JOUR. FAC. SCI., SHINSHU UNIV. VOl. 3, No. 1, pp. 35—60. June 1968

Exceptional Lie Group F_4 and its Representation Rings

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(Received April 30, 1968)

Introduction

The aim of this paper is to determine the real and complex representation rings $RO(F_4)$ and $R(F_4)$ of F_4, which is a simply connected compact Lie group of exceptional type F. Let $\mathfrak{J}$ denote the Jordan algebra of all 3-hermitian matrices over the division ring of Cayley numbers. We know that the group F_4 is obtained as the automorphism group of $\mathfrak{J}$. In Chapter I, we shall arrange some properties of F_4 : the subgroups Spin(8), Spin(9), maximal torus T, the Weyl group W and the Lie algebra $\mathfrak{F}_4$. The origin of the results of Chapter I are found in H. Freudenthal [1], however we rewrite them with some modifications. In Chapter II, we shall determine the ring structures of $RO(F_4)$ and $R(F_4)$. Let $\mathfrak{J}_0$ be the set of all elements of $\mathfrak{J}$ with zero trace and let $\mathfrak{F}_4$ be the Lie algebra of F_4. Then $\mathfrak{J}_0$ and $\mathfrak{F}_4$ are F_4-**R**-modules in the natural way. The results are follows : $RO(F_4)$ is a polynomial ring $\mathbf{Z}[\lambda_1, \lambda_2, \lambda_3, \kappa]$ with 4 variables $\lambda_1, \lambda_2, \lambda_3, \kappa$, where λ_i is the class of the exterior F_4-$\mathbf{R}$-module $\Lambda^i(\mathfrak{J}_0)$ in $RO(F_4)$ for $i=1, 2, 3$, and κ is the class of $\mathfrak{F}_4$ in $RO(F_4)$. $R(F_4)$ is also a polynomial ring $\mathbf{Z}[\lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C]$, where $\lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C$ are the complexification of $\lambda_1, \lambda_2, \lambda_3, \kappa$ respectively. In the final section, we consider the relationship between $R(F_4)$ and $R(\mathrm{Spin}(9))$, $R(\mathrm{Spin}(8))$.

CHAPTER I

1. Jordan algebra $\mathfrak{J}$

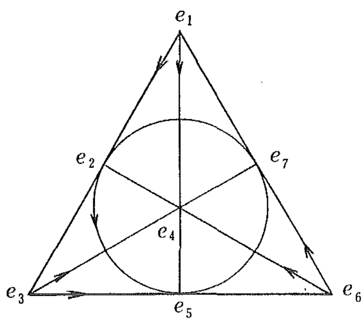
Let $\mathfrak{C}$ be the division ring of Cayley numbers. $\mathfrak{C}$ is an 8-dimensional $\mathbf{R}$[1]-module with a base $e_0, \cdots, e_7$ and the multiplications among them are given as follows;

e_0 is the unit of $\mathfrak{C}$ (which is often denoted by 1)

$$e_i^2 = -e_0 \qquad \text{for } i \neq 0,$$

$$e_i e_j = -e_j e_i \qquad \text{for } i, j \neq 0, \ i \neq j$$

and



(for example $e_1e_2 = e_3$, $e_2e_5 = e_7$, $e_2e_4 = -e_6$).

The conjugation $\bar{u}$ of $u \in \mathfrak{C}$ is defined by $\bar{u} = \overline{e_0u_0 + \sum_{i=1}^{7} e_iu_i} = e_0u_0 - \sum_{i=1}^{7} e_iu_i$ ($u_0, u_i \in \mathbf{R}$) and the real part $\mathrm{Re}\,u$ of u by $\frac{1}{2}(u+\bar{u})$. We define the inner product (u, v) of $u = \sum_{i=0}^{7} e_iu_i$, $v = \sum_{i=0}^{7} e_iv_i$ ($u_i, v_i \in \mathbf{R}$) by $\sum_{i=0}^{7} u_iv_i$ and the length of u by $|u| = \sqrt{(u, u)} = \sqrt{u\bar{u}}$.

We describe here some formulae in $\mathfrak{C}$ used in later.

1.1 For $u, v, a, b \in \mathfrak{C}$, we have

(1) $\bar{\bar{u}} = u$, $\quad \overline{uv} = \bar{v}\bar{u}$,

(2) $\mathrm{Re}(uv) = \mathrm{Re}(vu)$, $\quad \mathrm{Re}(u(vw)) = \mathrm{Re}((uv)w)$,

(3) $2(u, v) = u\bar{v} + v\bar{u} = \bar{u}v + \bar{v}u$, $\qquad |u|^2 = u\bar{u} = \bar{u}u$,

(4) $\bar{a}(bu) + \bar{b}(au) = 2(a, b)u$,

(5) $a(\bar{a}u) = (a\bar{a})u$, $\quad a(u\bar{a}) = (au)\bar{a}$, $\quad u(a\bar{a}) = (ua)\bar{a}$,

$\quad a(au) = (aa)u$, $\quad a(ua) = (au)a$, $\quad u(aa) = (ua)a$,

1) $\mathbf{R}$ is the field of real numbers.

(6) $(au)v + u(va) = a(uv) + (uv)a.$

Let $\mathfrak{J}$ denote the space of 3-hermitian matrix X over $\mathfrak{C}$

$$X = \begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix}, \qquad \xi_i \in \mathbf{R}, \ u_i \in \mathfrak{C}.$$

Such X is often denoted by $X(\xi, u)$. We define the Jordan product in $\mathfrak{J}$ by

$$X \circ Y = \frac{1}{2}(XY + YX)$$

where the product XY is the usual matrix product. Then $\mathfrak{J}$ is a non-associative commutative 27-dimensional $\mathbf{R}$-algebra.

We shall adopt the following notations;

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E = E_1 + E_2 + E_3,$$

$$F_1^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & \bar{u} & 0 \end{pmatrix}, \ F_2^u = \begin{pmatrix} 0 & 0 & \bar{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}, \ F_3^u = \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then E_i, F_i^u for $i = 1, 2, 3$, $u \in \mathfrak{C}$ generate $\mathfrak{J}$ additively and we have

$$\begin{cases} E_i \circ E_i = E_i, \quad E_i \circ E_j = 0 \quad \text{for } i \neq j, \\ E_i \circ F_i^u = 0, \quad 2E_i \circ F_j^u = F_j^u \quad \text{for } i \neq j, \\ F_i^u \circ F_i^v = (E - E_i)(u, v), \quad 2F_i^u \circ F_{i+1}^v = F_{i+2}^{\overline{uv}} \end{cases}$$

for i, $j = 1, 2, 3$ and suffixes are modulo 3.

In $\mathfrak{J}$, we define the trace, the inner product and triple inner product by

$$\begin{aligned} &\mathrm{tr}(X) = \xi_1 + \xi_2 + \xi_3, \\ &(X, Y) = \mathrm{tr}(X \circ Y), \\ &\mathrm{tr}(X, Y, Z) \overset{2)}{=} (X \circ Y, Z) \end{aligned}$$

respectively for $X = X(\xi, u)$, Y, $Z \in \mathfrak{J}$.

1.2 **Lemma.** *$(XY, A) = (YA, X)$ for X, Y, $A \in \mathfrak{J}$.*

Proof. Let $X = (x_{ij})$, $Y = (y_{ij})$, $A = (a_{ij})$, Then $(XY, A) = \mathrm{Re}(XY, A) = \frac{1}{2}\mathrm{Re}(\mathrm{tr}$

2) The notation $\mathrm{tr}(X, Y, Z,)$ is differ from that of [1] where (X, Y, Z) is used for this. We avoid here the notation (X, Y, Z) because this is used in another sense in almost every H. Freudenthal's papers (for example, Zur ebenen Oktavengeometrie, Indag. Math. 15, 1953).

$((XY)A+A(XY)))=\frac{1}{2}\mathrm{Re}(\sum_{i,k,l}((x_{ik}y_{kl})a_{li}+a_{ik}(x_{kl}y_{li}))) = \frac{1}{2}\mathrm{Re}(\sum_{i,k,l}((y_{kl}a_{li})x_{ik}+x_{kl}(y_{li}a_{ik})))$
$=\frac{1}{2}\mathrm{Re}(\mathrm{tr}((YA)X+X(YA)))=(YA,\ X).$

1.3 **Lemma.** *For* $X,\ X',\ Y,\ Z\in\mathfrak{J}$, *we have*

(1) $(X,\ Y)=(Y,\ X)$,

(2) $(X+X',\ Y)=(X,\ Y)+(X',\ Y)$, $\quad(X\xi,\ Y)=\xi(X,\ Y)$ *for* $\xi\in\mathbf{R}$,

(3) $(X,\ E)=\mathrm{tr}(X)$,

(4) $(\ \ ,\ \ )$ *is regular, i.e. if* $(X,\ Y)=0$ *for all* $Y\in\mathfrak{J}$, *then we have* $X=0$.

1.4 **Lemma.** *For* $X,\ X',\ Y,\ Z\in\mathfrak{J}$, *we have*

(1) $\mathrm{tr}(X,\ Y,\ Z)=\mathrm{tr}(Y,\ Z,\ X)=\mathrm{tr}(Z,\ X,\ Y)=\mathrm{tr}(X,\ Z,\ Y)=\mathrm{tr}(Z,\ Y,\ X)$
$=\mathrm{tr}(Y,\ Z,\ X)$,

(2) $\mathrm{tr}(X+X',\ Y,Z)=\mathrm{tr}(X,\ Y,\ Z)+\mathrm{tr}(X',\ Y,\ Z)$,
$\mathrm{tr}(X\xi,\ Y,\ Z)=\xi\mathrm{tr}(X,\ Y,\ Z)$ *for* $\xi\in\mathbf{R}$,

(3) $\mathrm{tr}(X,\ Y,\ E)=(X,\ Y)$.

Proof. (1) $\mathrm{tr}(X,\ Y,\ Z)=(X\circ Y,\ Z)=\frac{1}{2}(XY+YX,\ Z)=\frac{1}{2}((XY,\ Z)+(YX,\ Z))=$
$\frac{1}{2}((YZ,\ X)+(ZY,\ X))=(Y\circ Z,\ X)=\mathrm{tr}(Y,\ Z,\ X)$. (2), (3) are easily seen.

2. Definition of group F_4

2.1 **Definition.** Let F_4 denote the group of all automorphisms of $\mathfrak{J}$, that is, each $x\in F_4$ satisfies

(1) $x(X+Y)=xX+xY,\quad x(X\xi)=(xX)\xi$

(2) x is non-singular

(3) $x(X\circ Y)=xX\circ xY$

for $X,\ Y\in\mathfrak{J},\ \xi\in\mathbf{R}$.

Let F_4' denote the group of $\mathbf{R}$-homomorphisms $x:\mathfrak{J}\to\mathfrak{J}$ under which $(X,\ Y)$ and $\mathrm{tr}(X,\ Y,\ Z)$ are invariant, that is, each $x\in F_4'$ satisfies besides 2.1 (1),

(4) $(xX,\ xY)=(X,\ Y)$
(5) $\mathrm{tr}(xX,\ xY,\ xZ)=\mathrm{tr}(X,\ Y,\ Z)$ $\quad$ for $X,\ Y,\ Z\in\mathfrak{J}$.

2.2 **Lemma.** F_4' *is a subgroup of* F_4 : $F_4'\subset F_4$.

Proof. If $x\in F_4'$, $X,\ Y\in\mathfrak{J}$, then $(x(X\circ Y),\ xZ)=(X\circ Y,\ Z)=\mathrm{tr}(X,\ Y,\ Z)=\mathrm{tr}(xX,\ xY,\ xZ)=(xX\circ xY,\ xZ)$ for all $Z\in\mathfrak{J}$. This implies $x(X\circ Y)=xX\circ xY$, that is $x\in F_4$.

2.3 **Lemma.** (1) $xE=E$ *for* $x\in F_4$. (2) $\mathrm{tr}(xX)=\mathrm{tr}(X)$ *for* $x\in F_4'$, $X\in\mathfrak{J}$.

Proof. (1) We have $E\circ X=X$ for any $X\in\mathfrak{J}$. Operating x on $E\circ X=X$, then $xE\circ xX=xX$. Here put $X=x^{-1}E$, then $xE\circ E=E$. This implies $xE=E$. (2) $\mathrm{tr}(xX)=(xX,\ E)=(xX,\ xE)=(X,\ E)=\mathrm{tr}(X)$.

2.4 **Lemma.** F_4' *is the subgroup of* F_4 *consisting of all* $x\in F_4$ *under which the trace of every* $X\in\mathfrak{J}$ *is invariant.*

Proof. If the trace of every $X \in \mathfrak{J}$ is invariant under $x \in F_4$, *then* $(xX, xY) = \mathrm{tr}(xX \circ xY) = \mathrm{tr}(x(X \circ Y)) = \mathrm{tr}(X \circ Y) = (X, Y)$ and $\mathrm{tr}(xX, xY, xZ) = (xX \circ xY, xZ) = (x(X \circ Y), xZ) = (X \circ Y, Z) = \mathrm{tr}(X, Y, Z)$. Hence $x \in F_4'$. The converse follows from Lemmas 2.2, 2.3 (2).

We shall see that $F_4 = F_4'$ in Theorem 4.2, in particular, that the trace of every $X \in \mathfrak{J}$ is invariant under $x \in F_4$.

3. Deformation to diagonal form

3.1 **Lemma.** *F_4' is a compact group.*

Proof. Since the inner product $(X, Y) = \sum_{i=1}^{3}(\xi_i \eta_i + 2(u_i, v_i))$ is invariant under each element of F_4', where $X = X(\xi, u)$, $Y = Y(\eta, v) \in \mathfrak{J}$, F_4' is a closed subgroup of the orthogonal group $O(27)$ which is compact. Therefore F_4' is compact.

