JOUR. FAC. SCI., SHINSHU UNIV. Vol. 2, No. 2, pp. 125–138, Dec. 1967

# Representation Rings of Group $G_2$

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(Received Sept. 30, 1967)

## 1. Introduction

In this paper, we shall determine the real representation ring $RO(G_2)$ and the complex representation ring $R(G_2)$ of $G_2$, which is a simply connected compact simple Lie group of exceptional type $G$. $G_2$ is obtained as the group of all automorphisms in the division ring $\mathfrak{C}$ of Cayley numbers and $G_2$ invaries the set $L_1$ of all pure imaginary Cayley numbers, so that $L_1$ is a $G_2$-$\mathbf{R}$-module.[1] The result is as follows: $RO(G_2)$ is a polynomial ring $\mathbf{Z}[\lambda_1, \lambda_2]$ with two variables $\lambda_1$ and $\lambda_2$, where $\lambda_1$ is the class of $L_1$ in $RO(G_2)$ and $\lambda_2$ is the class of the exterior $G_2$-$\mathbf{R}$-module $\Lambda^2(L_1)$ in $RO(G_2)$. The structure of $R(G_2)$ is also a polynomial ring $\mathbf{Z}[\lambda_1^C, \lambda_2^C]$ with two variables $\lambda_1^C$ and $\lambda_2^C$, where $\lambda_1^C$, $\lambda_2^C$ are the complexification of $\lambda_1$, $\lambda_2$, respectively. In the final section, we consider the relations of $R(G_2)$ to $R(SO(7))$, $R(\mathrm{Spin}(7))$ and $R(SU(3))$.

## 2. Representation rings

Let $G$ be a topological group. By a $G$-$K$-module ($K=\mathbf{R}$ or $\mathbf{C}$)[2] is meant a finite dimensional right $K$-module $V$ together with a left action of $G$. That is, for each $x \in G$, $u \in V$ there should be defined an element $xu \in V$ depending continuously on $x$ and $u$, so that

(2. 1) $\quad x(u+v) = xu + xv$

(2. 2) $\quad x(u\lambda) = (xu)\lambda$

(2. 3) $\quad (xy)(u) = x(yu)$

(2. 4) $\quad eu = u$

for $x$, $y \in G$, $u$, $v \in V$, $\lambda \in K$ and $e$ denotes the identity of $G$.

Two $G$-$K$-modules $V_1$ and $V_2$ are $G$-$K$-isomorphic if there exists a $G$-$K$-isomorphism $f: V_1 \to V_2$, that is $f$ is a linear isomorphism such that $f(u\lambda) = f(u)\lambda$, $f(xu) = xf(u)$ for $u \in V_1$, $\lambda \in K$, $x \in G$.

Let $M_K(G)$ denote the set of $G$-$K$-isomorphism classes $[V]$ of $G$-$K$-modules $V$.

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1) $\mathbf{R}$ is the field of real numbers.
2) $\mathbf{C}$ is the field of complex numbers.

$[V]$ will also be denoted by $V$.

The direct sum $V_1 \oplus V_2$ and the tensor product $V_1 \otimes V_2$ of two $G$-$K$-modules $V_1$ and $V_2$ define a semiring structure on $M_K(G)$. The representation ring $R_K(G) = (R_K(G), \phi_G)$ is the universal ring associated with the semiring $M_K(G)$; that is, $\phi_G: M_K(G) \to R_K(G)$ is a semiring homomorphism and for any ring $A$ and any semiring homomorphism $\varphi : M_K(G) \to A$, there exists a unigue ring homomorphism $\tilde{\varphi} : R_k(G) \to A$ such that $\varphi = \tilde{\varphi}\phi_G$.

$R_K(G)$ is a commutative ring with the unit 1, where 1 is the class of $K$ with trivial group action.

Note that $M_K(G)$ has two further oprations: For each $G$-$K$-module $V$, there correspond the exterior $G$-$K$-module $\Lambda^r(V)$ $(0 \leqq r \leqq \dim V)$ and the dual $G$-$K$-module $\hat{V}$ ($\hat{V}$ is $\mathrm{Hom}_K(V, K)$ as $K$-module and group action is $(x\xi)(u) = \xi(x^{-1}u)$, for $x \in G$, $\xi \in \mathrm{Hom}_K(V, K)$, $u \in V$).

Let $H$ and $G$ be topological groups and $h: H \to G$ be a continuous homomorphism. Then, to every $G$-$K$-module $V$, there corresponds a $H$-$K$-module $h^{\#}(V)$ by the rule of group action

$$yu = h(y)u, \qquad \text{for } y \in H, \ u \in V.$$

The correspondence $V \to h^{\#}(V)$ gives rise to a ring homomorphism $h^* : R_K(G) \to R_K(H)$ such that the following diagram is commutative.

$$\begin{array}{ccc} M_K(G) & \xrightarrow{h^{\#}} & M_K(H) \\ \downarrow \phi_G & h^* & \downarrow \phi_H \\ R_K(G) & \longrightarrow & R_K(H). \end{array}$$

$M_{\mathbf{R}}(G)$, $R_{\mathbf{R}}(G)$ are denoted by $MO(G)$, $RO(G)$ and $M_{\mathbf{C}}(G)$, $R_{\mathbf{C}}(G)$ by $M(G)$, $R(G)$ respectively.

## 3. Cayley numbers $\mathfrak{C}$ and Group $G_2$

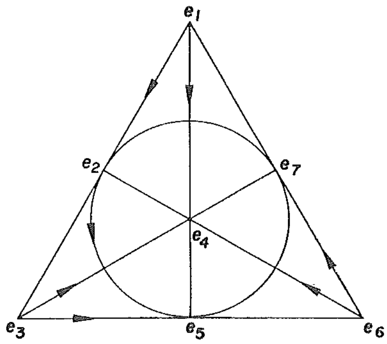
Let $\mathfrak{C}$ denote the division ring of Cayley numbers. $\mathfrak{C}$ is an 8-dimensional $\mathbf{R}$-module with an additive base $e_0$, $e_1, \cdots\cdots, e_7$, and ring structure is given as follows;

$$e_0 \text{ is the unit of } \mathfrak{C},$$

$$e_i^2 = -e_0, \qquad \text{for } i \neq 0,$$

$$e_ie_j = -e_je_i \qquad \text{for } i, j \neq 0, \ i \neq j,$$

and




(for example, $e_1e_2 = e_3$, $e_2e_5 = e_7$, $e_2e_4 = -e_6$)

Let $G_2$ be the group of all automorphisms in $\mathfrak{C}$, that is, each $x \in G_2$ satisfies

(3. 1) $x(u+v) = xu + xv$

(3. 2) $x(u\lambda) = (xu)\lambda$ for $u,\ v \in \mathfrak{C},\ \lambda \in \mathbf{R}$.

