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Characteristic Classes of Connections

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Introduction. It is known that if a vector bundle ξ over X has a connection θ with curvature 0, then ξ is induced from a representation of $\pi_1(X)$. ([1], [2], [5], [9]). This representation χ has been called the characteristic class of ξ ([10]). But χ may be regarded as a characteristic class of θ, because the classification of ξ by χ is too fine if we regard ξ as a differentiable vector bundle. (cf. [5]). In this point of view, it is natural to define the characteristic classes of connections which have the same curvature. The purpose of this note is to define this class. In fact, if θ and θ' have the same curvature Θ then the characteristic class $ch_\theta(\theta')$ of θ' with respect to θ is defined to be an element of $H^0(\pi_1(X_\Theta), \ker. m_\theta)$. (The detailed definitions of $H^0(\pi_1(X_\Theta), \ker. m_\theta)$ and X_Θ shall be given in § 2 and § 3). Moreover, we have

$$ch_\theta(\theta') = 1 \text{ if and only if}$$

$$\theta_U' = h_U f(\theta_U - (h_U f)^{-1} d(h_U f))(h_U f)^{-1},$$

where f is a regular matrix valued smooth function on X and h_U satisfies

$$dh_U = h_U\theta_U - \theta_U h_U$$

We note that by the uniqueness of the solutions of a linear differential equation, h_U is determined by its value at one point of U. Hence if $\xi = \{g_{UV}\}$ and $\xi' = \{g_{UV}'\}$ have connections $\theta = \{\theta_U\}$ and $\theta' = \{\theta_U'\}$ such that $ch_\theta(\theta') = 1$, then θ and θ' are equivalent as differentiable vector bundles. Moreover, this equivalence is finer than usual equivalence. In fact, if $\theta = 0$, then ξ and ξ' are equivalent in this sense, if and only if the representations of $\pi_1(X)$ by which ξ and ξ' are induced, are equivalent.

From this point of veiw, although the usual classification of the set of pairs $(\{g_{UV}\}, \{\theta_U\})$, where $\{g_{UV}\}$ is a vector bundle and $\{\theta_U\}$ is its connection, should be (cf. [3])

$$(\{g_{UV}\}, \{\theta_U\}) \sim (\{g_{UV}'\}, \{\theta_U'\}) \text{ if and only if there exists } h_U \text{ such that}$$

$$g_{UV}' = h_U g_{UV} h_V^{-1}, \quad \theta_U' = h_U(\theta_U - h_U^{-1} dh_U) h_U^{-1},$$

we define another classification of the set of pairs $(\{g_{UV}\}, \{\theta_U\})$ as follows :

$(\{g_{UV}\}, \{\theta_U\}) \underset{S}{\sim} (\{g_{UV}\}', \{\theta_U'\})$ if and only if there exist f and $\{h_U\}$ such that

$$g_{UA}' = h_U f g_{UV}(h_V f)^{-1}, \quad \theta_U' = h_U f(\theta_U - (hf)^{-1}d(h_U f))(h_U f)^{-1},$$

f is defined on X, $dh_U = h_U\theta_U - \theta_U h_U$.

Using these two equivalence relations, we define two types of Grothendieck groups $K_\theta(X)$ and $L_\theta(X)$ in § 5. (for the definition of $K_\theta(X)$, see also [3]). By the definition of the latter equivalence, the representation ring $R(\pi_1(X))$ of $\pi_1(X)$ is regarded as a subgroup of $L_\theta(X)$. Moreover, we have the following exact sequence.

$$0 \longrightarrow R.\text{-}H(X) \xrightarrow{\tau} L_\theta(X) \xrightarrow{j} K_\theta(X) \longrightarrow 0,$$

where j is the natural map from $L_\theta(X)$ onto $K_\theta(X)$, $R.\text{-}H(X)$ is the subgroup of $R(\pi_1(X))$ generated by those representations for which the Riemann -Hilbert's problem is solvable. (cf. [5]). The maps from $K_\theta(X)$ (and $L_\theta(X)$) onto $K(X)$ are also treated in § 6.

