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An Example of Differential Equation such that Every Solution is Asymptotically Vanishing, but the Equilibrium is Unstable

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1. Introduction

It is well known that, in the case of homogeneous linear differential systems, the boundedness of all solutions at the right[1)] is equivalent to the stability at the right of the equilibrium. Even in the nonlinear case, the stability of the equilibrium follows from the asymptotic vanishing of nonnull solutions under some mild conditions (cf. [3, p. 166]).

Counter examples for the circumstances above-mentioned were given by L. Cesari [1, p. 96] and R. É. Vinograd [2]. Another simple example will be provided in this note.

2. Example

Consider the system of differential equations[2)]:

$$\dot{x} = -(1+t)x\,|z| + \frac{x}{1+t} + y$$

$$\dot{y} = -(1+t)y\,|z| + \frac{y}{1+t} - x$$

$$\dot{z} = -\frac{z}{1+t}.$$

From the last equation we get $z = \frac{c}{1+t}$, where $c = z(0)$. For the sake of simplicity let us assume that c is positive.

1) A function $f(t)$ is bounded at the right if there exist positive numbers L, M such that $|f(t)| < M$ for $t > L$.

2) Differentiation with respect to the independent variable 't' is denoted by $\cdot$.

Then $$x\dot{x} + y\dot{y} = (x^2 + y^2)\left(\frac{1}{1+t} - c\right).$$

We have

$$U(t) \overset{\text{def}}{=} x^2(t) + y^2(t) = U(0)(1 + t^2)e^{-2ct},$$

which is asymptotically vanishing as $t \to \infty$, but

$$U\left(\frac{1}{c} - 1\right) > U(0)\,\frac{1}{(ce)^2},$$

which tends to infinity as $c \to 0$.

References

[1] L. CESARI : *Asymptotic behavior and stability problems in ordinary differential equations.* 2 nd edition Springer-Verlag, Berlin 1963.

[2] R. É. VINOGRAD : Inapplicability of the method of characteristic exponents to the study of nonlinear differential equations. Matem. Sbornik, t. 41 (83) (1957), pp. 431–438 (in Russian).

[2] I. G. PETROWSKI : *Vorlesungen über die Theorie der gewöhnlichen Differentialgleichungen.* Teubner, Berlin 1954.