3.2 **Lemma.** *For $a \in \mathfrak{C}$ $(a \neq 0)$, define an* $\mathbf{R}$*-homomorphism $x : \mathfrak{J} \to \mathfrak{J}$ by $xX(\xi, u) = Y(\eta, v)$, where*

$$\begin{cases} \eta_1 = \xi_1, \\ \eta_2 = \dfrac{(a, u_1)}{|a|}\sin 2|a| + \dfrac{\xi_2 - \xi_3}{2}\cos 2|a| + \dfrac{\xi_2 + \xi_3}{2}, \\ \eta_3 = -\dfrac{(a, u_1)}{|a|}\sin 2|a| - \dfrac{\xi_2 - \xi_3}{2}\cos 2|a| + \dfrac{\xi_2 + \xi_3}{2}, \end{cases}$$

$$\begin{cases} v_1 = u_1 - \dfrac{(\xi_2 - \xi_3)a}{2|a|}\sin 2|a| - \dfrac{2(a, u_1)a}{|a|^2}\sin^2 |a|, \\ v_2 = u_2 \cos |a| - \dfrac{\bar{a}\bar{u}_3}{|a|}\sin |a|, \\ v_3 = u_3 \cos |a| + \dfrac{\bar{u}_2 \bar{a}}{|a|}\sin |a|, \end{cases}$$

then we have $x \in F_4'$.

Proof. We shall show first that $x(X \circ X) = xX \circ xX$ by the direct computation.

$$X \circ X = \begin{pmatrix} \xi_1^2 + |u_2|^2 + |u_3|^2 & (\xi_1 + \xi_2)u_3 + \overline{u_1 u_2} & * \\ * & \xi_2^2 + |u_3|^2 + |u_1|^2 & (\xi_2 + \xi_3)u_1 + \overline{u_2 u_3} \\ (\xi_3 + \xi_1)u_2 + \overline{u_3 u_1} & * & \xi_3^2 + |u_1|^2 + |u_2|^2 \end{pmatrix}.$$

The (1, 1)-component of $xX \circ xX = \eta_1^2 + |v_2|^2 + |v_3|^2 = \xi_1^2 + |u_2 \cos|a| - \dfrac{\bar{a}\bar{u}_3}{|a|}\sin|a|\,|^2 + |u_3 \cos|a| + \dfrac{\bar{u}_2\bar{a}}{|a|}\sin|a|\,|^2 = \xi_1^2 + |u_2|^2 + |u_3|^2 =$ the (1, 1)-component of $x(X \circ X)$. The (2, 2)-component of $xX \circ xX = \eta_2^2 + |v_3|^2 + |v_1|^2 = \cdots = \dfrac{1}{|a|}(a, (\xi_2 + \xi_3)u_1 + \overline{u_2 u_3})\sin 2|a| + \dfrac{1}{2}(\xi_2^2 - \xi_3^2 + |u_3|^2 - |u_2|^2)\cos 2|a| + \dfrac{1}{2}(\xi_2^2 + \xi_3^2 + 2|u_1|^2 + |u_2|^2 + |u_3|^2) = \cdots =$ the (2, 2)-component of $x(X \circ X)$. About the (3, 3)-component, the computation is similar to the (2, 2)-component. The (2, 3)-component of $x(X\circ X)=(\xi_2+\xi_3)u_1+\overline{u_2u_3}-\frac{1}{2}(\xi_2{}^2-\xi_3{}^2+|u_3|^2-|u_2|^2)\frac{a}{|a|}\sin 2|a|-(a,\ (\xi_2+\xi_3)u_1+\overline{u_2u_3})\frac{2a}{|a|^2}\sin^2|a|=\cdots$ (we shall use the formula $2a(\bar{a},\ u_2u_3)=a(u_2u_3)a+u_2u_3|a|^2$ (cf. 1.1 (3))) $\cdots=(\eta_2+\eta_3)v_1+\overline{v_2v_3}=$ the (2, 3)-component of $xX\circ xX$. The (3, 1)-component of $x(X\circ X)=\cdots$ (we shall use $\bar{a}(u_1u_2)+\bar{u}_1(au_2)=2(a,\ u_1)u_2$ (cf. 1.1 (4))) $\cdots=$ the (3, 1)-component of $xX\circ xX$. The (1, 2)-com-ponent is similar to the (3, 1)-component. Thus we have $x(X\circ X)=xX\circ xX$ for any $X\in\mathfrak{J}$. By the polarization $X\to X+Y$, we have

$$x(X\circ Y)=xX\circ xY \qquad \text{for } X,\ Y\in\mathfrak{J}.$$

Hence $x\in F_4$. Finally, it is easily seen that $\mathrm{tr}(xX)=\eta_1+\eta_2+\eta_3=\xi_1+\xi_2+\xi_3=\mathrm{tr}(X)$. Therefore, by Lemma 2.4, we have $x\in F_4'$.

3.3 **Theorem.** [1]. *For $X_0\in\mathfrak{J}$, there exists $x\in F_4'$ such that xX_0 is of a diagonal form.*

Proof. For a fixed $X_0\in\mathfrak{J}$, $\mathfrak{X}_0=\{xX_0\mid x\in F_4'\}$ is a compact subset in $\mathfrak{J}$. Let X_1 be an element in $\mathfrak{X}_0$ which attains the maximum value of $\xi_1{}^2+\xi_2{}^2+\xi_3{}^2$ for $X=X(\xi,\ u)\in\mathfrak{X}_0$, then we shall show that X_1 is diagonal. Assume that $X_1=X_1(\xi,\ u)$ is not diagonal, for example, $u_1\neq 0$. Put $a(t)=\frac{u_1}{|u_1|}t$ for $t\in\mathbf{R}\ (t\neq 0)$ and construst an element $x(t)\in F_4'$ as in Lemma 3.2. Then the value of $\eta_1{}^2(t)+\eta_2{}^2(t)+\eta_3{}^2(t)$ in $x(t)X_1$ is $\xi_1{}^2+\frac{2(a(t),\ u_1)^2}{|a(t)|}\sin^2 2|a(t)|+2\left(\frac{\xi_2-\xi_3}{2}\right)^2\cos^2 2|a(t)|+2\left(\frac{\xi_2+\xi_3}{2}\right)^2+\frac{4(a(t),\ u_1)(\xi_2-\xi_3)}{|a(t)|}\sin 2|a(t)|\cos 2|a(t)|$

$$=\xi_1{}^2+\frac{(\xi_2+\xi_3)^2}{2}+2\left(\pm|u_1|\sin 2|t|+\frac{\xi_2-\xi_3}{2}\cos 2|t|\right)^2$$

$$\leqq \xi_1{}^2+\frac{(\xi_2+\xi_3)^2}{2}+2\left(|u_1|^2+\left(\frac{\xi_2-\xi_3}{2}\right)^2\right)$$

$$=\xi_1{}^2+\xi_2{}^2+\xi_3{}^2+2|u_1|^2 \text{ (its mximum value).}$$

This contradicts to the fact that $\xi_1^2+\xi_2^2+\xi_3^2$ in X_1 attains the maximum value.

4. Cayley projective plane $\mathfrak{C}P_2$

4.1 **Proposition.** [1]. *For $X\in\mathfrak{J}$, the following five statements are equivalent.*

(1) *$X\neq 0$ and X is an irreducible idempotent, i.e. $X\circ X=X$ and $X=X_1+X_2$, $X_i\circ X_i=X_i$, $X_i\in\mathfrak{J}\ (i=1,\ 2)$ imply $X_1=0$ or $X_2=0$.*

(2) *$X\circ X=X$ and $\mathrm{tr}(X)=1$.*

(3) $\mathrm{tr}(X)=(X,\ X)=\mathrm{tr}(X,\ X,\ X)=1$.

(4) *$X=xE_i$ for some $x\in F_4$ and for some $i=1,\ 2,\ 3$.*

(5) *$X=xE_1$ for some $x\in F_4$.*

Proof. (1)$\to$(2). For X, there exists $x\in F_4'\subset F_4$ such that $xX=\sum_{i=1}^{3}E_i\xi_i$. The idempotency of X induces that of xX, so that we have $\xi_i^2=\xi_i$, hence $\xi_i=0$ or 1 for $i=1, 2, 3$. We see that the only one ξ_i of them is 1 and the others are 0. In fact, if not, xX is reducible. Since the reducibility is invariant under $x\in F_4$ X is reducible. This contradicts to the hypothesis of X. Now, by Lemma 2.3 (2), $\mathrm{tr}(X)=\mathrm{tr}(xX)=\mathrm{tr}(E_i)=1$. (2)$\to$(3) is obvious. (3)$\to$(4). For X, there exists $x\in F_4$ such that xX is of a diagonal from $\sum_{i=1}^{3}E_i\xi_i$. The condition (3) means $\xi_1+\xi_2+\xi_3=\xi_1^2+\xi_2^2+\xi_3^2=\xi_1^3+\xi_2^3+\xi_3^3=1$. Hence we have the only one $\xi_i=1$ and the others are 0, i.e. $xX=E_i$ for some i. Therefore $X=x^{-1}E_i$ where $x^{-1}\in F_4$. (4)$\to$ (5). It suffices to show that E_2 (and E_3) can be deformed to E_1 by some $a\in F_4$. For the matrix $A=E_3+F_3^{e_0}$, we have $AE_2A=E_1$. Since A is a real matrix, $a: X\to AXA$ for $X\in\mathfrak{J}$ ($AA=E$ and associativity!!) is an element of F_4 and we have $aE_2=E_1$. (5)$\to$(1). We shall show first that E_1 is an irreducible idempotent. Assume $E_1=X_1+X_2$, $X_i\circ X_i=X_i$, $X_i\in\mathfrak{J}$ ($i=1, 2$). Then $X_1+X_2=E_1=E_1\circ E_1$ $=X_1+X_2+2X_1\circ X_2$, hence $X_1\circ X_2=0$. Multiply X_1 on $E_1=X_1+X_2$, then we have $E_1\circ X_1=X_1$. This shows that X_1 is of the form $E_1\xi_1+E_2\xi_2+E_3\xi_3+F_1{}^{u_1}$. From $E_1=X_1+X_2$, we have $X_2=E_1\eta_1-E_2\xi_2-E_3\xi_3-F_1{}^{u_1}$ where $\xi_1+\eta_1=1$. Since $X_1\circ X_2=E_1\xi_1\eta_1-E_2(\xi_2{}^2+|u_1|^2)-E_3(\xi_3{}^2+|u_1|^2)$, $X_1\circ X_2=0$ implies that $\xi_2=\xi_3=u_1=0$ and $\xi_1=0$ or $\eta_1=0$. Thus we have $X_1=0$ or $X_2=0$. Now, since the irreducibility and idempotency are invariant under $x\in F_4$, we see that xE_1 is an irreducible idempotent in $\mathfrak{J}$.

Let $\mathfrak{C}P_2$ denote the space of $X\in\mathfrak{J}$ satisfying one of the five conditions of Proposition 4.1. Then we remember that $\mathfrak{C}P_2$ is the projective plane over $\mathfrak{C}$ [1], [5].

4.2 **Theorem.** $F_4'=F_4$, *that is, the trace of every* $X\in\mathfrak{J}$ *is invariant under each* $x\in F_4$.

Proof. Note that the trace of an element of the form zE_i ($z\in F_4$, $i=1, 2, 3$) is 1 by (4)$\to$(2) of Proposition 4.1. Now, let $x\in F_4$ and $X\in\mathfrak{J}$. For this X, choose $y\in F_4'$ such that yX is of a diagonal form $\sum_{i=1}^{3}E_i\xi_i=X_1$. Then we have $xX=xy^{-1}X_1=zX_1$ (where $z=xy^{-1}\in F_4$) $=\sum_{i=1}^{3}(zE_i)\xi_i$, whence $\mathrm{tr}(xX)=\sum_{i=1}^{3}\mathrm{tr}(zE_i)\xi_i=\sum_{i=1}^{3}\xi_i=\mathrm{tr}(X_1)=\mathrm{tr}(yX)=\mathrm{tr}(X)$.

5. Principle of triality in $SO(8)$ and Spin(8)

For the results of this section, we refer to [1], [3], however we rewrite them with proofs.

Let $SO(8)$ denote the rotation group in $\mathfrak{C}$. Let $\mathfrak{d}_4$ be the Lie algebra of $SO(8)$, that is, the $\mathbf{R}$-module consisting of $\mathbf{R}$-homomorphisms $D:\mathfrak{C}\to\mathfrak{C}$ such that

$$(Du,\ v)+(u,\ Dv)=0 \qquad \text{for } u,\ v\in\mathfrak{C}.$$

5.1 **Proposition.** [1] (Principle of infinitesimal triality in $\mathfrak{d}_4$)
Ror every $D_1\in\mathfrak{d}_4$, there exist $D_2,\ D_3\in\mathfrak{d}_4$ such that

$$(D_1u)v+u(D_2v)=\overline{D_3(\overline{uv})} \qquad \textit{for } u,\ v\in\mathfrak{C},$$

and for D_1, such D_2, D_3 are unique.

5.2 **Proposition** [1]. (Principle of triality in $SO(8)$)
For every $d_1\in SO(8)$, there exist $d_2,\ d_3\in SO(8)$ such that

$$(d_1u)(d_2v)=\overline{d_3(\overline{uv})} \qquad \textit{for } u,\ v\in\mathfrak{C},$$

and for d_1, such d_2, d_3 are unique up to the sign.

Proof. As is well known, for $d_1\in SO(8)$, there exists $D_1\in\mathfrak{d}_4$ such that $d_1=\exp D_1=\sum_{n\geqq 0}\frac{D_1{}^n}{n!}$. By Proposition 5.1, there are $D_2,\ D_3\in\mathfrak{d}_4$ such that $(D_1u)v+u(D_2v)=\overline{D_3(\overline{uv})}$ for $u,\ v\in\mathfrak{C}$. Put $d_2=\exp D_2$ and $d_3=\exp D_3$, then $\overline{d_3(\overline{uv})}=\exp\overline{D_3(\overline{uv})}=\sum_{n\geqq 0}\frac{\overline{D_3{}^n(\overline{uv})}}{n!}=\sum_{n\geqq 0}\sum_{i+j=n}\frac{(D_1{}^iu)(D_2{}^jv)}{i!\quad j!}=\Big(\sum_{i\geqq 0}\frac{D_1{}^iu}{i!}\Big)\Big(\sum_{j=0}\frac{D_2{}^jv}{j!}\Big)=(\exp D_1)u(\exp D_2)v=(d_1u)(d_2v)$.
To prove the uniqueness, it is sufficient to show that for $d_1=e$ we have $d_2=\pm e$, $d_3=\pm e$ (where e is the identity of $SO(8)$). Assume that $u(d_2v)=\overline{d_3(\overline{uv})}$ for $u,\ v\in\mathfrak{C}$. Put $u=1$, then $d_2v=\overline{d_3\bar{v}}$ for all $v\in\mathfrak{C}$. Therefore $u(d_2v)=d_2(uv)$. Put $v=1$ and denote $d_21=c$, then $uc=d_2u$. This implies $u(vc)=(uv)c$ for $u,\ v\in\mathfrak{C}$. From this associativity we have $c\in\mathbf{R}$, whence $c=\pm 1$. Therefore $d_2u=\pm u$ for all $u\in\mathfrak{C}$. Thus, for $d_1=e$, only two cases $d_2=d_3=e$ and $d_2=d_3=-e$ occur.

