JOUR. FAC. SCI., SHINSHU UNIV. Vol. 2, No. 2, pp. 109—116, Dec. 1967

Connection of Flat Vector Bundles

By AKIRA ASADA

Depertment of Mathematics, Faculty of Science,
Shinshu University

(Received Sept. 30, 1967)

Introduction. A flat vector bundle over a smooth manifold X is, by definition, a bundle which admits a connection with curvature 0. It is a vector bundle induced from a representation of $\pi_1(X)$. (Asada [1], Nomizu [8]). Moreover, we can prove: "*A vector bundle which is induced from a representation of $\pi_1(X)$ is topologically trivial if and only if it admits a connection form $\{\theta_U\}$ such that $\theta_U=\theta \mid U$, where θ is a global matrix valued form on X and $d\theta+\theta_\wedge\theta=0$*". (This theorem is essentially proved by Röhrl in his study of Riemann-Hilbert's problem. (Röhrl [9], Gérard [4])). On the other hand, any connection form θ with curvature 0 of the trivial bundle ε^n satisfies $d\theta+\theta_\wedge\theta=0$ and this θ corresponds to a (unique) class $\chi=\chi(\theta)$ of Hom $(\pi_1(X),\ GL(n, \boldsymbol{F}))$. ($\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$). If $\xi(\chi)$ is the bundle induced from χ, then we have

$$\ker. D \text{ in } C^0(X,\ \varepsilon^n) \simeq \ker. d \text{ in } C^0(X,\ \xi),$$

where $D=d+\theta$ and $C^0(X,\ \xi)$ means the sheaf of germs of smooth cross-sections of ξ.

In this paper, we prove these theorems together with their relations between the representative functions of X. Their analogy for the differentiably trivial holomorphic vector bundles are also stated. In the last paragraph, we also prove "*$\chi(\theta)$ belongs in* Hom $(\pi_1(X),\ U(n))$ *if and only if there exists a positive definite matrix valued smooth functions A on X such that*

$$A\theta+(A\theta)^*=dA,$$

where B^* means ${}^t\bar{B}$". This theorem has some relations to the theorem of Fuchs.

§ 1. An exact sequence.

We start from the following two lemmas.

Lemma 1. *Let f be a matrix valued function and θ a matrix valued 1-form. Then the equation*

(1) $$df=f\theta, \quad (\text{resp. } df=\theta f),$$

has a local solution if and only if

$$(2) \qquad d\theta + \theta_\wedge\theta = 0,$$

and this solution is unique up to the multiple of constant matrices.

Lemma 2. *If the equation (1) is given on a simply connected manifold M, then (1) has a global solution on M. Moreover, we have*

$$(3) \qquad \text{det.}\, f(x) \neq 0 \text{ for all } x \in M \text{ if and only if } \text{det.}\, f(y) \neq 0 \text{ for some } y \in M.$$

Lemma 1 follows from Frobenius' theorem (Cartan [3] chap. 10). Moreover, setting

$$\theta = \sum_i h_i dx_i,$$

f is given by

$$f = \widehat{\int}(1 + h_i dx_i), \quad (i \text{ is arbitrary}).$$

Here $\widehat{\int}(1 + Adt)$ means the product integral (Sasaki [10], § 23). Then we have lemma 2.

On X, we set

$GL(n, \boldsymbol{F})_t$: *the sheaf of germs of constant $GL(n, \boldsymbol{F})$-valued functions.*

$GL(n, \boldsymbol{F})_d$: *the sheaf of germs of smooth $GL(n, \boldsymbol{F})$-valued functions.*

$\mathfrak{M}$: *the sheaf of germs of smooth $GL(n, \boldsymbol{F})$-valued 1-forms which satisfy (2).*

We note that $H^1(X, GL(n, \boldsymbol{F})_t)$ and $H^1(X, GL(n, \boldsymbol{F})_d)$ are the sets of equivalence classes of flat vector bundles and differentiable vector bundles respectively. (Hirzebruch [6] § 3).

For these sheaves, we obtain the following theorem by lemma 1.

