(A_N(j,\ m)-\frac{N}{m}\right)^2.$$

Theorem 1 follows from this relation at once.

The next result is due to NIVEN [2, Theorem 5.1] :

Theorem 2. *If a sequence* $A=(a_n)$ *of integers is uniformly distributed* (mod m), $m\geqq 2$, *then* A *is uniformly distributed* (mod d) *for all divisors* d *of* m, $d\geqq 2$.

In order to see this it will suffice, by virtue of Theorem 1, to observe that, for $d\,|\,m$, $d\geqq 2$, each of the fractions k/d $(1\leqq k\leqq d-1)$ appears at least once among the fractions h/m $(1\leqq h\leqq m-1)$.

2. On the other hand, W. NARKIEWICZ [1] introduced the concept of weakly uniform distribution of sequences of integers as follows. Let $A=(a_n)$ be an infinite sequence of integers. Let m be an integer $\geqq 3$ and suppose that the number $S_N(m)$ of terms a_n $(1\leqq n\leqq N)$ with $(a_n,\ m)=1$ tends to infinity with N. Then, the sequence A is said to be weakly uniformly distributed (mod m) if for every integer j with $(j,\ m)=1$ one has

$$\lim_{N\to\infty}\frac{A_N(j,\ m)}{S_N(m)}=\frac{1}{\phi(m)},$$

where $A_N(j,\ m)$ is as before the number of terms $a_n\equiv j$ (mod m), $1\leqq n\leqq N$, and $\phi(m)$ is the Euler totient function. If A is weakly uniformly distributed (mod m) for every integer $m\geqq 3$, we say that A is weakly uniformly distributed.

Thus the sequence (p_n) of all prime numbers (arranged in increasing order) is weakly uniformly distributed, as is readily seen from the prime number theorem for arithmetical progressions.

Now, let $A=(a_n)$ be a sequence of integers and put for any residue character χ (mod m), $m\geqq 3$,

$$S_N(A,\ \chi)=\sum_{n=1}^{N}\chi(a_n).$$

If $\chi=\chi_0$ is the principal character (mod m), then

$$S_N(A,\ \chi_0)=\sum_{\substack{n=1\\(a_n,m)=1}}^{N}1=S_N(m).$$

Since we have for each χ (mod m)

$$S_N(A,\ \chi)=\sum_{j=1}^{m}A_N(j,\ m)\chi(j),$$

we find

$$\sum_{\substack{\chi(\mathrm{mod}\ m) \\ \chi \neq \chi_0}} \left| S_N(A,\ \chi) \right|^2 = \phi(m) \sum_{\substack{j=1 \\ (j,\ m)=1}}^{m} \left(A_N(j,\ m) - \frac{S_N(m)}{\phi(m)} \right)^2 .$$

Thus we have proved the following result of NARKIEWICZ [1].

Theorem 3. *Let* $A=(a_n)$ *be a sequence of integers. A necessary and sufficient condition for the sequence* A *to be weakly uniformly distributed* (mod m), *where* $m \geqq 3$, *is that*

$$\lim_{N\to\infty} S_N(A,\ \chi_0) = \infty$$

and

$$\lim_{N\to\infty} \frac{S_N(A,\ \chi)}{S_N(A, \chi_0)} = 0$$

for all characters $\chi \neq \chi_0$ (mod m).

There is an analogue of Theorem 2 for weakly uniform distribution of integers. In order to state our result we first introduce the notion of k-weakly uniform distribution of sequences of integers, where k is a positive integer.

Let $A=(a_n)$ be a sequence of integers. Let k be a positive integer and let $A^{(k)}$ be the subsequence of A consisting of all a_n in A with $(a_n,\ k)=1$. The sequence A is said to be k-weakly uniformly distributed (mod m), where $m \geqq 3$, if the sequence $A^{(k)}$ is weakly uniformly distributed (mod m). If A is k-weakly uniformly distributed (mod m) for every $m \geqq 3$, A is said to be k-weakly uniformly distributed.