(3. 3) $x(uv) = x(u)x(v)$

(3. 4) $x$ is non-singular

As is well known, $G_2$ is a simply connected compact simple Lie group of exceptional type $G$ [3].

Obviously, $\mathfrak{C}$ is a $G_2$-$\mathbf{R}$-module. By (3, 3), (3, 4), we have $x(e_0) = e_0$ for $x \in G_2$. Therefore, if we denote by $L_1$ the $\mathbf{R}$-submodule of $\mathfrak{C}$ generated by $e_1$, ……, $e_7$ additively, then $L_1$ is also a $G_2$-$\mathbf{R}$-module and $\mathfrak{C}$ is decomposable into the direct sum of two $G_2$-$\mathbf{R}$-modules $\mathbf{R}$ (with trivial group action) and $L_1$; $\mathfrak{C} = \mathbf{R} \oplus L_1$. The complexification $L_1^C = L_1 \otimes_{\mathbf{R}} \mathbf{C}$ of $L_1$ is a $G_2$-$\mathbf{C}$-module and it will play an important role in the sequel.

## 4. Maximal torus $T$ and Weyl group $W$ of $G_2$

$G_2$ has a subgroup $SU(3)$ consisting of all elements $x$ of $G_2$ such that $x(e_1) = e_1$. Since the ranks of $G_2$ and $SU(3)$ are both 2, any maximal torus in $SU(3)$ is also a maximal torus in $G_2$.

Let $t_i : \mathbf{R} \to G_2$ for $i = 1, 2$ be the homomorphisms given by the relations

$$(4.1)\quad \begin{cases} t_1(\theta)(e_j) = e_j & \text{for } j = 0,\ 1,\ 4 \\ t_1(\theta)(e_2) = e_2 \cos\theta + e_3 \sin\theta \end{cases}$$

$$(4.2)\quad \begin{cases} t_2(\theta)(e_j) = e_j & \text{for } j = 0,\ 1,\ 2. \\ t_2(\theta)(e_4) = e_4 \cos\theta + e_5 \sin\theta \end{cases}$$

Let $t : \mathbf{R}^2 = \mathbf{R} \times \mathbf{R} \to G_2$ be defined by

$$t(\theta_1,\ \theta_2) = t_1(\theta_1)t_2(\theta_2)$$

for $(\theta_1,\ \theta_2) \in \mathbf{R}^2$. Define $T = t(\mathbf{R}^2)$, then $T$ is a maximal torus in $G_2$; $T \subset SU(3) \subset G_2$.

From (4. 1), we have

$$(4.3)\quad \begin{cases} t_1(\theta)(e_3) = t_1(\theta)(e_1e_2) = t_1(\theta)(e_1)t_1(\theta)(e_2) \\ \qquad = e_1(e_2\cos\theta + e_3\sin\theta) = -e_2\sin\theta + e_3\cos\theta \\ t_1(\theta)(e_5) = t_1(\theta)(e_1e_4) = t_1(\theta)(e_1)t_1(\theta)(e_4) = e_1e_4 = e_5 \\ t_1(\theta)(e_6) = t_1(\theta)(e_4e_2) = t_1(\theta)(e_4)t_1(\theta)(e_2) \\ \qquad = e_4(e_2\cos\theta + e_3\sin\theta) = e_6\cos\theta - e_7\sin\theta \\ t_1(\theta)(e_7) = t_1(\theta)(e_1e_6) = t_1(\theta)(e_1)t_1(\theta)(e_6) \\ \qquad = e_1(e_6\cos\theta - e_7\sin\theta) = e_6\sin\theta + e_7\cos\theta. \end{cases}$$

Similarly, from (4. 2) we have

$$(4.4)\quad \begin{cases} t_2(\theta)(e_3) = e_3 \\ t_2(\theta)(e_5) = -e_4\sin\theta + e_5\cos\theta \\ t_2(\theta)(e_6) = e_6\cos\theta - e_7\sin\theta \\ t_2(\theta)(e_7) = e_6\sin\theta + e_7\cos\theta. \end{cases}$$

The Weyl group $W=W(G_2)$ of $G_2$ is $N_T(G_2)/T$, where $N_T(G_2)$ is the normalizer of $T$ in $G_2$. If $x\in N_T(G_2)$, then $x(e_1)=\pm e_1$. In fact, since $x^{-1}tx\in T$ for any $t\in T$, we have $x^{-1}tx(e_1)=e_1$, hence $tx(e_1)=x(e_1)$. Put $x(e_1)=\sum_{i=1}^{7}e_ia_i$ ($a_i\in\mathbf{R}$, $\sum_{i=1}^{7}a_i^2=1$), then $t(\sum_{i=1}^{7}e_ia_i)=\sum_{i=1}^{7}e_ia_i$ for all $t\in T$. Using (4. 1)—(4. 4),

$$\begin{aligned} & e_1a_1 + (e_2\cos\theta_1 + e_3\sin\theta_1)a_2 + (-e_2\sin\theta_1 + e_3\cos\theta_1)a_3 \\ &\quad + (e_4\cos\theta_2 + e_5\sin\theta_2)a_4 + (-e_4\sin\theta_2 + e_5\cos\theta_2)a_5 \\ &\quad + (e_6\cos(\theta_1+\theta_2) - e_7\sin(\theta_1+\theta_2))a_6 + (e_6\sin(\theta_1+\theta_2) + e_7\cos(\theta_1+\theta_2))a_7 \\ &\quad = e_1a_1 + e_2a_2 + \cdots\cdots + e_7a_7 \qquad \text{for all } \theta_1,\ \theta_2\in\mathbf{R}. \end{aligned}$$

Hence we have

$$\begin{cases} a_2\cos\theta_1 - a_3\sin\theta_1 = a_2 \\ a_4\cos\theta_2 - a_5\sin\theta_2 = a_4 \\ a_6\cos\theta_3 - a_7\sin\theta_3 = a_6 \end{cases} \qquad \text{for all } \theta_1,\ \theta_2,\ \theta_3\in\mathbf{R},$$

where $\theta_3=-(\theta_1+\theta_2)$, so that we have $a_2=\cdots\cdots=a_7=0$. This implies that $x(e_1)=e_1a_1=\pm e_1$.