The definition of $ch_\theta(\theta')$ is based on the property of the equation

$$df = f\theta - \theta' f.$$

Therefore we study this equation in § 1 and § 2.

§ 1. A differential equation.

Let θ be a matrix valued 1-form over X, a smooth manifold, then we set

(1) $$m(\theta) = d\theta + \theta_\wedge\theta.$$

Using local coordinate $\{x_i\}$ of X, θ and $m(\theta)$ are given by

$$\theta = \sum\theta_i dx_i, \quad m(\theta) = \sum_{i<j}\Theta_{ij}dx_i{}_\wedge dx_j.$$

If h is a regular matrix valued smooth function, then (cf. [2]),

(2) $$m(h(\theta - h^{-1}dh)h^{-1}) = hm(\theta)h^{-1}.$$

We fix a coordinate neighborhood U of X and identify it with a bounded neighborhood of the origin of $\boldsymbol{R}^n$, the euclidean n-space.

For a boundded matrix valued function A on U, we set

$$I_0(A, \theta_i, \theta_i') = A,$$

$$I_m(A, \theta_i, \theta_i')$$

$$= \int_0^{x_i} \{I_{m-1}(A, \theta_i, \theta_i')\theta_i - \theta_i' I_{m-1}(A, \theta_i, \theta_i')\}_{x_i=t}\, dt.$$

For the simplicity, we often denote I_m instead of $I_m(A,\ \theta_i,\ \theta_i')$.

Then since we get

$$||I_m(A,\ \theta_i,\ \theta_i')|| \leqq CM^m/(m-1)! \quad \text{for suitable constants } C \text{ and } M,$$

$$(||B(x)|| = \max_{x\in U} \sqrt{\sum_{i,j} |B_{ij}(x)|^2}\ ),$$

the series $\sum_{m\geqq 0} I_m(A,\ \theta_i,\ \theta_i')$ converges absolutely and uniformly on U. Hence we may set

$$P(A,\ \theta_i,\ \theta_i') = \sum_{m\geqq 0} I_m(A,\ \theta_i,\ \theta_i').$$

Note. If $\partial A/\partial x_i = 0$, then $P(A,\ \theta_i,\ \theta_i')$ is the solution of the equation

$$\frac{\partial h}{\partial x_i} = h\theta_i - \theta_i' h$$

with data $A|_{x_i=0} = h|_{x_i=0}$. Therefore $P(0,\ \theta_i,\ \theta_i')$ is equal to 0 on U.

Lemma 1. *If* $\partial A/\partial x_i = \partial A/\partial x_j = 0$, then

$$\begin{aligned}(3)\qquad &\frac{\partial}{\partial x_j}(P(A,\ \theta_i,\ \theta_i')) \\ &= P(A,\ \theta_i,\ \theta_i')\theta_j - \theta_j' P(A,\ \theta_i,\ \theta_i') - \\ &\quad - P(P(A,\ \theta_i,\ \theta_i')\Theta_{ij} - \Theta_{ij}' P(A,\ \theta_i,\ \theta_i'),\ \theta_i,\ \theta_i').\end{aligned}$$

Proof, Since we know

$$\Theta_{ij} = \frac{\partial\theta_j}{\partial x_i} - \frac{\partial\theta_i}{\partial x_j} + [\theta_i,\ \theta_j],\ ([\theta_i,\ \theta_j] = \theta_i\theta_j - \theta_j\theta_i),$$

we get

$$\frac{\partial\theta_i}{\partial x_j} = \frac{\partial\theta_j}{\partial x_i} + [\theta_i,\ \theta_j] - \Theta_{ij}.$$