Again, the sequence (p_n) of all prime numbers is k-weakly uniformly distributed for every fixed $k \geqq 1$.

It will be clear that each of 1-weakly and m-weakly uniform distributions (mod m), $m \geqq 3$, is equivalent to weakly uniform distribution (mod m). Also, weakly uniform distribution (mod $[k,\ m]$), where $k \geqq 1$ and $m \geqq 3$, implies k-weakly uniform distribution (mod m), but not conversely. (Here we denote by $[a,\ b]$ the least common multiple of two integers a and b.)

For k-weakly uniform distribution of sequences of integers our Theorem 3 takes the following form :

Theorem 3 *bis. A necessary and sufficient condition that a sequence* $A=(a_n)$ *of integers be* k*-weakly uniformly distributed* (mod m), *where* $k \geqq 1$ and $m \geqq 3$, *is that*

$$\lim_{N\to\infty} S_N(A^{(k)}, \chi_0) = \infty$$

and

$$\lim_{N\to\infty}\frac{S_N(A^{(k)},\ \chi)}{S_N(A^{(k)},\chi_0)}=0$$

for all characters $\chi\neq\chi_0$ (mod m).

We are now able to formulate the analogue of Theorem 2.

Theorem 4. *If a sequence* $A=(a_n)$ *of integers is weakly uniformly distributed* (mod m), $m\geqq 3$, *then* A *is* m*-weakly uniformly distributed* (mod d) *for all divisors* d *of* m, $d\geqq 3$.

Note that for $d\mid m$ a reduced residue system (mod m) induces in the natural way a reduced residue system (mod d) with multiplicity $\phi(m)/\phi(d)$. Theorem 4 follows from this directly, or via Theorem 3.

3. We now study a little the relation between uniform distribution and weakly uniform distribution of sequences of integers.

Theorem 5. *If a sequence of integers* (a_n) *is uniformly distributed* (mod m), $m\geqq 3$, *then the sequence* (a_n+c) *is weakly uniformly distributed* (mod m) *for all integers* c.

Proof. By Theorem 2, it follows from the assumption on the sequence (a_n) that (a_n) is uniformly distributed (mod d) for all $d\mid m$, $d\geqq 2$ (and also for $d=1$ trivially), and, if we write for any integer j

$$A_N(j,\ d)=\frac{N}{d}+R_N(j,\ d),$$

then

$$R_N(j,\ d)=o(N)\ \text{for}\ N\to\infty.$$

We have, therefore, for any integer c

$$\begin{aligned}S_N(c\,;\ m)&\overset{\text{def}}{=}\sum_{\substack{n=1\\(a_n+c,m)=1}}^{N}1=\sum_{n=1}^{N}\sum_{d\mid(a_n+c,m)}\mu(d)\\&=\sum_{d\mid m}\mu(d)\sum_{\substack{n=1\\a_n+c\equiv0(d)}}^{N}1=\sum_{d\mid m}\mu(d)A_N(-c,\ d)\\&=\sum_{d,m}\mu(d)\left(\frac{N}{d}+R_N(-c,\ d)\right)\\&=\frac{\phi(m)}{m}N+\sum_{d\mid m}\mu(d)R_N(-c,\ d),\end{aligned}$$

where $\mu(d)$ is the Möbius function, so that

$$\frac{S_N(c;\ m)}{\phi(m)} = \frac{N}{m} + o(N) \quad \text{for } N \to \infty.$$

Hence, if we write $A_N(c;\ j,\ m)$ for the number of integers $a_n + c \equiv j \pmod{m}$, $1 \leqq n \leqq N$, then we have

$$\begin{aligned} A_N(c;\ j,\ m) &= A_N(j-c,\ m) \\ &= \frac{N}{m} + o(N) \\ &= \frac{S_N(c;\ m)}{\phi(m)} + o(S_N(c;\ m)) \end{aligned}$$

for $N \to \infty$. This last relation holds for any integer j and, *a fortiori*, for any integer j with $(j,\ m) = 1$. Thus the sequence $(a_n + c)$ is weakly uniformly distributed (mod m).