In case $x(e_1)=e_1$, $x$ is an element of the normalizer $N_T(SU(3))$ of $T$ in $SU(3)$. Therefore $W(G_2)$ contains the Weyl group $W(SU(3))=N_T(SU(3))/T$ of $SU(3)$, which is the symmetric group consisting of all permutations of 3 variables $\theta_1$, $\theta_2$, $\theta_3$. In case $x(e_1)=-e_1$, if we choose an element $y\in G_2$ such that $y(e_i)=e_i$ for $i=0$, 2, 4, 6 and $y(e_i)=-e_i$ for $i=1$, 3, 5, 7, then $yx\in N_T(SU(3))$ and $y$ induces the change of the sign $(\theta_1, \theta_2, \theta_3)\to(-\theta_1, -\theta_2, -\theta_3)$. $W(G_2)$ has 12 elements.

## 5. $G_2$-C-module $L_1^C = L_1 \otimes_R C$ and $R(T)$

Let $j_2 : T \to G_2$ denote the inclusion. *In* $j_2^\# : M(G_2) \to M(T)$, *we have*

$$j_2^\#(L_1^C) = \mathbf{C} \oplus W_1 \oplus \hat{W}_1 \oplus W_2 \oplus \hat{W} \oplus W_3 \oplus \hat{W}_3,$$

*where* $W_i$ *is* 1-*dimensional* $T$-$\mathbf{C}$-*module and* $\hat{W}_i$ *is the dual* $T$-$\mathbf{C}$-*module of* $W_i$, *for* $i = 1, 2, 3$. *And there exist relations*

$$W_i \otimes \hat{W}_i = \mathbf{C}, \qquad \textit{for } i = 1, 2, 3,$$
$$W_1 \otimes W_2 \otimes W_3 = \mathbf{C}.$$

In fact, let $\mathbf{C}$ be the $\mathbf{C}$-module with base $e_1$, and $W_i$, $\hat{W}_i$ be $\mathbf{C}$-modules with base $u_i = e_{2i} - e_{2i+1}\sqrt{-1}$, $\hat{u}_i = e_{2i} + e_{2i+1}\sqrt{-1}$ respectively for $i = 1, 2, 3$. For $t = t(\theta_1, \theta_2)$,

(5.1) $$te_1 = e_1,$$

(5.2) $$\begin{cases} tu_1 = t(\theta_1, \theta_2)(e_2 - e_3\sqrt{-1}) = t_1(\theta_1)(e_2 - e_3\sqrt{-1}) \\ \quad = (e_2 \cos\theta_1 + e_3 \sin\theta_1) - (-e_2 \sin\theta_1 + e_3 \cos\theta_1)\sqrt{-1} \\ \quad = (e_2 - e_3\sqrt{-1})(\cos\theta_1 + \sqrt{-1}\sin\theta_1) = u_1 \exp(\sqrt{-1}\,\theta_1) \\ t\hat{u}_1 = \hat{u}_1 \exp((-\sqrt{-1}\,\theta_1). \end{cases}$$

Similarly

(5.3) $$\begin{cases} tu_2 = u_2 \exp((\sqrt{-1}\,\theta_2) \\ t\hat{u}_2 = \hat{u}_2 \exp(-\sqrt{-1}\,\theta_2), \end{cases}$$

(5.4) $$\begin{cases} tu_3 = u_3 \exp(-\sqrt{-1}(\theta_1 + \theta_2)) \\ t\hat{u}_3 = \hat{u}_3 \exp(\sqrt{-1}(\theta_1 + \theta_2)). \end{cases}$$

Since $e_1$, $u_1$, $\hat{u}_1$, $u_2$, $\hat{u}_2$, $u_3$, and $\hat{u}_3$ are an additive base of $L_1^C$, formulae (5.1)—(5.4) yield the desired result.

Put $\phi_T(W_1) = \alpha$, $\phi_T(W_2) = \beta$, and $\phi_T(W_3) = \gamma$, then we have

$$R(T) = \mathbf{Z}[\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}]/(\alpha\beta\gamma - 1).$$

Define $\lambda_1^C = \phi_{G_2}(L_1^C)$ and $\lambda_2^C = \phi_{G_2}(\Lambda^2(L_1^C))$. Then we have

(5.5) $$\begin{aligned} j_2^*(\lambda_1^C) &= j_2^*(\phi_{G_2}(L_1^C)) = \phi_T(j_2^\#(L_1^C)) \\ &= \phi_T(\mathbf{C} \oplus W_1 \oplus \hat{W}_1 \oplus W_2 \oplus \hat{W}_2 \oplus W_3 \oplus \hat{W}_3) \\ &= 1 + \alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1}, \end{aligned}$$