Hence we obtain

$$\begin{aligned}\frac{\partial I_m}{\partial x_j} &= I_{m-1}\theta_i - \theta_j' I_{m-1} - \int_0^{x_i} (I_{m-1}\Theta_{ij} - \Theta_{ij}' I_{m-1})_{x_i=t}\, dt + \\ &+ \int_0^{x_i} (I_{m-1}[\theta_i,\ \theta_j] - [\theta_i',\ \theta_j'] I_{m-1} - I_{m-2}\theta_i\theta_j + \\ &+ \theta_i' I_{m-2}\theta_j + \theta_j' I_{m-2}\theta_i - \theta_j'\theta_i' I_{m-2})_{x_i=t} dt + \\ &+ \int_0^{x_i} \Big(\frac{\partial I_{m-1}}{\partial x_j}\theta_i - \theta_i' \frac{\partial I_{m-1}}{\partial x_j}\Big)_{x_i=t} dt.\end{aligned}$$

Then since $\partial A/\partial x_i = \partial A/\partial x_j = 0$, we have

$$\frac{\partial}{\partial x_j}\Big(\sum_{k=0}^{m} I_m(A,\ \theta_i,\ \theta_i')\Big)$$

$$= \sum_{k=0}^{m-1} (I_k(A, \theta_i, \theta_i')\theta_j - \theta_j' I_k(A, \theta_i, \theta_i')) -$$

$$- \sum_{s=0}^{m-1} \sum_{t=0}^{m-s-1} I_s(I_t\Theta_{ij} - \Theta_{ij}I'_t, \theta_j, \theta_i') +$$

$$+ \sum_{s+t=m-1} (-1)^s I_s(I_t[\theta_i, \theta_j] - [\theta_i', \theta_j']I_t, \theta_i', \theta_j').$$

Then, since we know

$$\| \sum_{s+t=m-1} (-1)^s I_s(I_t[\theta_i, \theta_j] - [\theta_i', \theta_j']I_t, \theta_i', \theta_j') \|$$

$$\leqq \sum_{s+t=m-1} BN^{m-1}/(s-1)!\,(t-1)! \qquad \text{(for suitable } B \text{ and } N)$$

$$\leqq mBN^{m-1}/[\frac{m-1}{2}]!, \qquad ([\ \ ] \text{ is the Gauss' notation}),$$

we have the lemma.

Corollary 1. *If U is simply connected, then the equation*

(4) $$dh = h\theta - \theta' h$$

has the solution on U for the data

$$h(0) = A, \ A \text{ is a constant matrix},$$

if and only if

$$P(A, \theta_i, \theta_i')m(\theta) = m(\theta')P(A, \theta i, \theta_i').$$

Corollary 2. (cf. [8], 7.4). *If $m(\theta) = m(\theta')$ on U, a simply connected open set of X, then there is a regular matrix valued amooth function h on U such that*

(5) $$\theta' = h(\theta - h^{-1}dh)h^{-1}, \ hm(\theta)h^{-1} = m(\theta).$$

§ 2. The case $\theta = \theta'$.

In this §, we consider the equation

(4)' $$df = f\theta - \theta f.$$

Lemma 2. (i). *If f, g are the solutions of* (4)' *then fg is also a solution of* (4)'.

(ii). *If f is a solution of* (4)' *and f^{-1} exists on U, then f^{-1} is also a solution of* (4)'.

Proof. Since we get

$$d(fg) = dfg + fdg = (f\theta - \theta f)g + f(g\theta - \theta g),$$

$$d(f^{-1}) = -f^{-1}dff^{-1} = -f^{-1}(f\theta - \theta f)f^{-1},$$

we obtain the lemma.

Corollary 1. *If U is simply connected and f is a solution of* (4)$'$ *such that $f(x)$ is regular at x, an arbitrary point of U, then f is regular on U.*

Proof. Since$f$ is continuous, f^{-1} exists on $V(x)$, a neighborhood of x. Then f^{-1} is extended on U, because it is a solution of (4)$'$ with the data $f^{-1}(x) = f(x)^{-1}$ and U is simply connected. We denote this extension by g. Since gf is a solution of (4)$'$ with the data 1, the identity matrix, at x, it must be equal to 1 on U by the uniqueness of the solution of (4)$'$. Hence we have the corollary.