5.3 **Lemma.** [3]. *Let $O(8)$ be the orthogonal group in $\mathfrak{C}$. Assume that for d_1, $d_2,\ d_3\in O(8)$*

$$(d_1u)(d_2v)=\overline{d_3(\overline{uv})} \qquad \textit{for all } u,\ v\in\mathfrak{C},$$

then we have

$$\left.\begin{aligned}(d_2u)(d_3v)&=\overline{d_1(\overline{uv})}\\(d_3u)(d_1v)&=\overline{d_2(\overline{uv})}\end{aligned}\right\} \qquad \textit{for all } u,\ v\in\mathfrak{C}.$$

Proof. Multiply $\overline{d_1u}$ on the left side and $d_3(\overline{uv})$ on the right side of the given formula, then we have $|u|^2(d_2v)(d_3(\overline{uv}))=\overline{d_1u}\,|uv|^2$, hence $(d_2v)(d_3(\overline{uv}))=\overline{d_1u}\,|v|^2$. Replace $\bar{u}$ by vw, then $(d_2v)(d_3(|v|^2w))=\overline{d_1(\overline{vw})}\,|v|^2$, hence we have $(d_2v)(d_3w)=\overline{d_1(\overline{vw})}$

5.4 **Lemma.** [3]. *Assume that for $d_1,\ d_2,\ d_3\in O(8)$,*

$$(d_1u)(d_2v)=\overline{d_3(\overline{uv})} \qquad \textit{for all } u,\ v\in\mathfrak{C},$$

then we have $d_1,\ d_2,\ d_3\in SO(8)$.

Proof. If $d_1\notin SO(8)$, then there exists $a_1\in SO(8)$ such that $a_1d_1u=\bar{u}$ for $u\in\mathfrak{C}$. Using the triality, for this a_1, there exist $a_2,\ a_3\in SO(8)$ such that $(a_1d_1u)(a_2d_2v)=$

$\overline{a_3((d_1u)(d_2v))} = \overline{a_3d_3(\overline{uv})}$. Denote $a_2d_2 = b_2$, $a_3d_3 = b_3$, then $\bar{u}(b_2v) = \overline{b_3(\overline{uv})}$. Put $u = 1$, then $b_2v = \overline{b_3\bar{v}}$. Thus we have $\bar{u}(b_2v) = b_2(uv)$. Put $v = 1$ and $b_21 = c$. then $\bar{u}c = b_2u$, hence $\bar{u}(\bar{v}c) = (\overline{uv})c$. Put $v = c$, $\bar{u}\bar{c}c = \bar{c}\bar{u}c$, hence $\bar{u}\bar{c} = \bar{c}\bar{u}$ for all $u \in \mathfrak{C}$. hence $c \in \mathbf{R}$. Therefore $\bar{u}\bar{v} = \overline{uv}$ for all u, $v \in \mathfrak{C}$. This is a contradiction. Hence we have $d_1 \in SO(8)$. d_2, $d_3 \in SO(8)$ follows from Lemma 5.3 and the above.

Let $\mathfrak{J}_i$ denote the space of F_i^u where $u \in \mathfrak{C}$ for $i = 1, 2, 3$. $\mathfrak{J}_i$ is an 8-dimensional $\mathbf{R}$-module and $X \in \mathfrak{J}_i$ is characterized by

5.5 $$2E_{i+1}\circ X = X,\quad 2E_{i+2}\circ X = X.$$

And we have

5.6 $$2X\circ Y = (E - E_i)(X, Y) \qquad \text{for } X,\ Y \in \mathfrak{J}_i.$$

Let Spin(8) be the subgroup of F_4 consisting of x such that $xE_i = E_i$ for $i = 1, 2, 3$. Moreover it is convenient to define the following group spin(8) : spin(8) is the subgroup of $SO(8) \times SO(8) \times SO(8)$ which consists of (d_1, d_2, d_3) such that $(d_1u)(d_2v) = \overline{d_3(\overline{uv})}$ for u, $v \in \mathfrak{C}$.

5.7 **Propositon.** [3]. spin(8) *and* Spin(8) *are isomorphic as group by the correspondence* $(d_1, d_2, d_3) \to d$;

5.8 $$d\begin{pmatrix} \xi_1 & u_3 & \overline{u_2} \\ \overline{u_3} & \xi_2 & u_1 \\ u_2 & \overline{u_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & d_3u_3 & \overline{d_2u_2} \\ \overline{d_3u_3} & \xi_2 & d_1u_1 \\ d_2u_2 & \overline{d_1u_1} & \xi_3 \end{pmatrix}.$$

Proof. Let $d \in$ Spin(8), then for X, $Y \in \mathfrak{J}_i$, we have $dX \in \mathfrak{J}_i$ and $(dX, dY) = (X, Y)$ for $i = 1, 2, 3$ by 5.5, 5.6, hence d induces an orthogonal $\mathbf{R}$-homomorphism d_i in $\mathfrak{J}_i$ such that $dF_i^u = F_i^{d_iu}$ for $i = 1, 2, 3$. $2F_1^u\circ F_2^v = F_3^{\overline{uv}}$ implies $(d_1u)(d_2v) = \overline{d_3(\overline{uv})}$ for u, $v \in \mathfrak{C}$. By Lemma 5.4 d_1, d_2, $d_3 \in SO(8)$, that is, $(d_1, d_2, d_3) \in$ spin(8).

In the sequel, we shall identify spin(8) and Spin(8) by the correspondence 5.8.

Spin(8) has a sequence of subgroups

$$\mathrm{Spin}(8) \supset \mathrm{Spin}(7) \supset G_2 \supset SU(3)$$

where Spin(7) is the subgroup of $SO(8)$ consisting of $\tilde{a}$ such that for some $a \in SO(7)$, $(au)(\tilde{a}v) = \tilde{a}(uv)$ for u, $v \in \mathfrak{C}$. (The projection p : Spin(7)$\to SO(7)$ is defined by $p(\tilde{a}) = a$). G_2 is the group of automorphisms in $\mathfrak{C}$, that is, the subgroup of $SO(7)$ consisting of a such that $(au)(av) = a(uv)$ for u, $v \in \mathfrak{C}$. $SU(3)$ is the subgroup of G_2 consisting of a such that $ae_1 = e_1$.

5.9 **Proposition.** Spin(8) *is a simply connected covering group of* $SO(8)$.

Proof. We identify $\mathfrak{J}_1$ with $\mathfrak{C}$ by $F_1^u \to u$ and let S^7 be the unit sphere in $\mathfrak{C}$. spin(8) operates on S^7 by $(d_1, d_2, d_3)u = d_1u$. This operation is transitive by the principle of triality and its isotropy group of e_0 is Spin(7). Thus we have spin(8) /Spin(7) $= S^7$. The fiberings $G_2/SU(3) = S^6$, Spin(7)$/G_2 = S^7$, spin(8)/Spin(7) $= S^7$ yield the connectivity of spin(8). Now, define p : spin(8)$\to SO(8)$ by $p(d_1, d_2, d_3) = d_1$, then p is an epimorphism and its kernel is (e, e, e), $(e, -e, -e)$ by the principle of triality. Hence p:spin(8) $\to SO(8)$ is a twofold covering of $SO(8)$.

6. Spin(9) and construction lemma

Let $\mathfrak{J}_{01}$ denote the subspace of $\mathfrak{J}$ consisting of X such that $E_1 \circ X = 0$ and $\mathrm{tr}(X) = 0$. Such X is of the form $(E_2 - E_3)\xi + F_1{}^u$ for $\xi \in \mathbf{R}$, $u \in \mathfrak{C}$. Hence $\mathfrak{J}_{01}$ is a 9-dimensional $\mathbf{R}$-module, and $(X, X') = 2(\xi\xi' + (u, u'))$ and $X \circ X' = (E_2 + E_3)(X, X')$ for $X, X' \in \mathfrak{J}_{01}$.

Let $\mathfrak{J}_{23}$ denote the subspace of $\mathfrak{J}$ consisting of Y such that $2E_1 \circ Y = Y$. Such Y is of the form $F_2{}^u + F_3{}^v$ for $u, v \in \mathfrak{C}$. Hence $\mathfrak{J}_{23}$ is a 16-dimensional $\mathbf{R}$-module and $(Y, Y') = 2((u, u') + (v, v'))$ for $Y, Y' \in \mathfrak{J}_{23}$.

Let $SO(9)$ denote the rotation group in $\mathfrak{J}_{01}$, i.e. $\alpha \in SO(9)$ is an $\mathbf{R}$-homomorphism of $\mathfrak{J}_{01}$ such that $(\alpha X, \alpha Y) = (X, Y)$ for $X, Y \in \mathfrak{J}_{01}$. Let Spin(9) be the subgroup of F_4 consisting of x such that $xE_1 = E_1$.

The following lemma is sometimes convenient to construct an element of Spin(9) satisfying the given conditions.

6.1 **Lemma.** (construction lemma)

For any given element $A \in \mathfrak{J}_{01}$ *such that* $(A, A) = 2$,
choose any element $X_0 \in \mathfrak{J}_{01}$ *such that* $(A, X_0) = 0$, $(X_0, X_0) = 2$,
choose any element $Y_0 \in \mathfrak{J}_{23}$ *such that* $2A \circ Y_0 = -Y_0$, $(Y_0, Y_0) = 2$
and put $Z_0 = 2X_0 \circ Y_0$.

Next choose any $X_1 \in \mathfrak{J}_{01}$ *such that* $(A, X_1) = (X_0, X_1) = 0$, $(X_1, X_1) = 2$,
choose any $X_2 \in \mathfrak{J}_{01}$ *such that* $(A, X_2) = (X_0, X_2) = (X_1, X_2) = 0$, $(X_2, X_2) = 2$
and put $Y_1 = -2Z_0 \circ X_1$, $Z_2 = -2X_2 \circ Y_0$, $X_3 = -2Y_1 \circ Z_2$.
Choose any $X_4 \in \mathfrak{J}_{01}$ *such that* $(A, X_4) = (X_0, X_4) = (X_1, X_4) = (X_2, X_4) = (X_3, X_4) = 0$, $(X_4, X_4) = 2$

and put $\quad Z_4 = -2X_4 \circ Y_0$, $Y_2 = -2Z_0 \circ X_2$, $Y_3 = -2Z_0 \circ X_3$,
$\qquad X_5 = -2Y_1 \circ Z_4$, $X_6 = 2Y_2 \circ Z_4$, $X_7 = -2Y_3 \circ Z_4$
and then put $Y_i = -2Z_0 \circ X_i$ *for* $i = 4, 5, 6, 7$,
$\qquad Z_i = -2X_i \circ Y_0$ *for* $i = 1, 3, 5, 6, 7$.

Now, let $a : \mathfrak{J} \to \mathfrak{J}$ *be the* $\mathbf{R}$*-homomorphism satisfying*

$$aE = E, \ aE_1 = E_1, \ a(E_2 - E_3) = A,$$
$$aF_1{}^{e_i} = X_i, \ aF_2{}^{e_i} = Y_i, \ aF_3{}^{e_i} = Z_i \text{ for } i = 0, 1, \cdots, 7,$$

then we have $a \in F_4$ *(apriori* $a \in \mathrm{Spin}(9)$).

The proof is not trivial, we don't however give the proof, because its calculation may be independent from the consideration of the present paper. It will appear in a forthcoming paper [7].

6.2 **Proposition.** Spin(9) *is a simply connected covering group of* $SO(9)$.

Proof. Let $a \in \mathrm{Spin}(9)$ and $X \in \mathfrak{J}_{01}$. Operate a on $E_1 \circ X = 0$ and $\mathrm{tr}(X) = 0$, then $E_1 \circ aX = 0$ and $\mathrm{tr}(aX) = 0$, hence $aX \in \mathfrak{J}_{01}$. And $(E - E_1)(aX, aX') = aX \circ aX' = a((E - E_1)(X, X')) = (E - E_1)(X, X')$, hence we have $(aX, aX') = (X, X')$. Thus a induces an orthogonal $\mathbf{R}$-homomorphism α in $\mathfrak{J}_{01}$. Let S^8 be the unit sphere in $\mathfrak{J}_{01}$, that is $S^8 = \{X \in \mathfrak{J}_{01} | (X, X) = 2\}$. Spin(9) operates on S^8 transitively; this transitivity follows from the construction lemma 6.1. We show that its isotropy group $G = \{a \in \mathrm{Spin}(9) | a(E_2 - E_3) = E_2 - E_3\}$ is Spin(8). For, since always $a(E_2 + E_3) = E_2 + E_3$ for $a \in \mathrm{Spin}(9)$, we have $aE_i = E_i$ $(i = 1, 2, 3)$ for any $a \in G$. Therefore $G = \mathrm{Spin}(8)$. Thus we have $\mathrm{Spin}(9)/\mathrm{Spin}(8) = S^8$, and this implies that Spin(9) is simply connected. Define the projection $p : \mathrm{Spin}(9) \to SO(9)$ by $p(a) = \alpha$, then p is a homomorphism and its kernel is (e, e, e) and $(e, -e, -e)$. In fact, let $a \in \mathrm{Spin}(9)$ satisfy $aX = X$ for all $X \in \mathfrak{J}_{01}$. First we shall see $a \in \mathrm{Spin}(8)$. Denote a by $(a_1, a_2, a_3) \in \mathrm{spin}(8)$. Since $F_1{}^u \in \mathfrak{J}_{01}$ we have $aF_1{}^u = F_1{}^u$. Hence, operating a on $2F_1{}^u \circ F_2{}^v = F_3{}^{\overline{uv}}$. then we have $u(a_2 v) = \overline{a_3(\overline{uv})}$. By the principle of triality, we have $a = (e, e, e)$ or $(e, -e, -e)$. Hence $p : \mathrm{Spin}(9) \to SO(9)$ is the twofold covering of $SO(9)$.