(5.6) $$\begin{aligned} j_2^*(\lambda_2^C) &= j_2^*(\phi_{G_2}(\Lambda^2(L_1^C))) = \phi_T(\Lambda^2(j_2^\#(L_1^C))) \\ &= \phi_T(\Lambda^2(\mathbf{C} \oplus W_1 \oplus \hat{W}_1 \oplus W_2 \oplus \hat{W}_2 \oplus W_3 \oplus \hat{W}_3)) \\ &= \phi_T(W_1 \oplus \hat{W}_1 \oplus W_2 \oplus \hat{W}_2 \oplus W_3 \oplus \hat{W}_3 \end{aligned}$$

$$\begin{aligned}
&\oplus \mathbf{C} \oplus W_1 \otimes W_2 \oplus W_1 \otimes \widehat{W}_2 \oplus W_1 \otimes W_3 \oplus W_1 \otimes \widehat{W}_3 \\
&\oplus \widehat{W}_1 \otimes W_2 \oplus \widehat{W}_1 \otimes \widehat{W}_2 \oplus \widehat{W}_1 \otimes W_3 \oplus \widehat{W}_1 \otimes \widehat{W}_3 \\
&\oplus \mathbf{C} \oplus W_2 \otimes W_3 \oplus W_2 \otimes \widehat{W}_3 \oplus \widehat{W}_2 \otimes W_3 \oplus \widehat{W}_2 \otimes \widehat{W}_3 \oplus \mathbf{C}) \\
&= \alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1} + 1 + \alpha\beta + \alpha\beta^{-1} + \alpha\gamma + \alpha\gamma^{-1} \\
&\quad + \alpha^{-1}\beta + \alpha^{-1}\beta^{-1} + \alpha^{-1}\gamma + \alpha^{-1}\gamma^{-1} + 1 + \beta\gamma + \beta\gamma^{-1} + \beta^{-1}\gamma + \beta^{-1}\gamma^{-1} + 1 \\
&= \alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1} + 1 + \gamma^{-1} + \alpha\beta^{-1} + \beta^{-1} + \alpha\gamma^{-1} \\
&\quad + \alpha^{-1}\beta + \gamma + \alpha^{-1}\gamma + \beta + 1 + \alpha^{-1} + \beta\gamma^{-1} + \beta^{-1}\gamma + \alpha + 1 \\
&= 3 + 2(\alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1}) \\
&\quad + \alpha\beta^{-1} + \alpha^{-1}\beta + \beta\gamma^{-1} + \beta^{-1}\gamma + \gamma\alpha^{-1} + \gamma^{-1}\alpha.
\end{aligned}$$

## 6. Ring structure of $R(T)^W$

Each element $w : T \to T$ in the Weyl group $W$ induces an automorphism $w^* : R(T) \to R(T)$ which permutes the 3 factors $\alpha$, $\beta$, $\gamma$ together with the map of the form $(\alpha, \beta, \gamma) \to (\alpha^{-1}, \beta^{-1}, \gamma^{-1})$. Let $R(T)^W$ denote the subring of $R(T)$ which is invariant under these operations $w^*$ (called $W$-invariant briefly). Since $j_2^* : R(G_2) \to R(T)$ is a ring monomorphism and the image of $j_2^*$ is contained in $R(T)^W$ [4], we will regard $R(G_2)$ as a subring of $R(T)^W$; $R(G_2) \subset R(T)^W$. We shall determine the ring structure of $R(T)^W$.

Put $\nu_1 = \alpha + \beta + \gamma$ and $\nu_2 = \beta\gamma + \gamma\alpha + \alpha\beta = \alpha^{-1} + \beta^{-1} + \gamma^{-1}$, (cf. section 8) then $\nu_1 + \nu_2 = \alpha + \beta + \gamma + \alpha^{-1} + \beta^{-1} + \gamma^{-1}$ and $\nu_1\nu_2 - 3 = \alpha\beta^{-1} + \alpha^{-1}\beta + \beta\gamma^{-1} + \beta^{-1}\gamma + \gamma\alpha^{-1} + \gamma^{-1}\alpha$ are $W$-invariant polynomials (the elementary $W$-invariant function !) and we have $\lambda_1^C = 1 + \nu_1 + \nu_2$ and $\lambda_2^C = 2(\nu_1 + \nu_2) + \nu_1\nu_2$ from (5.5), (5.6).

Let $f \in R(T)^W$. Case 1. If a monomial $\alpha^m\beta^n$ $(m > n > 0,\ m \neq 2n)$ appears in $f$, a polynomial $g + h$ also appears in $f$, where

$$g = \alpha^m\beta^n + \beta^m\alpha^n + \beta^m\gamma^n + \gamma^m\beta^n + \gamma^m\alpha^n + \alpha^m\gamma^n.$$
$$h = \alpha^{-m}\beta^{-n} + \beta^{-m}\alpha^{-n} + \beta^{-m}\gamma^{-n} + \gamma^{-m}\beta^{-n} + \gamma^{-m}\alpha^{-n} + \alpha^{-m}\gamma^{-n}.$$

Since $g$ is a symmetric function in 3 variables $\alpha$, $\beta$, $\gamma$, $g$ is representable as a polynomial in the elementary symmetric polynomials $\alpha + \beta + \gamma = \nu_1$, $\alpha\beta + \beta\gamma + \gamma\alpha = \nu_2$ and $\alpha\beta\gamma = 1$; that is, there exists a polynomial $P(X, Y) \in \mathbf{Z}[X, Y]$ with two variables $X$, $Y$ such that $g = P(\nu_1, \nu_2)$. On the other hand, $h$ can be represented by the same polynomial $P$ in the elementary symmetric polynomials $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = \nu_2$, $\alpha^{-1}\beta^{-1} + \beta^{-1}\gamma^{-1} + \gamma^{-1}\alpha^{-1} = \gamma + \alpha + \beta = \nu_1$ and $\alpha^{-1}\beta^{-1}\gamma^{-1} = 1$: $h = P(\nu_2, \nu_1)$. Therefore $g + h = P(\nu_1, \nu_2) + P(\nu_2, \nu_1)$ is symmetric in variables $\nu_1$ and $\nu_2$, so that $g + h$ is a polynomial in $\nu_1 + \nu_2 = \lambda_1^C - 1$ and $\nu_1\nu_2 = \lambda_2^C - 2\lambda_1^C + 2$. Hence $g + h$ is a polynomial in $\lambda_1^C$ and $\lambda_2^C$.

Case 2. If a monomial $\alpha^m$ $(m > 0)$ appears in $f$, a polynomial $g + h$ also appears in $f$, where $g = \alpha^m + \beta^m + \gamma^m$, $h = \alpha^{-m} + \beta^{-m} + \gamma^{-m}$. The same statements as in Case 1 hold.