Corollary 2. *If f is a solution of (4)$'$ such that $f(x)$ is regular at x, an arbitrary point of X, then f is regular on X.*

By lemma 2, (ii), the set of all regular solutions of (4)$'$ form a group. We denote this group by ker. m_θ, or ker.$_X$ m_θ, if the equation (4)$'$ in considered on X.

Note. If X is not simply connected, then the solutions of (4)$'$ may be multivalent and ker.$_X m_\theta$ may contain these multivalent solutions.

By definition, ker.$_X$ m_θ may considered to be a subgroup of GL$(n, \boldsymbol{F})$, $\boldsymbol{F}$ is $\boldsymbol{R}$ or $\boldsymbol{C}$, as an abstract group and has π_1 (X) to be ano perator group.

Definition. *A function χ from π_1 (X) into* ker.$_X$ m_θ *is called a* 0*-cocycle if it satisfies*

(6) $\chi_\sigma\sigma(\chi_\tau) = \chi_{\sigma\tau}$, χ_σ is the value of χ at σ and $\sigma(\chi_\tau)$ is the transform of χ_τ by σ.

Definition. *Two cocycles χ and χ' are said to be equivalent if and only if there exists an element f of ker.$_X$ m_θ such that*

(7) $$\chi_\sigma' = f^{-1}\chi_\sigma\sigma(f).$$

Definition. *The set of the all equivalence classes of 0-cocycles by the relation* (7) *is denoted by H^0 $(\pi_1(X)$,* ker.$_X$ $m_\theta)$.

Definition. *We call $m(\theta)$ and $m(\theta')$ are equivalent and denote*

$$m(\theta) \sim m(\theta'),$$

if there exists $\{h_U\}$, h_U is a regular matrix valued smooth function on U, such that

$$m(\theta_U') = h_U m(\theta_U) h_U^{-1}$$

for all U.

By corollary 2 of lemma 1, if $m(\theta) \sim m(\theta')$, then there exists $h' = \{h_U'\}$ such that

$$\theta_U' = h_U'(\theta_U - h_U'^{-1}dh_U')h_U'^{-1}.$$

Definition. *We call $m(\theta)$ and $m(\theta')$ are strongly equivalent on X or simply, strongly equivalent if*

$$m(\theta_U')=fm(\theta_U)f^{-1}$$

where f is a regular matrix valued smooth function on X and does not depend on X.

If $m(\theta)$ and $m(\theta')$ are strongly equivalent on X then we denote

$$m(\theta)\underset{S}{\sim}m(\theta').$$

Lemma 3. *If $m(\theta)\underset{S}{\sim}m(\theta')$, then there is a bijection of $H^0({}_1(\pi X)$, $\ker._X\ m_\theta)$ and $H^0\ (\pi_1(X)$, $\ker._X\ m_\theta')$.*

Proof. By assumption, there exists a regular matrix valued smooth function h such that

$$\theta'=h(\theta-h^{-1}dh)h^{-1}=\sigma(h)(\theta-\sigma(h)^{-1}d\sigma(h))\sigma(h)^{-1},\quad \sigma\in\pi_1(X).$$

Then since we get

$$\begin{aligned}&d(hf\sigma(h)^{-1})\\&=dhf\sigma(h)^{-1}+hf\theta\sigma(h)^{-1}-h\theta f\sigma(h)^{-1}-hf\sigma(h)^{-1}d\sigma(h)\sigma(h)^{-1}\\&=hf\sigma(h)^{-1}(\sigma(h)\theta\sigma(h)^{-1}-d\sigma(h)\sigma(h)^{-1}-(h\theta h^{-1}-dhh^{-1})hf\sigma(h)^{-1},\end{aligned}$$

the correspondence

$$h^{\#}:\ \ker._X m_\theta\to\ker._X m_\theta'$$

defined by

$$h^{\#}(f)=hf\sigma(h)^{-1}$$

is a bijection from $\ker._X m_\theta$ to $\ker._X\ m_\theta'$. Then by the definition of $h^{\#}$ and (7), $h^{\#}$ induces a bijection h^* from $H^0(\pi_1(X)$, $\ker._X\ m_\theta)$ to $H^0(\pi_1(X)$, $\ker._X\ m_\theta')$. Hence we have the lemma.