Theorem 1. *On X, we have the following exact sequence.*

$$H^0(X, GL(n, \boldsymbol{F})_d) \xrightarrow{m} H^0(X, \mathfrak{M}) \xrightarrow{\delta} H^1(X, GL(n, \boldsymbol{F})_t) \longrightarrow$$
$$\xrightarrow{i} H^1(X, GL(n, \boldsymbol{F})_d),$$

Here $m(f) = f^{-1}df$, $\delta(\theta)_{UV} = h_U h_V^{-1}$ were $h_U^{-1}dh_U = \theta \mid U$. Moreover, we get

$$\delta(\theta) = \delta(\theta') \text{ if and only if}$$
$$\theta' = f^{-1}\theta f + f^{-1}df, \quad f \in H^0(X, GL(n, \boldsymbol{F})_d).$$

Note. If X is a complex manifold, then by Koszul-Malgrange's theorem (Koszul-Malgrange [7]), we have the following exact sequence.

$$H^0(X, GL(n, \boldsymbol{C})_d) \xrightarrow{m''} H^0(X, \mathfrak{M}^{0,1}) \xrightarrow{\delta''} H^1(X, GL(n, \boldsymbol{C})_\omega) \longrightarrow$$
$$\xrightarrow{i} H^1(X, GL(n, \boldsymbol{C})_d).$$

Here $GL(n, \boldsymbol{C})_\omega$ is the sheaf of germs of holomorphic sections of $GL(n, \boldsymbol{C})$, $\mathfrak{M}^{0,1}$ is the sheaf of germs of matrix valued (0, 1)-type forms θ such that $\bar{\partial}\theta+\theta_\wedge\theta=0$ and $m''(f)=f^{-1}\bar{\partial}f$.

We denote the universal covering space of X by $\widetilde{X}$. The projection from $\widetilde{X}$ to X is denoted by p. Then by lemma 2, we get

$$p^*(\theta)=h^{-1}dh,\quad \theta\in H^0(X,\ \mathfrak{M}).$$

For a matrix valued function f on X, we set $f^\sigma(x)=f(\sigma^{-1}x)$, $\sigma\in\pi_1(X)$. Then we may set

$$h^\sigma=\chi_\sigma h,\ \ \chi \text{ is a homomorphism from } \pi_1\ (X) \text{ into } GL(n,\ \boldsymbol{F}),$$

because $p^*(\theta)$ is invariant under the operation of π_1 (X). And by the uniqueness of h, the equivalence class of χ is determined by θ. Moreover, we obtain

Lemma 3. *$\{g\mid g^\sigma=\chi_\sigma g,\ g$ is a smooth matrix valued function on $X\}$ forms a left $\boldsymbol{F}$ -right π^* $(H^0(X,\ \mathfrak{gl}(n,\ \boldsymbol{F})_d)$ -modul. Here π^* $(H^0(X,\ \mathfrak{gl}(n,\ \boldsymbol{F})_d)$ means the ring of all smooth matrix valued functions on X which are invariant under the operations of the elements of π_1 (X).*

Lemma 4. *χ becomes the characteristic homomorphism of a bundle in i-kernel if and only if there exists a $GL(n,\ \boldsymbol{F})$-valued smooth function h such that*

$$h^\sigma=\chi_\sigma h. \tag{4}$$

Note. Since the diagram

$$\begin{array}{ccc} H^1(X,\ GL(n,\ \boldsymbol{F})_t) & \xrightarrow{i} & H^1(X,\ GL(n,\ \boldsymbol{F})_d) \\ & \searrow^{i} & \downarrow \simeq \\ & & H^1(X,\ GL(n,\ \boldsymbol{F})_c) \end{array}$$

is commutative, if there exists a $GL(n,\ \boldsymbol{F})$-valued continuous function h' which satisfies (4), then there exists a smooth h which satisfies (4). Here $GL(n,\ \boldsymbol{F})_c$ means the sheaf of germs of continuous $GL(n,\ \boldsymbol{F})$-valued functions.