Case 3. If $f$ contains a monomial $\alpha^{2m}\beta^m$ $(m>0)$, then $f$ contains a polynomial $g=\alpha^{2m}\beta^m+\beta^{2m}\alpha^m+\beta^{2m}\gamma^m+\gamma^{2m}\beta^m+\gamma^{2m}\alpha^m+\alpha^{2m}\gamma^m$ (note that $\alpha^{-2m}\beta^{-m}=(\beta\gamma)^{2m}\beta^{-m}=\gamma^{2m}\beta^m$ etc.). We shall show that $g$ is also a polynomial in $\lambda_1^C$ and $\lambda_2^C$, by the induction with respect to $m$. First we have for $m=1$, $\alpha^2\beta+\beta^2\alpha+\beta^2\gamma+\gamma^2\beta+\gamma^2\alpha+\alpha^2\gamma=\alpha\gamma^{-1}+\beta\gamma^{-1}+\beta\alpha^{-1}+\gamma\alpha^{-1}+\gamma\beta^{-1}+\alpha\beta^{-1}=(\alpha+\beta+\gamma)\ (\alpha^{-1}+\beta^{-1}+\gamma^{-1})-3=\nu_1\nu_2-3=\ \lambda_2^C-2\lambda_1^C-1$. Suppose that the assertion is true for $k<m$. Now, for $m$, if we describe $g=(\alpha^2\beta+\beta^2\alpha+\beta^2\gamma+\gamma^2\beta+\gamma^2\alpha+\alpha^2\gamma)^m+h$, $h$ is a polynomial of Case 1, 2 or the lower degree than $m$ of Case 3. Hence by the induction, $g$ is a polynomial in $\lambda_1^C$ and $\lambda_2^C$.

We have thus proved that any polynomial in $R(T)^W$ is representable as a polynomial in $\lambda_1^C$, $\lambda_2^C$.

In addition, $\lambda_1^C$ and $\lambda_2^C$ are algebraically independent. In fact, $\nu_1$ and $\nu_2$ are algebraically independent in $\mathbf{Z}[\alpha,\alpha^{-1},\beta,\beta^{-1},\gamma,\gamma^{-1}]/(\alpha\beta\gamma-1)$. Therefore, $\nu_1+\nu_2$ and $\nu_1\nu_2$ are also algebraically independent. Since $\lambda_1^C=\nu_1+\nu_2+1$ and $\lambda_2^C=2(\nu_1+\nu_2)+\nu_1\nu_2$, we have that $\lambda_1^C$ and $\lambda_2^C$ are algebraically independent. And we have $\mathbf{Z}[\lambda_1^C,\lambda_2^C]\subset R(G_2)\subset R(T)^W=\mathbf{Z}[\lambda_1^C,\lambda_2^C]$. Thus, we have the following

**Theorem.** *The complex representation ring $R(G_2)$ of $G_2$ is a polynomial ring $\mathbf{Z}[\lambda_1^C,\lambda_2^C]$ with two variables $\lambda_1^C$ and $\lambda_2^C$, where $\lambda_1^C$ is the class of the $G_2$-$\mathbf{C}$-module $L_2^C$ in $R(G_2)$ and $\lambda_2^C$ is the class of the exterior $G_2$-$\mathbf{C}$-module $\Lambda^2(L_1^C)$ in $R(G_2)$.*

## 7. Real representation ring $RO(G_2)$

For a topological group $G$, we have the following correspondences:

$$c: RO(G)\to R(G),\qquad r: R(G)\to RO(G),$$

where $c$ is a ring homomorphism induced by the tensoring $c'$ with $\mathbf{C}$ (that is, $c'$ $: MO(G)\to M(G)$ is defined by $c'(V)=V\otimes_{\mathbf{R}}\mathbf{C})$) and $r$ is a homomorphism defined by the restricting scalars from $\mathbf{C}$ to $\mathbf{R}$. As is well known, relation $rc=2$ holds. If $G$ is a compact group, $RO(G)$ is the free module generated by the classes of irreducible $G$-$\mathbf{R}$-modules, so that relation $rc=2$ implies that $c$ is a ring monomorphism.

As for $G_2$, since we have obviously $c(\lambda_1)=\lambda_1^C$ and $c(\lambda_2)=\lambda_2^C$, (where $\lambda_1$ and $\lambda_2$ are the classes of $L_1$ and $\Lambda^2(L_1)$ in $RO(G_2)$ respectively), $c$ is an epimorphism. Hence $c$ is an isomorphism. Thus we have the following

**Theorem.** *The real representation ring $RO(G_2)$ is a polynomial ring $\mathbf{Z}[\lambda_1,\lambda_2]$ with two variables* $\lambda_1$ and $\lambda_2$.

## 8. Lie algebra $\mathfrak{g}_2$ and Element $\lambda_2-\lambda_1$

The Lie algebra $\mathfrak{so}(7)$ of $SO(7)$ (the rotation group in $L_1$) consists of all $\mathbf{R}$-homomorphisms $A$ of $\mathfrak{C}$ satisfying

$$\begin{cases} A(e_0)=0 \\ (A(u),\ v)^{3)}+(u,\ A(v))=0 \qquad \text{for } u,\ v\in\mathfrak{C}. \end{cases}$$

Let $G_{ij}$ $(i,\ j=1,\ \cdots\cdots,\ 7,\ i\neq j)$ be the $\mathbf{R}$-homomorphism given by

$$\begin{cases} G_{ij}(e_j)=e_i \\ G_{ij}(e_i)=-e_j \\ G_{ij}(e_k)=0 \qquad \text{for } k\neq i,\ j,\ 0\leqq k\leqq 7. \end{cases}$$

Then 21 elements $G_{ij}$ $(1\leqq i<j\leqq 7)$ are an additive base in $\mathfrak{so}(7)$.