§ 3. Connections which have the same curvature.

We assume the curvature forms of $\theta=\{\theta_U\}$ and $\theta'=\{\theta_U'\}$ are strongly equivalent. Then we may assume that they have the same curvature form $\Theta=\{\Theta_U\}$. We assume θ is a connection of ξ, θ' is a connection of ξ' and H is the (linear) Lie group whose Lie algebra is generated by Θ. We denote the associated H-bundle of ξ (or ξ') by X_Θ. Then by Ambrose -Singer's theorem ([1], [2], [9]), if we denote the universal covering space of X_Θ by $\tilde{X}_\Theta$ and the projection from $\tilde{X}_\Theta$ to X by p, $p^*\ (\xi)$ and $p^*\ (\xi')$ are both trivial bundles. Moreover θ and θ' induce global forms on X_Θ. These forms are also denoted by θ and θ'. We set

$$\theta' = \theta - \eta.$$

Then since $\tilde{X}_\Theta$ is simply connected, we may set

(8) $$\eta = dhh^{-1} - h\theta h^{-1} + \theta,$$

on X. Here η and θ mean the induced forms from η and θ.

If $\sigma \in \pi_1(X_\Theta)$, then by (8), we get

$$dhh^{-1} - h\theta h^{-1} = d\sigma(h)\sigma(h)^{-1} - \sigma(h)\theta\sigma(h)^{-1}.$$

Hence if we set

$$\sigma(h) = h\chi_\sigma,$$

we obtain

(9)$_{\text{i}}$ $$d\chi_\sigma = \chi_\sigma\theta - \theta\chi_\sigma$$

(9)$_{\text{ii}}$ $$\sigma\chi_\sigma(\chi_\tau) = \chi_{\sigma\tau},\ \sigma,\ \ \tau \in \pi_1(X_\Theta).$$

Moreover, if h' is another solution of (8), then setting

$$h' = hf,\ \ \sigma(h') = h'\chi_\sigma',$$

we have

(10)$_{\text{i}}$ $$df = f\theta - \theta f$$

(10)$_{\text{ii}}$ $$\chi_\sigma' = f^{-1}\chi_\sigma\sigma(f).$$

Hence η, therefore θ', defines an element of $H^0(\pi_1(X_\Theta),\ \ker._{X_\Theta} m_\theta)$.

Definition. *The above defined element is called the characteristic class of θ' with respect to θ, and denoted by $ch_\theta(\theta')$ or $ch(\theta')$.*

Note. $ch_\theta(\theta')$ is defined if $m(\theta) \underset{S}{\sim} m(\theta')$ on $\tilde{X}_\xi$. Here X_ξ means the associated principal bundle of ξ and $\tilde{X}_\xi$ is its universal covering space.

Note. If $\Theta = 0$, then $X_\Theta = X$ and $\ker._X m_0 = GL(n, F)$, hence the characteristic class of a connection with curvature 0 is the characteristic class of a flat bundle in the sense of Steenrod ([10]).

§ 4. Properties of $ch_\theta(\theta')$.

We call $\theta = \{\theta_U\}$ is a connection of $\xi = \{g_{UV}\}$ if

(11) $$dg_{UV} = g_{UV}\theta_V - \theta_U g_{UV}.$$

Definition. *The transformation of $\theta = \{\theta_U\}$ by $h = \{h_U\}$ is the connection of $h_U g_{UV} h_V^{-1}$ given by*

(12) $$h_U(\theta_U - h_U^{-1}dh_U)h_U^{-1}.$$

Then by the calculations of § 3 and the definition of the strong equivalence of curvature forms, we have

Theorem 1. *$ch_\theta(\theta')$ is equal to 1 if and only if θ' is a transformation of θ by $hf=\{h_U f\}$, where h_U satisfies*

$$dh_U = h_U\theta_U - \theta_U h_U,$$

and f is a smooth regular matrix valued function on X.