By theorem 1 and lemma 4, we obtain (cf. Gérard [4], Röhrl [9)],

Theorem 2. *The following conditions are equivalent :*

(i) *A flat vector bundle ξ is differentiably trivial.*

(ii) *ξ has a connection form $\{\theta_U\}$ such that $\theta_U=\theta\mid U$, where θ belongs in H^0 $(X,\ \mathfrak{M})$.*

(iii) *There exists a $GL(n,\ \boldsymbol{F})$-valued function h on X which satisfies (4), where $\chi\in$ Hom $(\pi_1\ (X),\ GL(n,\ \boldsymbol{F}))$ is the characteristic homomorphism of ξ. (Steenrod [11], § 13).*

Note. Similarly, if X is a complex manifold and ξ is a holomorphic vector bundle on X, then ξ is differentiably trivial if and only if ξ has a connection form $\{\theta_U\}$ such that $\{\theta_U{}^{0,1}\}=\{\theta^{0,1}\mid U\}$, where $\theta^{0,1}$ belongs in $H^0(X,\ \mathfrak{M}^{0,1})$. Moreover, by Grauert's theorem (Grauert [5]), if X is a Stein manifold, then we may write

$$p^*(\theta^{0,1}) = h^{-1}\bar{\partial}h, \quad \theta^{0,1} \in H^0(X, \mathfrak{M}^{0,1}).$$

Here h satisfies

$$h^\sigma = \chi_\sigma h, \quad \bar{\partial}\chi_\sigma = 0, \quad \chi_{\sigma\tau} = \chi_\sigma^\tau \chi_\tau,$$

and since $h^{-1}\bar{\partial}h = h'^{-1}\bar{\partial}h'$ implies $h' = gh$, $\bar{\partial}g = 0$, we may consider

$$\{\chi_\sigma\} \sim \{\chi_\sigma'\} \text{ if and only if } \chi_\sigma' = g^\sigma \chi_\sigma g^{-1}, \ \bar{\partial}g = 0.$$

Hence we may consider $\{\chi_\sigma\}$ to be an element of $H^1(\pi_1(X), GL(n, C_\omega(\tilde{X})))$. Here $C_\omega(\tilde{X})$ means the ring of holomorphic functions on X.

§ 2. Elliptic complexes associated with ε^n.

If ε^n is the n-dimensional trivial vector bundle over X then $\theta \in H^0(X, \mathfrak{M})$ becomes its connection form with curvature 0 and the converse is also true. Hence setting

$$D\varphi = D_\theta\varphi = (d+\theta)\varphi,$$

we have the following sequence. Here $C^p(X, \varepsilon^n) \simeq C^p(X) \otimes F^n$ is the group of smooth ε^n-valued p-forms on X for any p, $(p \geqq 0)$, and $m = \dim. X$.

$$(5) \qquad C^0(X, \varepsilon^n) \xrightarrow{D} C^1(X, \varepsilon^n) \xrightarrow{D} \cdots\cdots \xrightarrow{D} C^m(X, \varepsilon^n).$$

Since $DD = d\theta + \theta_\wedge\theta$ is equal to 0 and the symbol sequence of (5) is exact (Asada [2]), (5) is an elliptic complex. Moreover we have

Lemma 5. *If $D\varphi$ is equal to 0, then φ is written as $D\psi$ locally.*

Proof. We set $\theta \mid U = h_U^{-1}dh_U$. Then on U, $D\varphi$ is equal to 0 if and only if $d(h_U\varphi)$ is equal to 0. Hence by Poincaré's lemma, we may set (with suitable U)

$$h_U\varphi = d\omega.$$

From this, we get $\varphi = D(h_U^{-1}\omega)$, because $\theta \mid U = h_U^{-1}dh_U$.

By lemma 5, we have

$$(6) \qquad B_\theta{}^p(X, \varepsilon^n)/DC^{p-1}(X, \varepsilon^n) \simeq H^p(X, B_\theta{}^0(X, \varepsilon^n)).$$

Here $B_\theta{}^p(X, \varepsilon^n)$ is $\{\varphi \mid \varphi \in C^p(X, \varepsilon^n), D\varphi = 0\}$.

On the other hand, if ξ is a flat vector bundle, then we obtain the following elliptic complex

$$(7) \qquad C^0(X, \xi) \xrightarrow{d} C^1(X, \xi) \xrightarrow{d} \cdots\cdots \xrightarrow{d} C^m(X, \xi).$$

If ξ is in δ-image (i.e. in i-kernel), then setting $\xi = \delta(\theta)$, we get the following commutative diagram

$$\begin{array}{ccccccc}
C^0(X,\ \xi) & \xrightarrow{d} & C^1(X,\ \xi) & \xrightarrow{d} & \cdots\cdots & \xrightarrow{d} & C^m(X,\ \xi) \\
h^\sharp \uparrow & & h^\sharp \uparrow & & & & h^\sharp \uparrow \\
C^0(X,\ \varepsilon^n) & \xrightarrow{D} & C^1(X,\ \varepsilon^n) & \xrightarrow{D} & \cdots\cdots & \xrightarrow{D} & C^m(X,\ \varepsilon^n).
\end{array}$$