The Lie algebra $\mathfrak{g}_2$ of $G_2$ is a Lie subalgebra of $\mathfrak{so}(7)$ consisting of all $A$ such that

$$A(u)v+uA(v)=A(uv) \qquad \text{for } u,\ v\in\mathfrak{C}.$$

$\mathfrak{g}_2$ is a $G_2$-$\mathbf{R}$-module with the group operation given by

$$(xA)(u)=x(A(x^{-1}u)) \qquad \text{for } x\in G_2,\ A\in\mathfrak{g}_2,\ u\in\mathfrak{C}.$$

So that its complex form $\mathfrak{g}_2^C=\mathfrak{g}_2\otimes_{\mathbf{R}}\mathbf{C}$ is a $G_2$-$\mathbf{C}$-module. We shall show that $\phi_{G_2}(\mathfrak{g}_2^C)=\lambda_2^C-\lambda_1^C$. We choose an additive base in $\mathfrak{g}_2^C$ as follows:

$$\begin{aligned}
H_1 &= 2G_{23}-G_{45}-G_{67} \\
H_2 &= -G_{23}+2G_{45}-G_{67} \\
U_1 &= -2G_{13}+G_{46}+G_{57}-(2G_{12}-G_{47}-G_{56})\sqrt{-1} \\
\hat{U}_1 &= -2G_{13}+G_{46}+G_{57}+(2G_{12}-G_{47}-G_{56})\sqrt{-1} \\
U_2 &= -2G_{15}-G_{26}+G_{37}-(2G_{14}+G_{27}+G_{36})\sqrt{-1} \\
\hat{U}_2 &= -2G_{15}-G_{26}+G_{37}+(2G_{14}+G_{27}+G_{36})\sqrt{-1} \\
U_3 &= 2G_{17}-G_{24}+G_{35}-(-2G_{16}+G_{25}+G_{34})\sqrt{-1} \\
\hat{U}_3 &= 2G_{17}-G_{24}+G_{35}+(-2G_{16}+G_{25}+G_{34})\sqrt{-1} \\
U_{12} &= G_{24}+G_{35}-(-G_{25}+G_{34})\sqrt{-1} \\
\hat{U}_{12} &= G_{24}+G_{35}+(-G_{25}+G_{34})\sqrt{-1} \\
U_{23} &= G_{46}+G_{57}-(-G_{47}+G_{56})\sqrt{-1} \\
\hat{U}_{23} &= G_{46}+G_{57}+(-G_{47}+G_{56})\sqrt{-1} \\
U_{31} &= G_{26}+G_{37}-(G_{27}-G_{36})\sqrt{-1} \\
\hat{U}_{31} &= G_{26}+G_{37}+(G_{27}-G_{36})\sqrt{-1}.
\end{aligned}$$

3) The inner product $(u,\ v)$, where $u=\sum_{i=0}^{7}e_iu_i$, $v=\sum_{i=0}^{7}e_iv_i$, is meant by $\sum_{i=0}^{7}u_iv_i$.

Then, for $t = t(\theta_1, \theta_2, \theta_3)$ we have

$$\begin{aligned} tH_1 &= H_1, & tH_2 &= H_2 \\ tU_1 &= U_1 \exp(\sqrt{-1}\,\theta_1), & t\hat{U}_1 &= \hat{U}_1 \exp(-\sqrt{-1}\,\theta_1), \\ tU_2 &= U_2 \exp(\sqrt{-1}\,\theta_2), & t\hat{U}_2 &= \hat{U}_2 \exp(-\sqrt{-1}\,\theta_2), \\ tU_3 &= U_3 \exp(\sqrt{-1}\,\theta_3), & t\hat{U}_3 &= \hat{U}_3 \exp(-\sqrt{-1}\,\theta_3), \\ tU_{12} &= U_{12} \exp(\sqrt{-1}(\theta_1 - \theta_2)), & t\hat{U}_{12} &= \hat{U}_{12} \exp(\sqrt{-1}(\theta_2 - \theta_1)), \\ tU_{23} &= U_{23} \exp(\sqrt{-1}(\theta_2 - \theta_3)), & t\hat{U}_{23} &= \hat{U}_{23} \exp(\sqrt{-1}(\theta_3 - \theta_2)), \\ tU_{31} &= U_{31} \exp(\sqrt{-1}(\theta_3 - \theta_1)), & t\hat{U}_{31} &= \hat{U}_{31} \exp(\sqrt{-1}(\theta_1 - \theta_3)). \end{aligned}$$

One of them, for example, $tU_{12} = U_1 \exp(\sqrt{-1}(\theta_1 - \theta_2))$ will be proved. To do so, we need to show $(tU_{12})(e_i) = U_{12}(e_i)\exp(\sqrt{-1}(\theta_1 - \theta_2))$ for $i = 0, 1, \cdots\cdots, 7$. We shall show again one of them, for example, for $i = 4$. $(tU_{12})(e_4) = t(U_{12}(t^{-1}e_4))$ $= t(U_{12}(e_4 \cos\theta_2 - e_5 \sin\theta_2)) = t((e_2 - e_3\sqrt{-1})\cos\theta_2 - (e_3 + e_2\sqrt{-1})\sin\theta_2) = t(e_2$ $- e_3\sqrt{-1})(\cos\theta_2 - \sqrt{-1}\sin\theta_2) = ((e_2\cos\theta_1 + e_3\sin\theta_1) - (-e_2\sin\theta_1 + e_3\cos\theta_1)\sqrt{-1})$ $(\cos\theta_2 - \sqrt{-1}\sin\theta_2) = (e_2 - e_3\sqrt{-1})(\cos\theta_1 + \sqrt{-1}\sin\theta_1)(\cos\theta_2 - \sqrt{-1}\sin\theta_2)$ $= (e_2 - e_3\sqrt{-1})\exp(\sqrt{-1}(\theta_1 - \theta_2)) = U_{12}(e_4)\exp(\sqrt{-1}(\theta_1 - \theta_2))$. Thus we have proved these formulae. Hence, $\phi_{G_2}(\mathfrak{g}_2^C) = 2 + \alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1} + \alpha\beta^{-1} + \alpha^{-1}\beta + \beta\gamma^{-1} + \beta^{-1}\gamma + \gamma\alpha^{-1} + \gamma^{-1}\alpha = \lambda_2^C - \lambda_1^C$. Thus we have the following

**Theorem.** *The class of $G_2$-**C**-module $\mathfrak{g}_2^C$ in $R(G_2)$ is $\lambda_2^C - \lambda_1^C$, where $\mathfrak{g}_2$ is the Lie algebra of $G_2$.*

Since the complexification $c : RO(G_2) \to R(G_2)$ is an isomorphism and $\mathfrak{g}_2$ is a $G_2$-**R**-module, we have also in $RO(G_2)$ the following

**Theorem.** *The class of $G_2$-**R**-module $\mathfrak{g}_2$ in $RO(G_2)$ is $\lambda_2 - \lambda_1$.*

## 9. $SO(7)$ and Spin(7)

Let $SO(8)$ be the rotation group in $\mathfrak{C}$ and $SO(7)$ be the rotation group in $L_1$, namely $SO(7)$ is a subgroup of $SO(8)$ consisting of all elements $x \in SO(8)$ such that $x(e_0) = e_0$.