Here *1* means the map from $\pi_1(X)$ to *1*, the identity matrix. (*1* belongs in $\ker. X_\theta\ m_\theta$ because *1* is a solutionof (4)$'$ for any θ).

Theorem 2. *If θ and θ' are related by*

$$\theta' = h(\theta - h^{-1}dh)h^{-1},$$

then

$$ch_{\theta'}(\varphi) = h^{-1*}(ch_\theta(\varphi)). \tag{13}$$

Here φ is an arbitrary connection such that $m(\varphi)$ is strongly equivalent to $\Theta = m(\theta)$.

Corollary. *$ch_{\theta'}(\theta)$ is equal to $h^{-1*}(ch_\theta(\theta'))$.*

§ 5. The groups $K_\theta(X)$ and $L_\theta(X)$.

We assume that X is a smooth connected manifold. We denote by $E=\{g_{UV}\}$, an its vector bundle (real or complex), and by $\theta=\{\theta_U\}$, a conection of E. We note that

Lemma 4. *If $\theta^1=\{\theta_U{}^1\}$ and $\theta^2=\{\theta_U{}^2\}$ are connections of $E^1=\{g_{UV}{}^1\}$ and $E^2=\{g_{UV}{}^2\}$, then the matrix vblued form $\theta^1\oplus\theta^2$ given by*

$$\theta^1 \oplus \theta^2 = \{\theta_U{}^1 \oplus \theta_U{}^2\}$$

is a connection of $E^1\oplus E^2$.

We identify the pairs $(\{g_{UV}\}, \{\theta_U\})$ and $(\{g_{U'V'}{}'\}, \{\theta_{U'}{}'\})$ if there exists a common refinement $\{U''\}$ of $\{U\}$ and $\{U'\}$ such that

$$g_{UV}|\, U''\cap V'' = g_{U'V'}{}'|\, U''\cap V'', \quad \theta_U|\, U''=\theta_{U'}{}'|\, U''.$$

Here and in the rest, a pair $(\{g_{UV}\}, \{\theta_U\})$ means a pair of vector bundle $\{g_{UV}\}$ and its connection $\{\theta_U\}$ (cf. (11)).

Definition. *The pairs $(\{g_{UV}\}, \{\theta_U\})$ and $(\{g_{UV}{}'\}, \{\theta_U{}'\})$ are said to be equivalent if and only if $\{g_{UV}\}$ and $\{g_{UV}{}'\}$ are equivalent and $\{\theta_U{}'\}$ is the transformation of $\{\theta_U\}$ by h, where $h=\{h_U\}$ gives the equivalence of $\{g_{UV}\}$ and $\{g_{UV}{}'\}$.*

Definition. *The pairs $(\{g_{UV}\}, \{\theta_U\})$ and $(\{g_{UV}{}'\}, \{\theta_U{}'\})$ are said to be strongly equivalent if and only if $(\{g_{UV}\}, \{\theta_U\})$ and $(\{g_{UV}{}', \{\theta_U{}'\})$ are equivalent and $hf=\{h_U f\}$, by which the equivalence is given, satisfies*

$$dh_U = h_U\theta_U - \theta_U h_U,$$

f is a smooth regular matrix valued function on X.

Then by theorem 1, we get

Theorem 3. *$(\{g_{UV}\}, \{\theta_U\})$ and $(\{g_{UV}'\}, \{\theta_U'\})$ are strongly equivalent if and only if*

(14) $$m(\theta) \underset{S}{\sim} m(\theta') \text{ on } X_\Theta, \ \Theta = m(\theta),$$

$$ch_\varphi(\theta) = ch_\varphi(\theta'),$$

where $\theta = \{\theta_U\}$, $\theta' = \{\theta_U\}'$ and $\varphi = \{\varphi_U\}$ is an arbitrary connection with curvature $m(\theta)$.