6.3 **Remark.** Let S^{15} be the unit sphere in $\mathfrak{J}_{23}$, that is $S^{15} = \{Y \in \mathfrak{J}_{23} | (Y, Y) = 2\}$. Spin(9) operates on S^{15} transitively. The proof of the transitivity is as follows. Give a fixed element $F_2{}^{e_0}$ and any element $Y_0 \in \mathfrak{J}_{23}$. Choose any $A \in \mathfrak{J}_{01}$ such that $2A \circ Y_0 = -Y_0$, $(A, A) = 2$ and then take X_i, Y_i, Z_i for $i = 0, 1, \cdots, 7$ and construct $a \in \mathrm{Spin}(9)$ as well as in Lemma 6.1. Then $aF_2{}^{e_0} = Y_0$ for this a. Next it is easily verified that its isotropy group $\{a \in \mathrm{Spin}(9) | aF_2{}^{e_0} = F_2{}^{e_0}\}$ is Spin(7). Thus we have the well known fact

$$\mathrm{Spin}(9)/\mathrm{Spin}(7) = S^{15}.$$

F_4 operates on the Cayley projective plane $\mathfrak{C}P_2$ transitively by Proposition 4.1 (5) and its isotropy group of E_1 is Spin(9). Thus we have

$$F_4/\mathrm{Spin}(9) = \mathfrak{C}P_2.$$

Therefore, we have the following

6.4 **Theorem.** F_4 *is a* 52*-dimensional simply connected compact group.*

6.5 **Remark.** F_4 has 3 subgroups of type Spin(9); $\mathrm{Spin}^{(1)}(9) = \mathrm{Spin}(9)$, $\mathrm{Spin}^{(2)}(9)$ and $\mathrm{Spin}^{(3)}(9)$, where $\mathrm{Spin}^{(i)}(9) = \{a \in F_4 | aE_i = E_i\}$. And we have

$$\mathrm{Spin}(8) = \mathrm{Spin}^{(1)}(9) \cap \mathrm{Spin}^{(2)}(9) \cap \mathrm{Spin}^{(3)}(9).$$

7. Maximal torus T and Weyl group W

7.1 **Definition.** Let G be a (connected) topological group. A subgroup T of G is a maximal torus in G provided T is a torus with $G=\bigcup_{x\in G} xTx^{-1}$.

It is easy to see that maximal tori are conjugate to each other in G. The dimension of a maximal torus T is called the rank of G.

7.2 **Theorem**, *The rank of* F_4 *is* 4.

Proof. Let $x\in F_4$. Since the Cayley projective plane $\mathfrak{C}P_2$ is a homogeneous space $F_4/\mathrm{Spin}(9)$, x induces a homeomorphism f^x of $\mathfrak{C}P_2$ in the natural way ($X\to xX$, $X\in\mathfrak{C}P_2$). Hence f^x induces an isomorphism $f^x_{*i}: H_i(\mathfrak{C}P_2)\to H_i(\mathfrak{C}P_2)$ for all $i\geqq 0$. We shall calculate the Lefschetz number $L(f^x_*)=\sum_{i\geqq 0}(-1)^i\mathrm{tr}(f^x_{*i})$. For this, we recall that $\mathfrak{C}P_2$ is a CW-complex with 0, 8, 16-dimensional cells [5], so that its homology groups are $H_0(\mathfrak{C}P_2)=H_8(\mathfrak{C}P_2)=H_{16}(\mathfrak{C}P_2)=\mathbf{Z}$ and $H_i(\mathfrak{C}P_2)=0$ otherwise. Hence we have $L(f^x_*)=\mathrm{tr}(f^x_{*0})+\mathrm{tr}(f^x_{*8})+\mathrm{tr}(f^x_{*16})=\varepsilon_0+\varepsilon_8+\varepsilon_{16}$ (where ε_i is -1 or $1)\neq 0$. Therefore, by the fixed point theorem, there exists a point $Y\in\mathfrak{C}P_2$ such that $xY=Y$. For this Y, we can find $y\in F_4$ such that $Y=yE_1$ by Proposition 4.1 (5). $xyE_1=yE_1$, so $y^{-1}xyE_1=E_1$ and so that $y^{-1}xy\in\mathrm{Spin}(9)$. As is well known, the rank of Spin(9) is 4. Hence for a maximal torus T (dim $T=4$) in Spin(9), there exists $z\in\mathrm{Spin}(9)$ such that $z^{-1}(y^{-1}xy)z\in T$, so that $x\in(yz)T(yz)^{-1}$ where $yz\in F_4$. Hence we have $F_4=\bigcup_{y\in F_4} yTy^{-1}$, Thus the proof is completed.

We shall choose a maximal torus in Spin(8) = spin(8) as follows. Define a homomorphism $t:\mathbf{R}^4=\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathrm{spin}(8)$ ($t(\theta)=(t_1(\theta),\ t_2(\theta),\ t_3(\theta))$ where $\theta=(\theta_1,\ \theta_2,\ \theta_3,\ \theta_4)$ is denoted by $t=(t_1,\ t_2,\ t_3)$ briefly) by

7.3
$$\begin{cases} t_1e_0=e_0\cos\theta_1+e_1\sin\theta_1, & t_1e_1=-e_0\sin\theta_1+e_1\cos\theta_1,\\ t_1e_2=e_2\cos\theta_2+e_3\sin\theta_2, & t_1e_3=-e_2\sin\theta_2+e_3\cos\theta_2,\\ t_1e_4=e_4\cos\theta_3+e_5\sin\theta_3, & t_1e_5=-e_4\sin\theta_3+e_5\cos\theta_3,\\ t_1e_6=e_6\cos\theta_4+e_7\sin\theta_4, & t_1e_7=-e_6\sin\theta_4+e_7\cos\theta_4,\end{cases}$$

7.4
$$\begin{cases} t_2e_0=e_0\cos(-\theta_1+\theta_2+\theta_3+\theta_4)/2+e_1\sin(-\theta_1+\theta_2+\theta_3+\theta_4)/2,\\ \qquad t_2e_1=-e_0\sin(-\theta_1+\theta_2+\theta_3+\theta_4)/2+e_1\cos(-\theta_1+\theta_2+\theta_3+\theta_4)/2\\ t_2e_2=e_2\cos(-\theta_1+\theta_2-\theta_3-\theta_4)/2+e_3\sin(-\theta_1+\theta_2-\theta_3-\theta_4)/2,\\ \qquad t_2e_3=-e_2\sin(-\theta_1+\theta_2-\theta_3-\theta_4)/2+e_3\cos(-\theta_1+\theta_2-\theta_3-\theta_4)/2\\ t_2e_4=e_4\cos(-\theta_1-\theta_2+\theta_3-\theta_4)/2+e_5\sin(-\theta_1-\theta_2+\theta_3-\theta_4)/2\\ \qquad t_2e_5=-e_4\sin(-\theta_1-\theta_2+\theta_3-\theta_4)/2+e_5\cos(-\theta_1-\theta_2+\theta_3-\theta_4)/2\\ t_2e_6=e_6\cos(-\theta_1-\theta_2-\theta_3+\theta_4)/2+e_7\sin(-\theta_1-\theta_2-\theta_3+\theta_4)/2\\ \qquad t_2e_7=-e_6\sin(-\theta_1-\theta_2-\theta_3+\theta_4)/2+e_7\cos(-\theta_1-\theta_2-\theta_3+\theta_4)/2\end{cases}$$

$$\begin{cases} t_3e_0=e_0\cos(-\theta_1-\theta_2-\theta_3-\theta_4)/2+e_1\sin(-\theta_1-\theta_2-\theta_3-\theta_4)/2\\ \qquad t_3e_1=-e_0\sin(-\theta_1-\theta_2-\theta_3-\theta\ )/2+e_1\cos(-\theta_1-\theta_2-\theta_3-\theta_4)/2\\ t_3e_2=e_2\cos(\theta_1+\theta_2-\theta_3-\theta_4)/2+e_3\sin(\theta_1+\theta_2-\theta_3-\theta_4)/2\end{cases}$$

7.5
$$\begin{cases} t_3e_3 = -e_2\sin(\theta_1+\theta_2-\theta_3-\theta_4)/2 + e_3\cos(\theta_1+\theta_2-\theta_3-\theta_4)/2 \\ t_3e_4 = e_4\cos(\theta_1-\theta_2+\theta_3-\theta_4)/2 + e_5\sin(\theta_1-\theta_2+\theta_3-\theta_4)/2 \\ t_3e_5 = -e_4\sin(\theta_1-\theta_2+\theta_3-\theta_4)/2 + e_5\cos(\theta_1-\theta_2+\theta_3-\theta_4)/2 \\ t_3e_6 = e_6\cos(\theta_1-\theta_2-\theta_3+\theta_4)/2 + e_7\sin(\theta_1-\theta_2-\theta_3+\theta_4)/2 \\ t_3e_7 = -e_6\sin(\theta_1-\theta_2+\theta_3+\theta_4)/2 + e_7\cos(\theta_1-\theta_2-\theta_3+\theta_4)/2. \end{cases}$$

Then we can verify that

$$(t_1u)(t_2v) = \overline{t_3(\overline{uv})} \qquad \text{for } u,\ v \in \mathfrak{C}.$$

Hence the image $t(\mathbf{R}^4) = T$ is a maximal torus in Spin(8) (also in F_4).

7.6 **Definition**. Let G be a topological group with a maximal torus T. The Weyl group $W(G)$ of G is $N_T(G)/T$, where $N_T(G)$ is the normalizer of T in G.

7.7 **Lemma**. *If* $x \in N_T(F_4)$, *then* $xE_1 = E_{i_1}$, $xE_2 = E_{i_2}$, $xE_3 = E_{i_3}$ *where* (i_1, i_2, i_3) *is a substitution of* (1, 2, 3).

Proof. Let $x \in N_T(F_4)$, then $x^{-1}tx \in T \subset$ Spin(8) for all $t \in T$. So that we have $x^{-1}txE_1 = E_1$, hence $t(xE_1) = xE_1$. Put $xE_1 = \sum_{i=1}^{3}(E_i\xi_i + F_i^{u_i})$, then $t(xE_1) = xE_1$ shows $\sum_{i=1}^{3}(E_i\xi_i + F_i^{t_iu_i}) = \sum_{i=1}^{3}(E_i\xi_i + F_i^{u_i})$, therefore $t_1u_1 = u_1$, $t_2u_2 = u_2$, $t_3u_3 = u_3$ for all $t = (t_1, t_2, t_3) \in T$. By the formulae 7.3—7.4, these imply $u_1 = u_2 = u_3 = 0$. Therefore $xE_1 = \sum_{i=1}^{3}E_i\xi_i$. By Proposition 4.1 (1), xE_1 is an irreducible idempotent in $\mathfrak{J}$, hence xE_1 is E_1, E_2 or E_3. Similarly xE_2 and xE_3 are one of E_1, E_2, E_3 respectively. Obviously xE_1, xE_2, xE_3 are different to each other. Thus the proof is completed.

By Lemma 7.7, each $w \in N_T(F_4)/T$ induces a substitution among E_1, E_2, E_3. Thus we have a homomorphism

$$h : W(F_4) \longrightarrow \mathfrak{S}_3$$

where $\mathfrak{S}_3$ is the symmetric group of all permutations of E_1, E_2, E_3. We shall show that h is epimorphic. Since $\mathfrak{S}_3$ is generated by $\sigma = (1, 2, 3)$ and $\tau = (2, 3)$, it suffices to construct elements x, $y \in F_4$ which induce σ, τ respectively. Define $x = x(\sigma)$ by the $\mathbf{R}$-homomorphism of $\mathfrak{J}$ satisfying $xE_i = E_{i+1}$, $xF_i^u = F_{i+1}^u$ for $u \in \mathfrak{C}$, $i = 1, 2, 3$. Since $x^{-1}txE_i = E_i$, $x^{-1}txF_i^u = x^{-1}tF_{i+1}^u = x^{-1}F_{i+1}^{t_{i+1}u} = F_i^{t_{i+1}u}$ for $i = 1, 2, 3$, we have $x^{-1}(t_1, t_2, t_3)x = (t_2, t_3, t_1)$ (cf. Lemma 5.3), so that $x \in N_T(F_4)$ and x obviously induces σ. Next, let $y = y(\tau)$ be the $\mathbf{R}$-homomorphism given by

$$yE_1 = E_1 \qquad yE_2 = E_3 \qquad yE_3 = E_2$$

$$\begin{cases} yF_1^{e_0} = F_1^{e_0} \\ yF_1^{e_1} = -F_1^{e_1} \\ yF_1^{e_2} = F_1^{e_2} \end{cases} \qquad \begin{cases} yF_2^{e_0} = F_3^{e_1} \\ yF_2^{e_1} = F_3^{e_0} \\ yF_2^{e_2} = -F_3^{e_3} \end{cases} \qquad \begin{cases} yF_3^{e_0} = -F_2^{e_1} \\ yF_3^{e_1} = -F_2^{e_0} \\ yF_3^{e_2} = F_2^{e_3} \end{cases}$$

$$\begin{cases} yF_1{}^{e_3} = F_1{}^{e_3} \\ yF_1{}^{e_4} = F_1{}^{e_4} \\ yF_1{}^{e_5} = F_1{}^{e_5} \\ yF_1{}^{e_6} = F_1{}^{e_6} \\ yF_1{}^{e_7} = F_1{}^{e_7}, \end{cases} \qquad \begin{cases} yF_2{}^{e_3} = F_3{}^{e_2} \\ yF_2{}^{e_4} = -F_3{}^{e_5} \\ yF_2{}^{e_5} = F_3{}^{e_4} \\ yF_2{}^{e_6} = -F_3{}^{e_7} \\ yF_2{}^{e_7} = F_3{}^{e_6}, \end{cases} \qquad \begin{cases} yF_3{}^{e_3} = -F_2{}^{e_2} \\ yF_3{}^{e_4} = F_3{}^{e_5} \\ yF_3{}^{e_5} = -F_2{}^{e_4} \\ yF_3{}^{e_6} = F_2{}^{e_7} \\ yF_3{}^{e_7} = -F_2{}^{e_6}, \end{cases}$$

then y is in Spin(9), because it is easily verified that y satisfies the construction conditions of Lemma 6.1. And y induces τ obviously.