Here $h^\sharp(\varphi)=\{h_U\varphi\}$, where $h_U{}^{-1}dh_U=\theta\mid U$. By this diagram, we have

Theorem 3. *$B_\theta{}^0(X,\ \varepsilon^n)$ is isomorphic to $B^0(X,\ \xi)$, where $B^0(X,\ \xi)$ is $\{f\mid f\in C^0(X,\ \xi),\ df=0\}$. In general, we get*

$$B^p(X,\ \xi)/dC^{p-1}(X,\ \xi)\simeq B_\theta{}^p(X,\ \varepsilon^n)/DC^{p-1}(X,\ \varepsilon^n).$$

Moreover, the elliptic complexes (5) *and* (7) *are equivalent if $\xi=\delta(\theta)$.*

Proof. We need only to prove the equivalence of (5) and (7) which follows from the commutativity of the diagram

$$\begin{array}{ccccccccc}
0 \to & \pi^*(\xi) & \xrightarrow{\sigma(d)} & \pi^*(\xi\otimes T^*) & \xrightarrow{\sigma(d)} & \cdots\cdots & \xrightarrow{\sigma(d)} & \pi^*(\xi\otimes\Lambda^m T^*) & \to 0 \\
 & h^* \uparrow & & h^* \uparrow & & & & h^* \uparrow & \\
0 \to & \pi^*(\varepsilon^n) & \xrightarrow{\sigma(D)} & \pi^*(\varepsilon^n\otimes T^*) & \xrightarrow{\sigma(D)} & \cdots\cdots & \xrightarrow{\sigma(D)} & \pi^*(\varepsilon^n\otimes\Lambda^m T^*) & \to 0.
\end{array}$$

Here $\sigma(d)$ and $\sigma(D)$ mean the symbols of d and D.

In general, if $i(\xi)=i(\xi')$, then setting $\xi=\{g_{UV}\}$, $\xi'=\{h_U(x)g_{UV}h_V(x)^{-1}\}$, we have the elliptic complexes

(8) $$C^0(X,\ \xi')\xrightarrow{d}C^1(X,\ \xi')\xrightarrow{d}\cdots\cdots\xrightarrow{d}C^m(X,\ \xi'),$$

(9) $$C^0(X,\ \xi)\xrightarrow{D}C^1(X,\ \xi)\xrightarrow{D}\cdots\cdots\xrightarrow{D}C^m(X,\ \xi),$$

where $D=d+\theta_U$, $\theta_U=h_U{}^{-1}dh_U$. Then since the diagrams

$$\begin{array}{ccccccc}
C^0(X,\ \xi') & \xrightarrow{d} & C^1(X,\ \xi') & \xrightarrow{d} & \cdots\cdots & \xrightarrow{d} & C^m(X,\ \xi') \\
h_\sharp \uparrow & & h_\sharp \uparrow & & & & h_\sharp \uparrow \\
C^0(X,\ \xi) & \xrightarrow{D} & C^1(X,\ \xi) & \xrightarrow{D} & \cdots\cdots & \xrightarrow{D} & C^m(X,\ \xi),
\end{array}$$

$$\begin{array}{ccccccccc}
0 \to & \pi^*(\xi') & \xrightarrow{\sigma(d)} & \pi^*(\xi'\otimes T^*) & \xrightarrow{\sigma(d)} & \cdots\cdots & \xrightarrow{\sigma(d)} & \pi^*(\xi'\otimes\Lambda^m T^*) & \to 0 \\
 & h^* \uparrow & & h^* \uparrow & & & & h^* \uparrow & \\
0 \to & \pi^*(\xi) & \xrightarrow{\sigma(D)} & \pi^*(\xi\otimes T^*) & \xrightarrow{\sigma(D)} & \cdots\cdots & \xrightarrow{\sigma(D)} & \pi^*(\xi\otimes\Lambda^m T^*) & \to 0,
\end{array}$$

are commutative, the elliptic complexes (8) and (9) are equivalent.