We remember the principle of triality in $SO(8)$ [2].

*For every $x_1 \in SO(8)$, there exist $x_2, x_3 \in SO(8)$ such that*

(9.1) $\qquad x_1(u)x_2(v) = x_3(uv). \qquad$ *for $u, v \in \mathfrak{C}$,*

*and for $x_1$, such $x_2$, $x_3$ are unique up to the sign.*

If $x_1 \in SO(7)$, we have $x_2 = x_3$. For, if we put $u = e_0$ in (9.1), we have $x_1(e_0)\cdot x_2(v) = x_3(v)$, hence $x_2(v) = x_3(v)$ for all $v \in \mathfrak{C}$. This implies that $x_2 = x_3$.

Consider the subgroup of $SO(8)$, denoted by Spin(7), consisting of all elements $x \in SO(8)$ such that for some $x \in$ SO(7)

$$x(u)\tilde{x}(v) = \tilde{x}(uv) \qquad \text{for all } u,\ v \in \mathfrak{C}.$$

Spin(7) is a simply connected group and the projection $p : \mathrm{Spin}(7) \to SO(7)$ defined by $p(\tilde{x}) = x$ is a twofold covering of $SO(7)$.

$G_2$ is a subgroup of $SO(7)$ and Spin(7) and

$$G_2 = SO(7) \cap \mathrm{Spin}(7),$$

and so we have the commutative diagram

$$\begin{array}{ccccc} SU(3) & \xrightarrow{i} & G_2 & \xrightarrow{k} & \mathrm{Spin}(7) \\ & & & \searrow^{l} & \downarrow p \\ & & & & SO(7) \end{array}$$

where $i$, $k$, $l$ are inclusions.

We shall choose maximal tori $T'$ and $\widetilde{T}'$ in $SO(7)$ and Spin(7) respectively as follows. Let $\tau_i : \mathbf{R} \to SO(7)$ be the homomorphism given by the relations

$$\text{(9. 2)} \qquad \begin{cases} \tau_i(\theta)(e_{2i}) = e_{2i}\cos\theta + e_{2i+1}\sin\theta \\ \tau_i(\theta)(e_{2i+1}) = -e_{2i}\sin\theta + e_{2i+1}\cos\theta \\ \tau_i(\theta)(e_j) = e_j \qquad\qquad \text{for } j \neq 2i,\ 2i+1 \end{cases}$$

for $i = 1, 2, 3$. Let $\tau : \mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to SO(7)$ be defined by $\tau(\theta_1, \theta_2, \theta_3) = \tau_1(\theta_1)\tau_2(\theta_2)\tau_3(\theta_3)$ for $(\theta_1, \theta_2, \theta_3) \in \mathbf{R}^3$. Then $T' = \tau(\mathbf{R}^3)$ is a maximal torus in $SO(7)$.

Obviously $\widetilde{T}' = p^{-1}(\widetilde{T}')$ is a maximal torus in Spin(7). However we need to describe $\widetilde{T}'$ explicitly. Let $\tilde{\tau}_i : \mathbf{R} \to \mathrm{Spin}(7)$ be the homomorphism defined by formulae

$$\text{(9. 3)} \qquad \begin{cases} \tilde{\tau}_1(\theta)(e_0) = e_0\cos\theta/2 + e_1\sin\theta/2, & \tilde{\tau}_1(\theta)(e_1) = -e_0\sin\theta/2 + e_1\cos\theta/2 \\ \tilde{\tau}_1(\theta)(e_2) = e_2\cos\theta/2 + e_3\sin\theta/2, & \tilde{\tau}_1(\theta)(e_3) = -e_2\sin\theta/2 + e_3\cos\theta/2 \\ \tilde{\tau}_1(\theta)(e_4) = e_4\cos\theta/2 - e_5\sin\theta/2, & \tilde{\tau}_1(\theta)(e_5) = e_4\sin\theta/2 + e_5\cos\theta/2 \\ \tilde{\tau}_1(\theta)(e_6) = e_6\cos\theta/2 - e_7\sin\theta/2, & \tilde{\tau}_1(\theta)(e_7) = e_6\sin\theta/2 + e_7\cos\theta/2 \end{cases}$$

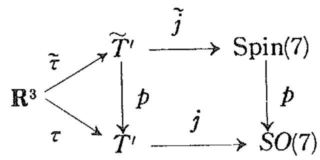
$$\text{(9. 4)} \qquad \begin{cases} \tilde{\tau}_2(\theta)(e_0) = e_0\cos\theta/2 + e_1\sin\theta/2, & \tilde{\tau}_2(\theta)(e_1) = -e_0\sin\theta/2 + e_1\cos\theta/2, \\ \tilde{\tau}_2(\theta)(e_2) = e_2\cos\theta/2 - e_3\sin\theta/2, & \tilde{\tau}_2(\theta)(e_3) = e_2\sin\theta/2 + e_3\cos\theta/2, \\ \tilde{\tau}_2(\theta)(e_4) = e_4\cos\theta/2 + e_5\sin\theta/2, & \tilde{\tau}_2(\theta)(e_5) = -e_4\sin\theta/2 + e_5\cos\theta/2, \\ \tilde{\tau}_2(\theta)(e_6) = e_6\cos\theta/2 - e_7\sin\theta/2, & \tilde{\tau}_2(\theta)(e_7) = e_6\sin\theta/2 + e_7\cos\theta/2, \end{cases}$$

$$\text{(9. 5)} \qquad \begin{cases} \tilde{\tau}_3(\theta)(e_0) = e_0\cos\theta/2 + e_1\sin\theta/2, & \tilde{\tau}_3(\theta)(e_1) = -e_0\sin\theta/2 + e_1\cos\theta/2, \\ \tilde{\tau}_3(\theta)(e_2) = e_2\cos\theta/2 - e_3\sin\theta/2, & \tilde{\tau}_3(\theta)(e_3) = e_2\sin\theta/2 + e_3\cos\theta/2, \\ \tilde{\tau}_3(\theta)(e_4) = e_4\cos\theta/2 - e_5\sin\theta/2, & \tilde{\tau}_3(\theta)(e_5) = e_4\sin\theta/2 + e_5\cos\theta/2, \\ \tilde{\tau}_3(\theta)(e_6) = e_6\cos\theta/2 + e_7\sin\theta/2, & \tilde{\tau}_3(\theta)(e_7) = -e_6\sin\theta/2 + e_7\cos\theta/2. \end{cases}$$