A converse of Theorem 5 is the following

Theorem 6. *Let m be an integer $\geqq 3$ and suppose that a sequence of integers (a_n) be such that the sequence $(a_n + c)$ is weakly uniformly distributed* (mod m) *for all integers c, $0 \leqq c \leqq m-1$. Then, if m is odd, the sequence (a_n) is uniformly distributed* (mod m); *whereas if m is even, the sequence (a_n) is not necessarily uniformly distributed* (mod m).

Proof. Case of odd m. From the assumption on the sequence (a_n) it follows that the sequence (a_n+c) is weakly uniformly distributed (mod m) for any integer c. Let c be an arbitrary integer and j be any integer with $(j,\ m)=1$. Again, we denote by $A_N(c;j,\ m)$ the number of integers $a_n + c \equiv j \pmod{m}$, $1 \leqq n \leqq N$. Then we have, by assumption,

$$A_N(c;\ j,\ m) = \frac{S_N(c;\ m)}{\phi(m)} + Q_N(c;\ j,\ m),$$

where

$$S_N(c;\ m) = \sum_{\substack{n=1 \\ (a_n+c,\, m)=1}}^{N} 1 \leqq N$$

and

$$Q_N(c;\ j,\ m) = o(S_N(c;\ m)) = o(N) \quad \text{for } N \to \infty.$$

Now we have for $N \to \infty$

$$A_N(c;\ 1,\ m) = \frac{S_N(c;\ m)}{\phi(m)} + o(N)$$

and, since $(2, m) = 1$,

$$A_N(c+1;\ 2,\ m) = \frac{S_N(c+1;\ m)}{\phi(m)} + o(N).$$

Hence, on noticing that $A_N(c;\ 1,\ m) = A_N(c+1;\ 2,\ m)$ for all N we find

$$S_N(c;\ m) = S_N(c+1;\ m) + o(N),$$

so that

$$\sum_{a=0}^{m-1} S_N(a;\ m) = mS_N(c;\ m) + o(N)$$

for any particular c. Therefore, it follows from the relation

$$N = \sum_{a=0}^{m-1} A_N(a;\ j,\ m) = \frac{1}{\phi(m)} \sum_{a=0}^{m-1} S_N(a;\ m) + o(N),$$

where j is any integer with $(j, m) = 1$, that

$$\frac{S_N(c;\ m)}{\phi(m)} = \frac{N}{m} + o(N)$$

or

$$A_N(c;\ j,\ m) = \frac{N}{m} + o(N).$$

Since the integer c is arbitrary and $A_N(c;\ j,\ m) = A_N(j-c,\ m)$, we obtain for $N \to \infty$

$$A_N(j,\ m) = \frac{N}{m} + o(N)$$

for all j, $1 \leqq j \leqq m$. Thus the sequence (a_n) is uniformly distributed (mod m).

Case of even m. Consider the sequence (a_n) of integers obtained by arranging in increasing order from the set of non-negative integers $M_0 \cup M_1$, where

$$M_0 = \{a \equiv 0,\ 2,\ 4, \cdots,\ m-2 \pmod{m}\}$$

and

$$M_1 = \{a \equiv 1,\ 3,\ 5,\ \cdots,\ m-1 \pmod{2m}\}.$$

One may easily verify that the sequnce $(a_n + c)$ is weakly uniformly distributed (mod m) for all integers c, but (a_n) is not uniformly distributed (mod m).

References

[1] NARKIEWICZ, W. : On distribution of values of multiplicative functions in residue classes. Acta Arith., **12** (1967), 269-279.

[2] NIVEN, I. : Uniform distribution of sequences of integers. Trans. Amer. Math. Soc., **98** (1961), 52-61.

[3] UCHIYAMA, S. : On the uniform distribution of sequences of integers. Proc. Japan Acad., **37** (1961), 605-609.