Note. Similarly, if ξ and ξ' are differentiably equivalent holomorphic bundles, then the elliptic complexes

$$C^0(X,\ \xi')\xrightarrow{\bar\partial}C^{0,1}(X,\ \xi')\xrightarrow{\bar\partial}\cdots\cdots\xrightarrow{\bar\partial}C^{0,m}(X,\ \xi'),$$

$$C_0(X,\ \xi)\xrightarrow{D''}D^{0,1}(X,\ \xi)\xrightarrow{D''}\cdots\cdots\xrightarrow{D''}C^{0,m}(X,\ \xi),$$

are equivalent each other. Here, setting $\xi' = \{h_U g_{UV} h_V^{-1}\}$, $\xi = \{g_{UV}\}$, D'' means $\bar{\partial} + h_U^{-1}\bar{\partial}h_U$, $2m = \dim. X$ (i. e. $m =$ *the complex dimension of* X) and $C^{0,p}(X, \xi)$ is the group of smooth $(0, p)$-type forms with coefficients in ξ.

§ 3 Bundles induced from the representation of $\pi_1(X)$ *in* $U(n)$.

Since $\pi_1(X)$ may not be a finite group in general, Hom $(\pi_1(X), U(n))$ (*or* Hom $(\pi_1(X), O(n))$) may be different from Hom $(\pi_1(X), GL(n, \boldsymbol{C}))$ (resp. Hom $(\pi_1(X), GL(n, \boldsymbol{R}))$). But since $U(n)$ is a subgroup of $GL(n, \boldsymbol{C})$, there is a map

$$\iota = \iota_U : \text{Hom}\,(\pi_1(X), U(n)) \to \text{Hom}\,(\pi_1(X), GL(n, \boldsymbol{C})).$$

In this §, we characterize these bundles ξ that belong to i-kernel and whose characteristic homomorphisms belong to ι-image.

Theorem 4. *The characteristic homomorphism of* $\xi = \delta(\theta)$ *belongs to* ι*-image if and only if there exists a positive definite matrix valued smooth function* A *on* X *such that*

$$A\theta + (A\theta)^* = dA. \tag{10}$$

Proof. If $\xi = \delta(\theta)$, then setting

$$p^*(\theta) = h^{-1}dh, \quad h^\sigma = \chi_\sigma h,$$

the characteristic homomorphism of ξ is the class of χ. If χ is in ι-image, then by lemma 4, we get

$$h^* = Ah^{-1}, \quad A^\sigma = A, \quad \sigma \in \pi_1(X). \tag{11}$$

By (11), A must be a function on X and as $A = h^*h$, A is a smooth positive definite matrix valued function. Furthermore, since $\theta^* = dh^*h^{*-1}$, we have

$$\theta^* = dAA^{-1} - A\theta A^{-1},$$

which is equivalent to (10). This proves the necessity.

On the other hand, if θ satisfies (10), then we obtain

$$h^* = Ah^{-1}C, \tag{12}$$

where C is a constant matrix and since $C = hA^{-1}h^*$, C is a positive definite matrix. Therefore we may set

$$C = BB^*, \quad B \text{ is a (constant) regular matrix.} \tag{13}$$

Using this B, we set $h' = B^{-1}h$. Then we have

$$h'^{-1}dh' = p^*(\theta), \quad (h')^\sigma = B^{-1}\chi_\sigma B h',$$

and by (12), (13), we get

$$(h')^* = Ah^{-1}B, \quad ((h')^*)^\sigma = (h')^*(B^{-1}\chi_\sigma B)^{-1}.$$

Hence $(B^{-1}\chi_\sigma B)^*$ is equal to $(B^{-1}\chi_\sigma B)^{-1}$. This proves the sufficiency.

Corollary. *If $\xi=\delta(\theta)$ is a complex line bundle, then the characteristic homomorphism of ξ is induced from a representation of $\pi_1(X)$ in $U(1)$ if and only if*

(14) $$\theta+\bar{\theta}=df,$$

where f is an arbitrary smooth real valued function on X.

Proof. Since $n=1$, the condition (10) is rewritten as

$$\theta+\theta^*=A^{-1}dA=d\log A.$$

Then as $\theta^*=\bar{\theta}$, and $\log A$ is an arbitrary smooth function on X, we have the corollary.