We denote

$(\{g_{UV}\}, \{\theta_U\}) \sim (\{g_{UV}'\}, \{\theta_U\}')$ if they are strongly equivalent,

$(\{g_{UV}\}, \{\theta_U\}) \underset{S}{\sim} (\{g_{UV}\}', \{\theta_U\}')$ if they are strongly equivalent.

Then we get

(15) $$(\{g_{UV}^1 \oplus g_{UV}^2\}, \{\theta_U^1 \oplus \theta_U^2\}) \sim (\{g_{UV}^{1'} \oplus g_{UV}^{2'}\}, \{\theta_U^{1'} \oplus \theta_U^{2'}),$$

if $(\{g_{UV}^1\}, \{\theta_U^1\}) \sim (\{g_{UV}^{1'}\}, \{\theta_U^{1'}\})$

and $(\{g_{UV}^2\}, \{\theta_U^2\}) \sim (\{g_{UV}^{2'}\}, \{\theta_U^{2'}\})$.

$(15)_S$ $$(\{g_{UV}^1 \oplus g_{UV}^2\}, \{\theta_U^1 \oplus \theta_U^2\}) \underset{S}{\sim} (\{g_{UV}^{1'} \oplus g_{UV}^{2'}\}, \{\theta_U^{1'} \oplus \theta_U^{2'}\}),$$

if $(g_{UV}^1\}, \{\theta_U^1\}) \underset{S}{\sim} (\{g_{UV}^{1'}\}, \{\theta_U^{1'}\})$

and $(\{g_{UV}^2\}, \{\theta_U^2\}) \sim (\{g_{UV}^{2'}\}, \{\theta_U^{2'}\})$.

By (15) and $(15)_S$, we can define the Grothendieck groups generated by the equivalence (or strong equivalence) classes of $(\{g_{UV}\}, \{\theta_U\})$.

Definition. *The Grothendieck group generated by the equivalence classes of $(\{g_{UV}\}, \{\theta_U\})$ is denoted by $K_\theta(X)$.*

Definition. *The Grothendieck group generated by the strong equivalence classes of $(\{g_{UV}\}, \{\theta_U\})$ is denoted by $L_\theta(X)$.*

§ 6. The mappings ρ, σ, $\hat{\rho}$, $\hat{\sigma}$ and j.

By the definitions of $L_\theta(X)$ and $K_\theta(X)$, there is a homomorphism of $L_\theta(X)$ onto $K_\theta(X)$ defined by

(16) $$j((\{g_{UV}\}, \{\theta_U\})) = (\{g_{UV}\}, \{\theta_U\}).$$

Here the $(\{g_{UV}\}, \{\theta_U\})$ at the left hand side means its class in $L_\theta(X)$ and at the right hand side, it means the clas of $(\{g_{UV}\},\ \{\theta_U\})$ in $K_\theta(X)$.

We denote the representation ring of $\pi_1(X)$ over $\boldsymbol{F}$ ($\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$) by $R_F(\pi_1(X))$ or $R(\pi_1((X))$.

By the definition of the strong equivalence, we get

Lemma 5. *There is an isomorphism τ of $R_F(\pi_1(X))$ into $L_\theta(X)$. ($\boldsymbol{F}$ is $\boldsymbol{R}$ if $L_\theta(X)$ is generated by real vector bundles and their connnections and $\boldsymbol{F}$ is $\boldsymbol{C}$ if $L_\theta(X)$ is generated by complex vector bundles and their connectios).*

Definition. *The subgroup of $R_F(\pi_1(X))$ generated by those representations such that the Riemann -Hilbert's problem given by them are solvable, is called the Riemann -Hilbert group of X and denoted by $R.\text{-}H._F(X)$ or $R.\text{-}H.(X)$.*

Since we know that a vector bundle which is defined by a representation χ of π_1 (X), is trivial if and only if the Riemann -Hilbert's problem given by χ is solvable (cf. [5]), we obtain

Theorem 4. *We have*

$$\text{ker. } j=\tau(R.\text{-}H.(X)). \tag{17}$$

Note. By Chern's theorem and Peterson's theorem, we get

$$R.\text{-}H._C(X)=R_C(\pi_1(X)),$$

if X has no torsion.