The kernel of h is W(Spin(8)) which is the Weyl group of Spin(8). In fact, suppose $h(w) = 1$ where $w \in W(F_4)$, then any representative $x \in N_T(F_4)$ of w satisfies $xE_i = F_i$ $(i = 1. \ 2. \ 3)$ so that we have $x \in$ Spin(8) (apriori, $x \in N_T(\text{Spin}(8))$). Therefore $w \in W$(Spin(8)). Thus we have an exact sepuence

$$1 \longrightarrow W(\text{Spin}(8)) \longrightarrow W(F_4) \longrightarrow \mathfrak{S}_3 \longrightarrow 1.$$

And it splits by $\sigma \to x(\sigma)$, $\tau \to y(\tau)$. Thus we have the following

7.8 **Theorem.** *The Weyl group $W(F_4)$ of F_4 is a semidirect product of $\mathfrak{S}_3$ and W(Spin(8)). That is,*

$$W(F_4) = \mathfrak{S}_3 W(\text{Spin}(8)), \quad \mathfrak{S}_3 \cap W(\text{Spin}(8)) = 1.$$

We remember that W(Spin(8)) consists of $2^3 4! = 192$ permutations of 4 variables $(\theta_1, \ \theta_2, \ \theta_3, \ \theta_4)$ composed with substitutions $(\theta_1, \ \theta_2, \ \theta_3, \ \theta_4) \to (\varepsilon_1\theta_1, \ \varepsilon_2\theta_2, \ \varepsilon_3\theta_3, \ \varepsilon_4\theta_4)$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1$.

7.9 **Remark.** Let $\mathbf{Z}_3$ denote the subgroup of $\mathfrak{S}_3$ generaetd by σ. Then we have a splitting exact sequence

$$1 \longrightarrow W(\text{Spin}(9)) \longrightarrow W(F_4) \longrightarrow \mathbf{Z}_3 \longrightarrow 1.$$

7.10 Since it is easy to see that $x(\sigma)\mathfrak{J}_i = \mathfrak{J}_{i+1}$ for $i = 1, \ 2, \ 3$ and $y(\tau)\mathfrak{J}_1 = \mathfrak{J}_1$, $y(\tau)\mathfrak{J}_2 = \mathfrak{J}_3$, $y(\tau)\mathfrak{J}_3 = \mathfrak{J}_2$ by 5.5, any element of $\mathfrak{S}_3$ induces a substitution among $\mathfrak{J}_1, \ \mathfrak{J}_2, \ \mathfrak{J}_3$.

8. Lie algebra $\mathfrak{F}_4$

Let $\mathfrak{M}$ denote the space of 3-matrices over $\mathfrak{C}$ and $\mathfrak{M}^-$ denote the space of 3-skew-hermitian matrices over $\mathfrak{C}$ (skew-hermitian matrix X is meant by X^*[3] $=$ $-X$). We extend the inner product of $\mathfrak{J}$ to $\mathfrak{M}$ by

$$(X, \ Y) = \frac{1}{2}\text{tr}(XY + Y^*X^*).$$

8.1 **Lemma.** *$(XY, \ A) = (YA, \ X)$ for $X, \ Y, \ A \in \mathfrak{M}$.*

The proof is the same as Lemma 1.2.

We define the bracket product by

3) X^* is ${}^t\overline{X}$.

$$[A,\ X] = AX - XA \qquad \text{for } A,\ X \in \mathfrak{M}.$$

If $A \in \mathfrak{M}^-$, $X \in \mathfrak{J}$, then $[A,\ X]$ is often denoted by $\widetilde{A}X$. Obviously we have

8.2 **Lemma.** $[\mathfrak{M}^-,\ \mathfrak{J}] \subset \mathfrak{J}$, $[\mathfrak{J},\ \mathfrak{J}] \subset \mathfrak{M}^-$.

8.3 **Lemma.** [1]. *For $X \in \mathfrak{J}$, there exists a pure imaginary Cayley number u such that*

$$[X,\ XX] = Eu.$$

Proof. Let $X = (u_{ij})$ where $\bar{u}_{ji} = u_{ij}$ and v_{ij} be the (i, j)-component of $[X,\ XX]$, namely $v_{ij} = \sum_{k,l=1}^{3} (x_{ik}(x_{kl}x_{lj}) - (x_{ik}x_{kl})x_{lj})$. Note that the parenthesis containing a u_{ss} (with double suffixes) is zero. If $i \neq j$, using 1.1 (5) we have $v_{ij} = 0$. For the case $i = j$, $v_{11} = v_{22} = v_{33}$ by 1.1 (6). Thus we have $[X,\ XX] = Eu$ for some $u \in \mathfrak{C}$. Since $X,\ XX \in \mathfrak{J}$, we have $[X,\ XX] \in \mathfrak{M}^-$, hence $\bar{u} = -u$, whence $\mathrm{Re}\,u = 0$.

8.4 **Lemma.** (1) *For $A,\ X,\ Y \in \mathfrak{M}$,* we have

$$([A,\ X],\ Y) + (X,\ [A,\ Y]) = 0.$$

(2) *For $A \in \mathfrak{M}$ such that* $\mathrm{tr}(A) = 0$, *we have*

$$\mathrm{tr}([A,\ X],\ Y,\ Z) + \mathrm{tr}(X,\ [A,\ Y],\ Z) + \mathrm{tr}(X,\ Y,\ [A,\ Z]) = 0$$

for $X,\ Y,\ Z \in \mathfrak{J}$.

Proof. (1) is obvious by Lemma 8.1. (2) $(A,\ [X,\ XX]) = (A,\ Eu)$ (where $\bar{u} = -u) = \frac{1}{2}\mathrm{tr}(Au + \bar{u}A^*) = \frac{1}{2}\mathrm{tr}(Au + uA) = 0$. Hence $(A,\ X(XX)) = (A,\ (XX)X)$. Thus $(AX,\ XX) = (XA,\ XX)$ by Lemma 8.1. Hence$([A,\ X], XX]) = 0$. By the polarization $X \to X + Y + Z$, $([A,\ X],\ YZ + ZY) + ([A,\ Y],\ XZ + ZX) + ([A,\ Z],\ XY + YX) = 0$. This means (2).

8.5 **Definition.** Let $\mathfrak{F}_4$ denote the set of $\mathbf{R}$-homomorphisms $\varphi : \mathfrak{J} \to \mathfrak{J}$ such that

$$\varphi(X \circ Y) = \varphi X \circ Y + X \circ \varphi Y.$$

Let F_4' denote the set of $\mathbf{R}$-homomorphisms $\varphi : \mathfrak{J} \to \mathfrak{J}$ satisfying

$$\begin{cases} (\varphi X,\ Y) + (X,\ \varphi Y) = 0, \\ \mathrm{tr}(\varphi X,\ Y,\ Z) + \mathrm{tr}(X,\ \varphi Y,\ Z) + \mathrm{tr}(X,\ Y,\ \varphi Z) = 0. \end{cases}$$

$\mathfrak{F}_4$ and $\mathfrak{F}_4'$ are Lie $\mathbf{R}$-algebra by the bracket multiplication

$$[\varphi,\ \psi]X = \varphi(\psi X) - \psi(\varphi X) \qquad \text{for } X \in \mathfrak{J}.$$

8.6 **Lamma.** *$\mathfrak{F}_4'$ is a Lie subalgebra of $\mathfrak{F}_4$; $\mathfrak{F}_4' \subset \mathfrak{F}_4$.*

Proof. For $\varphi \in \mathfrak{F}_4'$, $X,\ Y,\ Z \in \mathfrak{J}$, $(\varphi X \circ Y,\ Z) + (X \circ \varphi Y,\ Z) = \mathrm{tr}(\varphi X,\ Y,\ Z) + \mathrm{tr}(X,\ \varphi Y,\ Z) = -\mathrm{tr}(X,\ Y,\ \varphi Z) = -(X \circ Y,\ \varphi Z) = (\varphi(X \circ Y),\ Z)$. Hence we have $\varphi X \circ Y + X \circ \varphi Y = \varphi(X \circ Y)$.

The lemma 8.4 shows that for $A \in \mathfrak{M}^-$ such that $\mathrm{tr}(A) = 0$, $\widetilde{A} \in \mathfrak{F}_4$.

8.7 **Remark.** We see that $\mathfrak{F}_4' = \mathfrak{F}_4$. It will be remained to the readers.

We shall use the following notations; for $a \in \mathfrak{J}$

$$A_1^a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -\bar{a} & 0 \end{pmatrix}, \quad A_2^a = \begin{pmatrix} 0 & 0 & -\bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad A_3^a = \begin{pmatrix} 0 & a & 0 \\ -\bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have

$$\begin{cases} \widetilde{A}_i^a E_i = 0, & \widetilde{A}_i^a F_i^u = (E_{i+1} - E_{i+2})2(a,\ u), \\ \widetilde{A}_i^a E_{i+1} = -F_i^a, & \widetilde{A}_i^a F_{i+1}^u = F_{i+2}^{\bar{a}\bar{u}}, \\ \widetilde{A}_i^a E_{i+2} = F_i^a, & \widetilde{A}_i^a F_{i+2}^u = -F_{i+1}^{\overline{\bar{a}u}}. \end{cases}$$

Let $\mathfrak{D}_4$ denote the Lie subalgebra of $\mathfrak{F}_4$ consisting of D such that $DE_i = 0$ for $i = 1,\ 2,\ 3$.

Let $D \in \mathfrak{D}_4$. $E_i \circ F_i^u = 0$, $2E_j \circ F_i^u = F_i^u\ (i \neq j)$ imply $E_i \circ DF_i^u = 0$, $2E_j \circ DF_i^u = DF_i^u$. Thus we can set $DF_i^u = F_i^{D_i u}$. And $F_i^u \circ F_i^v = (E_{i+1} + E_{i+2})(u,\ v)$ and $2F_i^u \circ F_{i+1}^v = F_{i+2}^{\overline{vu}}$ imply $(D_i u,\ v) + (u,\ D_i v) = 0$ and $(D_i u)v + u(D_{i+1} v) = \overline{D_{i+2}(\overline{uv})}$. Hence we have

8.8 **Proposition.** $\mathfrak{d}_4$ *and* $\mathfrak{D}_4$ *are isomorphic as Lie algebra by the correspondence* $D_1 \in \mathfrak{d}_4 \to D \in \mathfrak{D}_4$;

$$D \begin{pmatrix} \xi_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \xi_2 & u_1 \\ u_2 & \bar{u}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & D_3 u_3 & \overline{D_2 u_2} \\ \overline{D_3 u_3} & 0 & D_1 u_1 \\ D_2 u_2 & \overline{D_1 u_1} & 0 \end{pmatrix}$$

where $D_2,\ D_3 \in \mathfrak{d}_4$ *are given by the infinitesimal triality for* D_1.

We shall identify $\mathfrak{d}_4$ and $\mathfrak{D}_4$ by the above correspondence in later.

CHAPTER II

9. Representation rings

Let G be a topological group. By a G-K-module ($K = \mathbf{R}$ or $\mathbf{C}$[4]) is meant a finite dimensional right K-module V together with a left action of G. That is for each $x \in G$, $u \in V$, there should be defined an element $xu \in V$ depending continuously x and u so that

9.1
$$\begin{cases} x(u + v) = xu + xv, & x(u\xi) = (xu)\xi, \\ (xy)u = x(yu), & eu = u \end{cases}$$

for $x,\ y \in G$, $u,\ v \in V$, $\xi \in K$ and e denotes the identity of G.

4) $\mathbf{C}$ is the field of complex numbers.

Two G-K-modules V_1 and V_2 are G-K-isomorphic if there exists a G-K-isomorphism $f: V_1 \to V_2$, that is, f is a K-isomorphism such that $f(xu) = xf(u)$ for $x \in G$, $u \in V_1$.

Let $M_K(G)$ denote the set of G-K-isomorphism classes $[V]$ of G-K-modules V. $[V]$ will be denoted by V simply.

The direct sum $V_1 \oplus V_2$, the tensor product $V_1 \otimes V_2$ of two G-K-modules V_1, V_2 and the exterior G-K-modules $\Lambda^i(V)$ $(0 \leqq i \leqq \dim V)$ for a G-K-module V define a λ-semiring structure on $M_K(G)$. That is, $\Lambda^i : M_K(G) \to M_K(G)$ for $i \geqq 0$ satisfy

9.2
$$\begin{cases} \Lambda_0(V) = K, & \Lambda^1(V) = V, \\ \Lambda^k(V_1 \oplus V_2) = \bigoplus\limits_{i+j=k} (\Lambda^i(V_1) \otimes \Lambda^j(V_2)). \end{cases}$$

In particular, we have

9.3 **Lemma.** *Let* $V_1, \cdots\cdots, V_n$ *be 1-dimensional G-K-modules. Then* $\Lambda^k(V_1 \oplus \cdots \oplus V_n)$ *and* $\bigoplus\limits_{i_1 < \cdots < i_k} V_{i_1} \otimes \cdots \otimes V_{i_k}$ *are G-K-isomorphic.*

The representation ring $R_K(G) = (R_K(G), \phi_G)$ is the universal λ-ring associated with the λ-semiring $M_K(G)$. The λ-ring $R_K(G)$ is meant a commutative ring with the unit 1 and functions $\lambda^i : R_K(G) \to R_K(G)$ for $i \geqq 0$ satisfying the following properties

9.4
$$\begin{cases} \lambda^0(\alpha) = 1, & \lambda^1(\alpha) = \alpha, \\ \lambda^k(\alpha + \beta) = \sum\limits_{i+j=k} \lambda^i(\alpha)\lambda^j(\beta). \end{cases}$$

The universality is as follows: $\phi_G : M_K(G) \to R_K(G)$ is a λ-semiring homomorphism and for any λ-ring A and any semiring homomorphism $\varphi : M_K(G) \to A$, there exists a unique λ-ring homomorphism $\tilde{\varphi} : R_K(G) \to A$ such that $\varphi = \tilde{\varphi}\phi_G$.