Note. If $n=1$, then $GL(1, \boldsymbol{C})=\boldsymbol{R}^+\times U(1)$, where $\boldsymbol{R}^+$ is the multiplicative group of all positive real numbers, and we can prove the characteristic homomorphism of $\xi=\delta(\varphi)$ belongs in $\iota_R(\mathrm{Hom}(\pi_1(X), GL(1, \boldsymbol{R})))$ if and only if

(15) $$\varphi g+\overline{g\varphi}=dg, \quad |g|=1.$$

On the other hand, since we get

$$\ker. i=H^1(X, C)/i^*(H^1(X, Z)), \text{ if } n=1,$$

Setting $\chi=\chi_1\chi_2$, $\chi_1\in\mathrm{Hom}(\pi_1(X), \boldsymbol{R}^+)$, $\chi_2\in\mathrm{Hom}(\pi_1(X), U(1))$, there are functions f_1, f_2 on X such that

$$f_1^\sigma=\chi_{1\sigma}f_1,\ f_2^\sigma=\chi_{2\sigma}f_2,\ f_1\neq 0,\ f_2\neq 0 \text{ on } X,$$

if χ is a characteristic homomorphism of a bundle $\xi=\delta(\theta)$ in $\ker. i$. Hence we have

(16) $$\theta=p^{*-1}(d\log f_1)+p^{*-1}(d\log f_2)+d\log g,$$
$$g \text{ is a smooth function on } X \text{ and } g\neq 0 \text{ on } X.$$

Since $H^0(X, \mathfrak{M})$ is the group of all closed 1-forms on X, we have by (14), (15) and (16),

(17) $\omega=\varphi+\theta+df$, ω is a closed 1-form on X, φ satisfies (15), θ satisfies (14).

In the same way, we can prove

Theorem 5. *The characteristic homomorphism of $\xi=\delta(\theta)$ belongs to $\iota(\mathrm{Hom}(\pi_1(X), O(n))$ if and only if there exists a positive definite smooth matrix valued function A on X such that*

(18) $$A\theta+{}^t(A\theta)=dA.$$

Example. We assume $X=C^1-\{z_1\cdots\cdots z_m\}$, then the characteristic homomorphism of $\delta(\theta)$ belongs in $\iota_U(\mathrm{Hom}(\pi_1(X), U(1))$ if

$$\theta = \sum_{i=1}^{m} \frac{\alpha_i}{z - z_i} + d \log f, \ \alpha_1 \text{ are real numbers.}$$

Similarly, the chracteristic homomorphism of $\delta(\theta)$ belongs to ι_R (Hom $(\pi_1(X), GL(1, \boldsymbol{R})))$ if

$$\theta = \sum_{i=1}^{m} \frac{\sqrt{-1}\beta_i}{z - z_1} + d \log f, \ \beta_i \text{ are real numbers.}$$

Hence θ has poles of order at most 1 on $\{z_1, \cdots\cdots\cdots, z_m\}$ if there exist $\rho_1, \cdots\cdots, \rho_m$ such that

$$\lim_{z \to z_i} |z - z_i|^{\rho_i} f(z) = 0, \ i = 1, \cdots\cdots, m.$$

References

[1]. ASADA, A. : Connection of topological vector bundles, Proc. J. Acad., 41 (1965), 919–922.

[2]. ASADA, A. : Elliptic semi-complexes, J. Fac. Sci. Shinshu Univ., 2 (1967), 1–17.

[3]. CARTAN, E. : *Leçons sur les invariants intégraux*, Paris, 1922.

[4]. GÉRARD, R. : Le problème de Riemann-Hilbert sur une variété analytique complexe, C. R. Acad. Sc. Paris, 264 (1967), 1133–1136.

[5]. GRAUERT, H. : Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, Math. Ann., 133 (1957), 450–472.

[6]. HIRZEBRUCH, F. : *Neue topologische Methoden in der algebraischen Geometrie*, Berlin, 1956.

[7]. KOSZUL, J. L. -MALGRANGE, B. : Sur certaines structures fibrées complexes, Arkiv der Math., 9 (1958), 102–109.

[8]. NOMIZU, K. : *Lie groups and differential geometry*, Tokyo, 1956.

[9]. RÖHRL, H. : Das Riemann-Hilbertsche Problem der Theorie der Linearen Differentialgleichungen, Math. Ann., 133 (1957), 1–25.

[10]. SASAKI, S. : *Riemannian geometry* (in Japanese), Tokyo, 1957.

[11]. STEENROD, N. E. : *The topology of fibre bundles*, Princeton, 1951.