Definition. *We denote*

$$K_\Theta(X)=K_\theta(X)/j_0\tau(R_F(\pi_1(X)))\simeq L_\theta(X)/\tau(R_F(\pi_1(X))). \tag{18}$$

By definition, $K_\Theta(X)$ is the Grothendieck group generated by the equivalence classes of curvatures.

Definitino. *We define a homomorphism of $K_\theta(X)$ onto $K(X)$ by*

$$\rho((\{g_{UV}\},\ \{\theta_U\}))=(\{g_{UV}\}).$$

Here $(\{g_{UV}\})$ means the class of $\{g_{UV}\}$ in $K(X)$.

We also set

$$\hat{\rho}=\rho_0 j.$$

By definition, $\hat{\rho}$ is a homomorphism of $L_\theta(X)$ onto $K(X)$.

On the other hand, if we denote the cotangent bundle of X by T^*, then we have the sequence

$$\Gamma(E)\xrightarrow{D}\Gamma(E\otimes T^*)D\xrightarrow{D}\cdots\cdots\xrightarrow{D}\Gamma(E\otimes\Lambda^n T^*), \tag{19}$$

$$D=d+\theta,\ \ n=\text{dim. } X$$

X. Here $\Gamma(E)$, etc. mean the groups of C^∞-cross -sections of bundles E, etc. and E means $\{g_{UV}\}$.

Although the sequence (19) is not a differential complex, its symbol sequence

$$0 \to \pi^*(E) \xrightarrow{\rho(D)} \pi^*(E \otimes T^*) \xrightarrow{\rho(D)} \cdots\cdots \xrightarrow{\rho(D)} \pi^*(E \otimes \Lambda^n T^*) \to 0$$

is exact. ([4], [5]). Here π^* means the projection from the Thom complex $M(X)$ of T^* to X and $\rho(D)$ is the symbol of D. Then setting

$$\begin{aligned} &\sigma((g_{UV},\ \theta_U)) \\ &= d(\pi^*(E),\ \pi^*(E \otimes T^*),\ \cdots\cdots, \pi^*(E \otimes \Lambda^n T^*),\ \sigma(D), \cdots\cdots,\ \sigma(D)), \end{aligned} \tag{20}$$

σ defines a homomorphism of $K_\theta(X)$ onto $\tilde{K}(M(X))$. Here $d(F_1, \cdots\cdots, F_k, \sigma_1, \cdots\cdots, \sigma_k)$ means the difference bundle of $F_1, \cdots\cdots, F_k$. ([6]).

Definition. *σ is called the symbol homomorphism.*

We also set

$$\hat{\sigma} = j_0 \sigma.$$

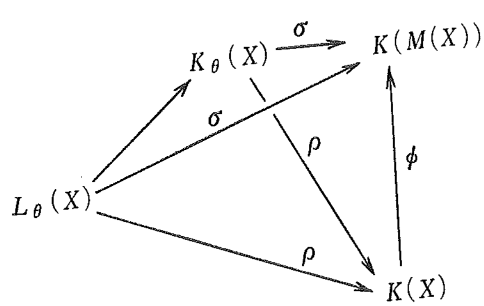
By definition, $\hat{\sigma}$ is a homomorphism of $L_\theta(X)$ onto $\tilde{K}(M(X))$.

We denote the Thom isomorphism from $K(X)$ onto $\tilde{K}(M(X))$ by ϕ. Then we know (cf. [7], [4]),

$$\begin{aligned} &\phi([E]) \\ &= [d(\pi^*(E),\ \pi^*(E\otimes T^*), \cdots\cdots, \pi^*(E\otimes\Lambda^n T^*),\ \sigma(D), \cdots\cdots, \sigma(D))], \end{aligned}$$

where $[E]$ is the class of E in $K(X)$. Hence we obtain the following

Theorem 5. *The following diagram is commutative.*



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