$M_K(G)$ has one more operation so called conjugation: for each G-K-module V, there corresponds the dual G-K-module $\hat{V}$ ($\hat{V}$ is $\mathrm{Hom}_K(V, K)$ as K-module and group action is $(x\omega)u = \omega(x^{-1}u)$ for $x \in G$, $\omega \in \mathrm{Hom}_K(V, K)$, $u \in V$). If W is a 1-dimensional G-K-module, then we have $W \otimes \hat{W} = K$, so that $\hat{W}$ is often denoted by W^{-1}.

Let H and G be topological groups and $h : H \to G$ be a continuous homomorphism. Then to every G-K-module V, there corresponds an H-K-module $h^{\#}(V)$ by the rule of group action

$$yu = h(y)u \qquad \text{for } y \in H, \ u \in V.$$

The correspondence $V \to h^{\#}(V)$ gives rise to a λ-ring homomorphism $h^* : R_K(G) \to R_K(H)$ such that the following diagram is commutative

$$
\begin{array}{ccc}
M_K(G) & \xrightarrow{h^\#} & M_K(H) \\
\downarrow{\scriptstyle \phi_G} & & \downarrow{\scriptstyle \phi_H} \\
R_K(G) & \xrightarrow{h^*} & R_K(H).
\end{array}
$$

$M_{\mathbf{R}}(G)$, $R_{\mathbf{R}}(G)$ are denoted by $MO(G)$, $RO(G)$ and $M_{\mathbf{C}}(G)$, $R_{\mathbf{C}}(G)$ are denoted by $M(G)$, $R(G)$ respectively.

10. Spin(8)-C-module $\mathfrak{J}_i^{\mathbf{C}}$ and Spin(9)-C-moules $\mathfrak{J}_{01}^{\mathbf{C}}$, $\mathfrak{J}_{23}^{\mathbf{C}}$

Since for $d \in \mathrm{Spin}(8)$, $X \in \mathfrak{J}_i$ we have $dX \in \mathfrak{J}_i$ by 5.5, each $d \in \mathrm{Spin}(8)$ induces a $\mathbf{R}$-homomorphism of $\mathfrak{J}_i$. Hence $\mathfrak{J}_i$ is a Spin(8)-$\mathbf{R}$-module and $\mathfrak{J}_i^C = \mathfrak{J}_i \otimes_{\mathbf{R}} \mathbf{C}$ is a Spin(8)-C-module for $i=1, 2, 3$.

Let T be the maximal torus in Spin(8) which is indicated in the section 7 and let $j_2 : T \to \mathrm{Spin}(8)$ be the inlcusion.

10.1 **Lemma.** *In* $j_2{}^\# : M(\mathrm{Spin}(8)) \to M(T)$, *we have*

$$j_2{}^\#(\mathfrak{J}_2^C) = \bigoplus_{j=1}^{4} (W_j \oplus W_j{}^{-1}),$$

$$j_2{}^\#(\mathfrak{J}_2^C) = \bigoplus_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = -1} W_1{}^{\varepsilon_1/2} \otimes W_2{}^{\varepsilon_1/2} \otimes W_3{}^{\varepsilon_3/2} \otimes W_4{}^{\varepsilon_4/2},$$

$$j_2{}^\#(\mathfrak{J}_2^C) = \bigoplus_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1} W_1{}^{\varepsilon_1/2} \otimes W_2{}^{\varepsilon_2/2} \otimes W_3{}^{\varepsilon_3/2} \otimes W_4{}^{\varepsilon_4/2}$$

where $W_j{}^{1/2}$ *is a* 1*-dimensional* T-C*-module,* $W_j{}^{-1/2}$ *is the dual* T-C*-module of* $W_j{}^{1/2}$ *and* $W_j{}^{\varepsilon}$ *is* $W_j{}^{\varepsilon/2} \otimes W_j{}^{\varepsilon/2}$ *for* $j=1, 2, 3, 4$ (ε_j, $\varepsilon = \pm 1$).

Proof. Choose an additive base in $\mathfrak{J}_i{}^C$ as follows;

10.2 $$X_j = F_1^{e_{2j-2} - e_{2j-1}\sqrt{-1}}, \qquad \hat{X}_j = F_1^{e_{2j-2} + e_{2j-1}\sqrt{-1}} \qquad \text{in } \mathfrak{J}_1^C,$$

10.3 $$Y_j = F_2^{e_{2j-2} - e_{2j-1}\sqrt{-1}}, \qquad \hat{Y}_j = F_2^{e_{2j-2} + e_{2j-1}\sqrt{-1}} \qquad \text{in } \mathfrak{J}_2^C,$$

10.4 $$Z_j = F_3^{e_{2j-2} - e_{2j-1}\sqrt{-1}}, \qquad \hat{Z}_j = F_3^{e_{2j-2} + e_{2j-1}\sqrt{-1}} \qquad \text{in } \mathfrak{J}_3^C$$

for $j = 1, 2, 3, 4$. For $t = t(\theta) = (t_1(\theta), t_2(\theta), t_3(\theta)) \in T$ where $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbf{R}^4$, then we have

10.5 $$tX_j = X_j \exp(\sqrt{-1}\theta_j), \qquad t\hat{X}_j = \hat{X}_j \exp(-\sqrt{-1}\theta_j)$$

for $j = 1, 2, 3, 4$. In fact, $t_1(e_{2j-2} - e_{2j-1}\sqrt{-1}) = t_1 e_{2j-2} - t_1 e_{2j-1}\sqrt{-1} = (e_{2j-2}\cos\theta_j + e_{2j-1}\sin\theta_j) - (-e_{2j-2}\sin\theta_j + e_{2j-1}\cos\theta_j)\sqrt{-1} = (e_{2j-2} - e_{2j-1}\sqrt{-1})(\cos\theta_j + \sqrt{-1}\sin\theta_j) = (e_{2j-2} - e_{2j-1}\sqrt{-1})\exp(\sqrt{-1}\theta_j)$, and $t_1(e_{2j-2} + e_{2j-1}\sqrt{-1}) = (e_{2j-2} + e_{2j-1}\sqrt{-1})\exp(-\sqrt{-1}\theta_j)$. Similarly we have by 7.4, 7.5,

10.6
$$\begin{cases} tY_1 = Y_1\exp(\sqrt{-1}(-\theta_1+\theta_2+\theta_3+\theta_4)/2), \\ \qquad t\hat{Y}_1 = \hat{Y}_1\exp(-\sqrt{-1}(-\theta_1+\theta_2+\theta_3+\theta_4)/2), \\ tY_2 = Y_2\exp(\sqrt{-1}(-\theta_1+\theta_2-\theta_3-\theta_4)/2), \\ \qquad t\hat{Y}_2 = \hat{Y}_2\exp(-\sqrt{-1}(-\theta_1+\theta_2-\theta_3-\theta_4)/2), \\ tY_3 = Y_3\exp(\sqrt{-1}(-\theta_1-\theta_2+\theta_3-\theta_4)2), \\ \qquad t\hat{Y}_3 = \hat{Y}_3\exp(-\sqrt{-1}(-\theta_1-\theta_2+\theta_3-\theta_4)/2), \\ tY_4 = Y_4\exp(\sqrt{-1}(-\theta_1-\theta_2-\theta_3+\theta_4)/2), \\ \qquad t\hat{Y}_4 = \hat{Y}_4\exp(-\sqrt{-1}(-\theta_1-\theta_2-\theta_3+\theta_4)/2), \end{cases}$$

10.7
$$\begin{cases} tZ_1 = Z_1\exp(\sqrt{-1}(-\theta_1-\theta_2-\theta_3-\theta_4)/2), \\ \qquad t\hat{Z}_1 = \hat{Z}_1\exp(-\sqrt{-1}(-\theta_1-\theta_2-\theta_3-\theta_4)/2), \\ tZ_2 = Z_2\exp(\sqrt{-1}(\theta_1+\theta_2-\theta_3-\theta_4)/2), \\ \qquad t\hat{Z}_2 = \hat{Z}_2\exp(-\sqrt{-1}(\theta_1+\theta_2-\theta_3-\theta_4)/2), \\ tZ_3 = Z_3\exp(\sqrt{-1}(\theta_1-\theta_2+\theta_3-\theta_4)/2), \\ \qquad t\hat{Z}_3 = \hat{Z}_3\exp(-\sqrt{-1}(\theta_1-\theta_2+\theta_3-\theta_4)/2), \\ tZ_4 = Z_4\exp(\sqrt{-1}(\theta_1-\theta_2-\theta_3+\theta_4)/2), \\ \qquad t\hat{Z}_4 = \hat{Z}_4\exp(-\sqrt{-1}(\theta_1-\theta_2-\theta_3+\theta_4)/2). \end{cases}$$

These formulae 10.5–10.7 give the proof of the lemma.

Putting $\phi_T(W_j^{1/2}) = \alpha_j^{1/2}$ for $j=1, 2, 3, 4$, then we have (cf. [2], [4])

$$R(T) = \mathbf{Z}[\alpha_1, \alpha_1^{-1}, \alpha_2, \alpha_2^{-1}, \alpha_3\ \alpha_3^{-1}, \alpha_4, \alpha_4^{-1}, (\alpha_1\alpha_2\alpha_3\alpha_4)^{1/2}].$$

Put $\nu_1^C = \phi_{\mathrm{Spin}(8)}(\mathfrak{J}_1^C)$, $\Delta_-^C = \phi_{\mathrm{Spin}(8)}(\mathfrak{J}_2^C)$, $\Delta_+^C = \phi_{\mathrm{Spin}(8)}(\mathfrak{J}_3^C)$, $\nu_2^C = \phi_{\mathrm{Spin}(8)}(\Lambda^2(\mathfrak{J}_1^C))$ in $R(\mathrm{Spin}(8))$ and denote $\boldsymbol{a} = j_2^*(\nu_1^C)$, $\boldsymbol{b} = j_2^*(\Delta_-^C)$, $\boldsymbol{c} = j_2^*(\Delta_C^+)$, $\boldsymbol{d} = j_2^*(\nu_2^C)$ in $R(T)$.

10.8 **Lemma.**

$$\boldsymbol{a} = j_2^*(\nu_1^C) = \sum_{j=1}^{4}(\alpha_j+\alpha_j^{-1}),$$

$$\boldsymbol{b} = j_2^*(\Delta_-^C) = \sum_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=-1} \alpha_1^{\varepsilon_1/2}\alpha_2^{\varepsilon_2/2}\alpha_3^{\varepsilon_3/2}\alpha_4^{\varepsilon_4/2},$$

$$\boldsymbol{c} = j_2^*(\Delta_+^C) = \sum_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=1} \alpha_1^{\varepsilon_1/2}\alpha_2^{\varepsilon_2/2}\alpha_3^{\varepsilon_3/2}\alpha_4^{\varepsilon_4/2},$$

$$\boldsymbol{d} = j_2^*(\lambda^2(\nu_1^C)) = j_2^*(\lambda^2(\Delta_-^C)) = j_2^*(\lambda^2(\Delta_+^C)) = 4 + \sum_{i\neq j}\alpha_i^{\varepsilon_i}\alpha_j^{\varepsilon_j}$$

Proof. The first three formulae are the direct consequences of Lemma 10.1. To prove the last formula, we shall use Lemma 9.3. Pick up two different monomials from 8 monomials in $\boldsymbol{a} = \sum_{j=1}^{4}(\alpha_j+\alpha_j^{-1})$, multiply them and sum up ${}_8C_2 = 28$ monomials (the result polynomial is denoted by $\boldsymbol{a}_2$). Then we have $\boldsymbol{a}_2 = 4 + \sum_{i\neq j}\alpha_1^{\varepsilon_i}\alpha_j^{\varepsilon_j} = \boldsymbol{d}$. Similarly we have $\boldsymbol{b}_2 = \boldsymbol{c}_2 = \boldsymbol{d}$. These show the last of the lemma.

Recall that we have (cf. [2], [4]) by using Lemma 10.8

10.9 $$R(\mathrm{Spin}(8)) = \mathbf{Z}[\nu_1^C,\ \nu_2^C,\ \Delta_-^C,\ \Delta_+^C].$$

We have seen that $\mathfrak{J}_{01}$ and $\mathfrak{J}_{23}$ are Spin(9)-$\mathbf{R}$-modules (cf. 6). Hence we have two Spin(9)-$\mathbf{C}$-modules $\mathfrak{J}_{01}^C = \mathfrak{J}_{01} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathfrak{J}_C^{23} = \mathfrak{J}_{23} \otimes_{\mathbf{R}} \mathbf{C}$. Put $\mu_1^C = \phi_{\mathrm{Spin}(9)}(\mathfrak{J}_{01}^C)$, $\mu_2^C = \phi_{\mathrm{Spin}(9)}(\Lambda^2(\mathfrak{J}_{01}^C))$, $\mu_3^C = \phi_{\mathrm{Spin}(9)}(\Lambda^3(\mathfrak{J}_{01}^C))$ and $\Delta^C = \phi_{\mathrm{Spin}(9)}(\mathfrak{J}_{23}^C)$. And let $j_1 : T \to \mathrm{Spin}(9)$ be the inclusion. Then we have easily the following

10.10 **Lemma.** *As a* Spin(8)-$\mathbf{R}$*-modules,*

$$\mathfrak{J}_{01} = \mathbf{R} \oplus \mathfrak{J}_1,$$
$$\mathfrak{J}_{23} = \mathfrak{J}_2 \oplus \mathfrak{J}_3.$$

Hence we have in $R(T)$

$$j_1^*(\mu_1^C) = 1 + \boldsymbol{a} = 1 + \sum_{j=1}^{4}(\alpha_j + \alpha_j^{-1}),$$
$$j_1^*(\Delta^C) = \boldsymbol{b} + \boldsymbol{c} = \prod_{j=1}^{4}(\alpha_j^{1/2} + \alpha_j^{-1/2}),$$
$$(\, j_1^*(\mu_2^C) = \boldsymbol{a} + \boldsymbol{d},\quad j_1^*(\mu_3^C) = -\boldsymbol{a} + \boldsymbol{d} + \boldsymbol{bc}\,).$$

Therefore we see (cf. [2], [4]) that

10.11 $$R(\mathrm{Spin}(9)) = \mathbf{Z}[\mu_1^C,\ \mu_2^C,\ \mu_3^C,\ \Delta^C],$$

11. F_4-C-module $\mathfrak{J}_0^C$

Since F_4 is the automorphism group of $\mathfrak{J}$, $\mathfrak{J}$ is obviously an F_4-$\mathbf{R}$-module. Remember that the trace of every $X \in \mathfrak{J}$ is invariant under the operation of F_4 by Theorem 4.2. Let $\mathfrak{J}_0$ denote the set of $X \in \mathfrak{J}$ such that $\mathrm{tr}(X) = 0$. Then $\mathfrak{J}_0$ is invariant under F_4, so that $\mathfrak{J}_0$ is an F_4-$\mathbf{R}$-module and $\mathfrak{J}$ is decomposable into the direct sum of two F_4-$\mathbf{R}$-module $\mathbf{R}$ (which is spanned by E with the trivial group action) and $\mathfrak{J}_0$; $\mathfrak{J} = \mathbf{R} \oplus \mathfrak{J}_0$ by

$$X = E\frac{1}{3}\mathrm{tr}(X) + (X - E\frac{1}{3}\mathrm{tr}(X)).$$

And we have an F_4-$\mathbf{C}$-module $\mathfrak{J}_0^C = \mathfrak{J}_0 \otimes_{\mathbf{R}} \mathbf{C}$.

Let T be the same maximal torus in F_4 as in the sections 7, 10 and let $j : T \to F_4$ be the inclusion.

11.1 **Lemma.** *As a* Spin(8)-$\mathbf{R}$*-module, we have*

$$\mathfrak{J}_0 = \mathbf{R} \oplus \mathbf{R} \oplus \mathfrak{J}_1 \oplus \mathfrak{J}_2 \oplus \mathfrak{J}_3.$$

Putting $\lambda_1^C = \phi_{F_4}(\mathfrak{J}_0^C)$, $\lambda_2^C = \phi_{F_4}(\Lambda^2(\mathfrak{J}_0^C))$ and $\lambda_3^C = \phi_{F_4}(\Lambda^3(\mathfrak{J}_0^C))$, then we have the following

11.2 **Proposition.**

$$j^*(\lambda_1^C) = 2 + (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}),$$

$$j^*(\lambda_2^C) = 1 + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) + (\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + 3\boldsymbol{d},$$

$$j^*(\lambda_3^C) = 3(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + \boldsymbol{abc} + 6\boldsymbol{d} + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\boldsymbol{d}.$$

Proof. The first formula is the direct consequence of Lemma 11.1. To prove the second formula, we shall use the result $\boldsymbol{a}_2 = \boldsymbol{b}_2 = \boldsymbol{c}_2 = \boldsymbol{d}$ in Lemma 10.8. Now,

$$\begin{aligned} j^*(\lambda_2^C) &= 1 + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) + (\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + \boldsymbol{a}_2 + \boldsymbol{b}_2 + \boldsymbol{c}_2 \\ &= 1 + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) + (\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + 3\boldsymbol{d}. \end{aligned}$$

To prove the last, we shall apply the same technique as the above. Pick up 3 different monomials from $\boldsymbol{a}$, multiply and sum up them (the result polynomial is denoted by $\boldsymbol{a}_3$). Then we have $\boldsymbol{a}_3 = \boldsymbol{bc} - \boldsymbol{a}$ by the direct calculation. Similarly we have $\boldsymbol{b}_3 = \boldsymbol{ca} - \boldsymbol{b}$, $\boldsymbol{c}_3 = \boldsymbol{ab} - \boldsymbol{c}$. Hence

$$\begin{aligned} j^*(\lambda_3^C) &= (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) + 2(\boldsymbol{a}_2 + \boldsymbol{b}_2 + \boldsymbol{c}_2) + 2(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) \\ &\quad + (\boldsymbol{a}_2\boldsymbol{b} + \boldsymbol{a}_2\boldsymbol{c} + \boldsymbol{b}_2\boldsymbol{c} + \boldsymbol{b}_2\boldsymbol{a} + \boldsymbol{c}_2\boldsymbol{a} + \boldsymbol{c}_2\boldsymbol{b}) + \boldsymbol{abc} + (\boldsymbol{a}_3 + \boldsymbol{b}_3 + \boldsymbol{c}_3) \\ &= (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) + 6\boldsymbol{d} + 2(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\boldsymbol{d} \\ &\quad + \boldsymbol{abc} + (\boldsymbol{bc} - \boldsymbol{a} + \boldsymbol{ca} - \boldsymbol{b} + \boldsymbol{ab} - \boldsymbol{c}) \\ &= 3(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}) + \boldsymbol{abc} + 6\boldsymbol{d} + 2(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\boldsymbol{d}. \end{aligned}$$

11.3 **Remark.** $(1+t)^2 \prod_{j=1}^{4} (1 + \alpha_j t)(1 + \alpha_j^{-1} t) \prod_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = \pm 1} (1 + \alpha_1^{\varepsilon_1/2} \alpha_2^{\varepsilon_2/2} \alpha_3^{\varepsilon_3/2} \alpha_4^{\varepsilon_4/2} t)$

$= 1 + j^*(\lambda_1^C)t + j^*(\lambda_2^C)t^2 + j^*(\lambda_3^C)t^3 + \cdots$

12. F_4-C-module $\mathfrak{F}_4^C$

The group F_4 operates on its Lie algebra $\mathfrak{F}_4$ in the natural way, that is, for $x \in F_4$ and $\varphi \in \mathfrak{F}_4$, $x\varphi \in \mathfrak{F}_4$ is defined by

$$(x\varphi)X = x(\varphi(x^{-1}X)) \qquad \text{for } X \in \mathfrak{J}.$$

Thus $\mathfrak{F}_4$ is an F_4-**R**-module, whence its complex form $\mathfrak{F}_4^C = \mathfrak{F}_4 \otimes_{\mathbf{R}} \mathbf{C}$ is an F_4-**C**-module.

To decompose $j^*(\mathfrak{F}_4^C)$, we shall extend the operation of Spin(8). Let $\mathfrak{M}^r$ denote the space of $X \in \mathfrak{M}$ with real diagonal elements. For $d = (d_1, d_2, d_3) \in \mathrm{spin}(8)$ and $X \in \mathfrak{M}^r$, we define dX by

12.1 $$d \begin{pmatrix} \xi_1 & u_{12} & u_{13} \\ u_{21} & \xi_2 & u_{23} \\ u_{31} & u_{32} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & d_3 u_{12} & \overline{d_2 \bar{u}_{13}} \\ \overline{d_3 \bar{u}_{21}} & \xi_2 & d_1 u_{23} \\ d_2 u_{31} & \overline{d_1 \bar{u}_{32}} & \xi_3 \end{pmatrix}.$$

12.2 **Lemma.** *For* $d \in \mathrm{Spin}(8)$, *we have*

$$d(X \circ Y) = dX \circ dY \qquad \text{for } X,\ Y \in \mathfrak{M}^r$$

where $X \circ Y = \dfrac{1}{2}(XY + Y^*X^*)$.

Proof. We shall show $d(X \circ X) = dX \circ dX$ for $X \in \mathfrak{M}^r$. The $(1,1)$-component of $dX \circ dX = \frac{1}{2}(\xi_1{}^2 + (d_3u_{12})(\overline{d_3\bar{u}_{21}}) + (\overline{d_2u_{13}})(d_2\bar{u}_{31}) + \xi_1{}^2 + (d_3\bar{u}_{21})(\overline{d_3u_{12}}) + (\overline{d_2u_{31}})(d_2\bar{u}_{13})) =$ (use 1.1 (3)) $= \xi_1{}^2 + (d_3u_{12},\ d_3\bar{u}_{21}) + (d_2u_{31},\ d_2\bar{u}_{13}) = \xi_1{}^2 + (u_{12},\ \bar{u}_{21}) + (u_{31},\ \bar{u}_{13}) = \frac{1}{2}(\xi_1{}^2 + u_{12}u_{21} + u_{31}u_{13} + \xi_1{}^2 + \bar{u}_{21}\bar{u}_{12} + \bar{u}_{13}\bar{u}_{31}) =$ the $(1,1)$-component of $d(X \circ X)$. The $(2,3)$-component of $2dX \circ dX = (\overline{d_3\bar{u}_{21}})(\overline{d_2\bar{u}_{13}}) + \xi_2d_1u_{23} + \xi_3d_1u_{23} + (\overline{d_3u_{12}})(\overline{d_2u_{31}}) + \xi_2d_1\bar{u}_{32} + \xi_3d_1\bar{u}_{32}$ (since $(\overline{d_3\bar{u}_{21}})(\overline{d_2\bar{u}_{13}}) = (\overline{d_2\bar{u}_{13}})(\overline{d_3\bar{u}_{21}}) = d_1(u_{21}u_{13})$ and similarly $(\overline{d_3u_{12}})(\overline{d_3u_{31}}) = d_1(\bar{u}_{12}\bar{u}_{31})) = d_1(u_{21}u_{13} + \xi_2u_{23} + \xi_3u_{23} + \bar{u}_{12}\bar{u}_{31} + \xi_2\bar{u}_{32} + \xi_3\bar{u}_{32}) =$ the $(2,3)$-component of $2d(X \circ X)$. About the other components the calculations are similar. Thus we have $d(X \circ X) = dX \circ dX$. By the polarization $X \to X + Y$, we have $d(X \circ Y) = dX \circ dY$.

12.3 **Lemma.** *For* $d \in \mathrm{Spin}(8)$ *and* $A \in \mathfrak{M}^- \cap \mathfrak{M}^r$, *we have*

$$d\widetilde{A} = \widetilde{dA}.$$

Proof. By Lemma 12.2, $d(A \circ X) = dA \circ dX$ for any $X \in \mathfrak{J}$. This shows that $d(AX - XA) = (dA)(dX) - (dX)(dA)$, i.e. $d(\widetilde{A}X) = \widetilde{dA}(dX)$. Replacing X by $d^{-1}X$, then we have $d(\widetilde{A}(d^{-1}X)) = \widetilde{dA}X$ for all $X \in \mathfrak{J}$. This proves the lemma.

Now, in $j^\# : M(F_4) \to M(T)$, we have

12.4 **Lemma.** $j^*(\mathfrak{F}_4^C) = \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \oplus \bigoplus_{j=1}^{4} (W_j + W_j{}^{-1}) \oplus \bigoplus_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = -1} W_1^{\varepsilon_1/2} \oplus W_2^{\varepsilon_2/2} \otimes W_3^{\varepsilon_3/2} \otimes W_4^{\varepsilon_4/2} \otimes \bigoplus_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1} W_1^{\varepsilon_1/2} \oplus W_2^{\varepsilon_2/2} \otimes W_3^{\varepsilon_3/2} \otimes W_4^{\varepsilon_4/2} \otimes \bigoplus_{\substack{i,j=1 \\ i \neq j}}^{4} W_1{}^{\varepsilon_i} \oplus W_j{}^{\varepsilon_j}$.

Proof. We shall use the following notations G_{ij} for $0 \leqq i < j \leqq 7 : G_{ij}$ is the $\mathbf{R}$-homomorphism of $\mathfrak{C}$ satisfying

$$\begin{cases} G_{ij}(e_j) = e_i \\ G_{ij}(e_i) = -e_j \\ G_{ij}(e_k) = 0 \qquad \text{for } k \neq i,\ j. \end{cases}$$

(These G_{ij} form an additive base of $\mathfrak{D}$). We choose now an additive base in $\mathfrak{F}_4^C$ as follows:

$$\begin{array}{llll} H_1 = G_{01}, & H_2 = G_{23}, & H_3 = G_{45}, & H_4 = G_{67}, \\ X_j = \widetilde{A}_1^{e_{2j-2}-e_{2j-1}\sqrt{-1}}, & \hat{X}_j = \widetilde{A}_1^{e_{2j-2}+e_{2j-1}\sqrt{-1}}, & & \\ Y_j = \widetilde{A}_2^{e_{2j-2}-e_{2j-1}\sqrt{-1}}, & \hat{Y}_j = \widetilde{A}_2^{e_{2j-2}+e_{2j-1}\sqrt{-1}}, & & \\ Z_j = \widetilde{A}_3^{e_{2j-2}-e_{2j-1}\sqrt{-1}}, & \hat{Z}_j = \widetilde{A}_3^{e_{2j-2}+e_{2j-1}\sqrt{-1}} & & \end{array}$$

for $j = 1,\ 2,\ 3,\ 4$ and

$$\begin{aligned}
S_{12} &= G_{02} - G_{13} - (G_{03} + G_{12})\sqrt{-1}, & \widehat{S}_{12} &= G_{02} - G_{13} + (G_{03} + G_{12})\sqrt{-1},\\
S_{13} &= G_{04} - G_{15} - (G_{05} + G_{14})\sqrt{-1}, & \widehat{S}_{13} &= G_{04} - G_{15} + (G_{05} + G_{14})\sqrt{-1},\\
S_{14} &= G_{06} - G_{17} - (G_{07} + G_{16})\sqrt{-1}, & \widehat{S}_{14} &= G_{06} - G_{17} + (G_{07} + G_{16})\sqrt{-1},\\
S_{23} &= G_{24} - G_{35} - (G_{25} + G_{34})\sqrt{-1}, & \widehat{S}_{23} &= G_{24} - G_{35} + (G_{25} + G_{34})\sqrt{-1},\\
S_{24} &= G_{26} - G_{37} - (G_{27} + G_{36})\sqrt{-1}, & \widehat{S}_{24} &= G_{26} - G_{37} + (G_{27} + G_{36})\sqrt{-1},\\
S_{34} &= G_{46} - G_{57} - (G_{47} + G_{56})\sqrt{-1}, & \widehat{S}_{34} &= G_{46} - G_{57} + (G_{47} + G_{56})\sqrt{-1},
\end{aligned}$$

$$\begin{aligned}
T_{12} &= G_{02} + G_{13} + (G_{03} - G_{12})\sqrt{-1}, & \widehat{T}_{12} &= G_{02} + G_{13} - (G_{03} - G_{12})\sqrt{-1},\\
T_{13} &= G_{04} + G_{15} + (G_{05} - G_{14})\sqrt{-1}, & \widehat{T}_{13} &= G_{04} + G_{15} - (G_{05} - G_{14})\sqrt{-1},\\
T_{14} &= G_{06} + G_{17} + (G_{07} - G_{16})\sqrt{-1}, & \widehat{T}_{14} &= G_{06} + G_{17} - (G_{07} - G_{16})\sqrt{-1},\\
T_{23} &= G_{24} + G_{35} + (G_{25} - G_{34})\sqrt{-1}, & \widehat{T}_{23} &= G_{24} + G_{35} - (G_{25} - G_{34})\sqrt{-1},\\
T_{24} &= G_{26} + G_{37} + (G_{27} - G_{36})\sqrt{-1}, & \widehat{T}_{24} &= G_{26} + G_{37} - (G_{27} - G_{36})\sqrt{-1},\\
T_{34} &= G_{46} + G_{57} + (G_{47} - G_{56})\sqrt{-1}, & \widehat{T}_{34} &= G_{46} + G_{57} - (G_{47} - G_{56})\sqrt{-1}.
\end{aligned}$$

Then, for $t = t(\theta_1, \theta_2, \theta_3, \theta_4) \in T$, we have

$$tH_j = H_j \qquad \text{for } j = 1, 2, 3, 4.$$

As for X_j, $\widehat{X}_j$, Y_j, $\widehat{Y}_j$, Z_j, $\widehat{Z}_j$, we have the same formulae as 10.5—10.7 for $j = 1, 2, 3, 4$ and

$$\begin{aligned}
tS_{ij} &= S_{ij}\exp(\sqrt{-1}(\theta_i + \theta_j)), & t\widehat{S}_{ij} &= \widehat{S}_{ij}\exp(-\sqrt{-1}(\theta_i + \theta_j)),\\
tT_{ij} &= T_{ij}\exp(\sqrt{-1}(\theta_i - \theta_j)), & t\widehat{T}_{ij} &= \widehat{T}_{ij}\exp(-\sqrt{-1}(\theta_i - \theta_j)).
\end{aligned}$$

Some of them will be proved. For example, $tY_1 = t\widetilde{A}_2^{e_0 - e_1\sqrt{-1}} = t\widetilde{A}_2^{e_0 - e_1\sqrt{-1}} = \widetilde{A}_2^{t_2(e_0 - e_1\sqrt{-1})} = \widetilde{A}_2^{(e_0 - e_1\sqrt{-1})\exp(\sqrt{-1}(-\theta_1+\theta_2+\theta_3+\theta_4)/2)} = Y_1(\exp\sqrt{-1}(-\theta_1 + \theta_2 + \theta_3 + \theta_4)/2)$.

An another example $tT_{12} = T_{12}\exp(\sqrt{-1}(\theta_1 - \theta_2))$ will be proved. To do so, it is sufficient to show that $(t_1T_{12})e_i = (T_{12}e_i)\exp(\sqrt{-1}(\theta_1 - \theta_2))$ for $i = 0, 1, \cdots, 7$. For $i=0$, $(t_1T_{12})e_0 = t_1(T_{12}(t_1^{-1}e_0)) = t_1(T_{12}(e_0\cos\theta_1 - e_1\sin\theta_1)) = t_1((-e_2 - e_3\sqrt{-1})\cos\theta_1 - (-e_3 + e_2\sqrt{-1})\sin\theta_1) = t_1(-e_2 - e_3\sqrt{-1})\exp(\sqrt{-1}\theta_1) = (-e_2 - e_3\sqrt{-1})\exp(-\sqrt{-1}\theta_2)\exp(\sqrt{-1}\theta_1) = (T_{12}e_0)\exp(\sqrt{-1}(\theta_1 - \theta_2))$. For $i = 1, \cdots, 7$, it will be also veirfied analogously. Thus the proposition is proved.

Putting $\kappa^C = \phi_{F_4}(\mathfrak{F}_4^C)$, then we have by Lemma 12.4

12.5 **Proposition.** $j^*(\kappa^C) = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}$.

13. Complex representation ring $R(F_4)$

Each element $w : T \to T$ in the Weyl group $W(F_4)$ induces an automorphism $w^* : R(T) \to R(T)$. Let $R(T)^W$ denote the subring of $R(T)$ which is invariant elementwise under these operation w^*. Since $j^* : R(F_4) \to R(T)$ is a ring monomorphism and the image of j^* is contained in $R(T)^W$([3], [5]), we will regard $R(F_4)$ as a subring of $R(T)^W$; $R(F_4) \subset R(T)^W$. We shall determine the ring structure of $R(T)^W$. From Propositions 11.2 and 12.5, we have

13.1. **Lemma.** $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca}$, $\boldsymbol{abc}$ *and* $\boldsymbol{d}$ *are polynomials in* λ_1^C. λ_2^C, λ_3^C, *and* κ^C. *In fact,*

$$\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \lambda_1^C - 2,$$

$$\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca} = \lambda_1^C + \lambda_2^C - 3\kappa^C - 3,$$

$$\boldsymbol{abc} = -5\lambda_1^C - 3\lambda_2^C + \lambda_3^C + 7\kappa^C + 2(\lambda_1^C)^2 - 2\lambda_1^C\kappa^C + 5,$$

$$\boldsymbol{d} = -\lambda_1^C + \kappa^C - 2.$$

Let $f \in R(T)^W$, that is, f be *a* $W(F_4)$-invariant polynomial. We know that any W(Spin(8))-invaiant polynomial is representable as a polynomials in λ_1^C, λ_2^C, Δ_-^C, Δ_+^C (cf. 10) namely as a polynomial in $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. Recall that the Weyl group $W(F_4)$ is the semidirect product of W(Spin(8)) and $\mathfrak{S}_3$, and each element of $\mathfrak{S}_3$ induces a substitution of 3 factors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ (cf. 7.10). Hence, $f \in R(T)^W$ is a polynomial in the elementary symmetric functions $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca}$, $\boldsymbol{abc}$ and $\boldsymbol{d}$. Thus, from Lemma 13.1, f can be represented as a polynomial in λ_1^C, λ_2^C, λ_3^C, κ^C.

Next we have to show that λ_1^C, λ_2^C, λ_3^C, and κ^C are algebraically independent. In fact, we know that $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ algebraically independent because R(Spin (8)) $= \mathbf{Z}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]$. Hence $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca}$, $\boldsymbol{abc}$ and $\boldsymbol{d}$ are also algebraically independent. Using propositions 11.2, 12.5, a non-trivial algebraic relation among λ_1^C, λ_2^C, λ_3^C and κ^C yields a non-trivial algebraic relation among $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{ab}+\boldsymbol{bc}+\boldsymbol{ca}$, $\boldsymbol{abc}$ and $\boldsymbol{d}$. Therefore λ_1^C, λ_2^C, λ_3^C and κ^C are algebraically independent. And we have $\mathbf{Z}[\lambda_1^C, \lambda_2^C, \gamma_3^C, \kappa^C] \subset R(F_4) \subset R(T)^W \subset \mathbf{Z}[\lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C]$. Thus we can proved the following

13.2 **Theorem.** *The complex representation ring* $R(F_4)$ *of* F_4 *is a polynomial ring* $\mathbf{Z}[\lambda_1^C, \lambda_2^C, \lambda_3^C, \kappa^C]$, *where* λ_i^C *is the calss of the* F_4-C*-module* $\Lambda^i(\mathfrak{J}_0^C)$ *for* $i=1$, 2, 3 *and* κ^C *is the class of the Lie* C*-algebra* $\mathfrak{F}_4^C$ *in* $R(F_4)$.

14. Real representation ring $RO(F_4)$

For a topological group G, we have two correspondence :

$$c : RO(G) \to R(G), \qquad r : R(G) \to RO(G),$$

where c is a ring homomorphism induced by the tensoring c' with C (that is,

$c' : MO(G) \to M(G)$ is defined by $c'(V) = V \otimes_{\mathbf{R}} \mathbf{C}$) and r is a homomorphism defined by restricting scalars from $\mathbf{C}$ to $\mathbf{R}$. As is well known, the relation $rc = 2$ holds. If G is a compact group, then $RO(G)$ is a free module generated by the classes of irreducible G-$\mathbf{R}$-modules, so that, the relation $rc = 2$ implies that c is a ring monomorphsim.

Let λ_1, λ_2, λ_3 and κ be the classes of F_4-$\mathbf{R}$-modules $\mathfrak{J}_0$, $\Lambda^2(\mathfrak{J}_0)$, $\Lambda^3(\mathfrak{J}_0)$ and $\mathfrak{F}_4$ respectively. Since we have obviously $c(\lambda_i) = \lambda_i^C$ for $i = 1, 2, 3$ and $c(\kappa) = \kappa^C$, c is an epimorphism, so that c is an isomorphism. Thus we have the following

14.1 **Theorem.** *The real representation ring $RO(F_4)$ is a polynomial ring $\mathbf{Z}[\lambda_1, \lambda_2, \lambda_3, \kappa]$ with 4 variables λ_1, λ_2, λ_3 and κ.*

As for $RO(\mathrm{Spin}(9))$ and $RO(\mathrm{Spin}(8))$, we can discuss in the real range. Using the fact that c is an isomorphism, then we have by 10.9, 10.11.

14.2 $$RO(\mathrm{Spin}(9)) = \mathbf{Z}[\mu_1, \mu_2, \mu_3, \Delta]$$

where μ_i is the class of $\Lambda^i(\mathfrak{J}_{01})$ for $i = 1, 2, 3$, and Δ is the class of $\mathfrak{J}_{23}$.

14.3 $$RO(\mathrm{Spin}(8)) = \mathbf{Z}[\nu_1, \nu_2, \Delta_-, \Delta_+]$$

where ν_i is the class of $\Lambda^i(\mathfrak{J}_1)$ for $i = 1, 2$ and Δ_-, Δ_+ are the classes of $\mathfrak{J}_2$, $\mathfrak{J}_3$ respectively.

15. Relations of $R(F_4)$ to $R(\mathrm{Spin}(9))$ and $R(\mathrm{Spin}(8))$

Let
$$\begin{tikzcd} \mathrm{Spin}(8) \arrow[r, "k"] \arrow[dr, "i"'] & \mathrm{Spin}(9) \arrow[d, "l"] \\ & F_4 \end{tikzcd}$$
be the inclusions.

15.1 **Theorem.** *In the diagram*

$$\begin{tikzcd} & RO(F_4) \arrow[dl, "l^*"'] \arrow[dr, "i^*"] & \\ RO(\mathrm{Spin}(9)) \arrow[rr, "k^*"] & & RO(\mathrm{Spin}(8)) \end{tikzcd}$$

namely, in

$$\begin{tikzcd} & \mathbf{Z}[\lambda_1, \lambda_2, \lambda_3, \kappa] \arrow[dl, "l^*"'] \arrow[dr, "i^*"] & \\ \mathbf{Z}[\mu_1, \mu_2, \mu_3, \Delta] \arrow[rr, "k^*"] & & \mathbf{Z}[\nu_1, \nu_2, \Delta_-, \Delta_+] \end{tikzcd}$$

we have

15.2 $$\begin{cases} l^*(\lambda_1) = 1 + \mu_1 + \Delta \\ l^*(\lambda_2) = \mu_1 + 2\mu_2 + \mu_3 + \Delta + \mu_1\Delta \\ l^*(\lambda_3) = 2\mu_1 + 2\mu_2 - \Delta + \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\Delta + 2\mu_2\Delta \\ l^*(\kappa) = \mu_2 + \Delta, \end{cases}$$

15.3
$$\begin{cases} k^*(\mu_1) = 1 + \nu_1 \\ k^*(\mu_2) = \nu_1 + \nu_2 \\ k^*(\mu_3) = -\nu_1 + \nu_2 + \varDelta_-\varDelta_+ \\ k^*(\varDelta) = \varDelta_- + \varDelta_+ \end{cases}$$

15.4
$$\begin{cases} i^*(\lambda_1) = 2 + \nu_1 + \varDelta_- + \varDelta_+ \\ i^*(\lambda_2) = 1 + 2(\nu_1 + \varDelta_- + \varDelta_+) + \nu_1\varDelta_- + \varDelta_-\varDelta_+ + \varDelta_+\nu_1 + 3\nu_2 \\ i^*(\lambda_3) = 3\,(\nu_1\varDelta_- + \varDelta_-\varDelta_+ + \varDelta_+\nu_1) + \nu_1\varDelta_-\varDelta_+ + 6\nu_2 + 2(\nu_1 + \varDelta_- + \varDelta_+)\nu_2 \\ i^*(\kappa) = \nu_1 + \nu_2 + \varDelta_- + \varDelta_+. \end{cases}$$

In the complex case, the relations between $R(F_4)$, $R(\mathrm{Spin}(9))$ *and* $R(\mathrm{Spin}(8))$ *are quite analogous to the real case (add the upper suffix* $\mathbf{C}$*).*

Proof. It suffices to show in the complex case. Since from Lemmas 11.1, 10.10 we have $\mathfrak{J}_0 = \mathbf{R} \oplus \mathfrak{J}_{01} \oplus \mathfrak{J}_{23}$ as a Spin(8)-$\mathbf{R}$-module we have obviously the first of 15.2. Using 9.2. the 2nd and 3rd of 15.2 are obtained from the first of 15.2. Since the j_1^*-image of two sides of the last of 15.2 are both $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}$ in $R(T)$ and j_1^* is an isomorphism, we see that the last of 15.2 is true. 15.3 and 15.4 are the direct consequences of Lemma 10.10 and Propositions 11.2, 12.5 respectively.

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