Let $\tilde{\tau} : \mathbf{R}^3 \to \mathrm{Spin}(7)$ be a map defined by $\tilde{\tau}(\theta_1, \theta_2, \theta_3) = \tilde{\tau}_1(\theta_1)\tilde{\tau}_2(\theta_2)\tilde{\tau}_3(\theta_3)$ for $(\theta_1, \theta_2, \theta_3) \in \mathbf{R}^3$. Then we have

$$\tau(\theta_1, \theta_2, \theta_3)(u)\tilde{\tau}(\theta_1, \theta_2, \theta_3)(v) = \tilde{\tau}(\theta_1, \theta_2, \theta_3)(uv)$$

for $(\theta_1, \theta_2, \theta_3) \in \mathbf{R}^3$ and $u, v \in \mathfrak{C}$. Therefore $\tilde{\tau}(\theta_1, \theta_2, \theta_3)$ covers $\tau(\theta_1, \theta_2, \theta_3)$ by the projection $p$, so that $\tilde{\tau}(\mathbf{R}^3) = p^{-1}(T')$ (which was denoted by $\widetilde{T'}$). Hence we have the following commutative diagram



where $j$, $\tilde{j}$ are inclusions.

## 10. Relations of $R(G_2)$ to $R(SO(7))$, $R(\mathrm{Spin}(7))$ and $R(SU(3))$

Since $SO(7)$ is the rotation group in $L_1$, $L_1$ is an $SO(7)$-$\mathbf{R}$-module, so that we have an $SO(7)$-$\mathbf{C}$-module $M_1^C = L_1 \otimes_{\mathbf{R}} \mathbf{C}$.

We show that in $M(SO(7))$

$$(10.1) \qquad j^{\#}(M_1^C) = \mathbf{C} \oplus W_1 \oplus \widehat{W}_1 \oplus W_2 \oplus \widehat{W}_2 \oplus W_3 \oplus \widehat{W}_3,$$

where $W_i$ is a 1-dimensional $T'$-$\mathbf{C}$-module and $\widehat{W}_i$ is its dual $T'$-$\mathbf{C}$-module for $i = 1, 2, 3$.

In fact, let $\mathbf{C}$, $W_i$ and $\widehat{W}_i$ $(i = 1, 2, 3)$ be the same $\mathbf{C}$-modules as in the section 5. Then, for $t' = \tau(\theta_1, \theta_2, \theta_3) \in T'$, we have $t'u_i = u_i \exp(\sqrt{-1}\,\theta_i)$ and $t'\hat{u}_i = \hat{u}_i \exp(-\sqrt{-1}\,\theta_i)$ for $i = 1, 2, 3$. These prove the above result (10.1).

Put $\phi_{T'}(W_1) = a$, $\phi_{T'}(W_2) = b$ and $\phi_{T'}(W_3) = c$, then we have

$$R(T') = \mathbf{Z}[a, a^{-1}, b, b^{-1}, c, c^{-1}].$$

In $R(SO(7))$, put $\phi_{SO(7)}(M_1^C) = \mu_1^C$, $\phi_{SO(7)}(\Lambda^2(M_1^C)) = \mu_2^C$ and $\phi_{SO(7)}(\Lambda^3(M_1^C)) = \mu_3^C$, then by (10.1)

$$j^*(\mu_1^C) = 1 + (a + a^{-1}) + (b + b^{-1}) + (c + c^{-1})$$

$$\begin{aligned} j^*(\mu_2^C) = 3 &+ (a + a^{-1}) + (b + b^{-1}) + (c + c^{-1}) \\ &+ (a + a^{-1})(b + b^{-1}) + (b + b^{-1})(c + c^{-1}) + (c + c^{-1})(a + a^{-1}) \end{aligned}$$

$$\begin{aligned} j^*(\mu_1^C) = 3 &+ 2(a + a^{-1}) + 2(b + b^{-1}) + 2(c + c^{-1}) \\ &+ (a + a^{-1})(b + b^{-1}) + (b + b^{-1})(c + c^{-1}) + (c + c^{-1})(a + a^{-1}) \\ &+ (a + a^{-1})(b + b^{-1})(c + c^{-1}). \end{aligned}$$

And we have [4]

$$R(SO(7)) = \mathbf{Z}[\mu_1^C, \mu_2^C, \mu_3^C].$$

Next, since $\mathrm{Spin}(7)$ is a subgroup of $SO(8)$, $\mathrm{Spin}(7)$ operates on $\mathfrak{C}$. Thus $\mathfrak{C}$ is a Spin(7)-**R**-module, whence we have a Spin(7)-**C**-module $\varDelta^C = \mathfrak{C}\otimes_{\mathbf{R}}\mathbf{C}$.

We have in $M$(Spin(7))

$$(10.2)\qquad \tilde{j}^*(\varDelta^C) = \widetilde{W}_1 \otimes \widetilde{W}_2 \otimes \widetilde{W}_3 \oplus \hat{\widetilde{W}}_1 \otimes \hat{\widetilde{W}}_2 \otimes \hat{\widetilde{W}}_3 \oplus \widetilde{W}_1 \otimes \hat{\widetilde{W}}_2 \otimes \hat{\widetilde{W}}_3$$
$$\oplus \hat{\widetilde{W}}_1 \otimes \widetilde{W}_2 \otimes \widetilde{W}_3 \oplus \hat{\widetilde{W}}_1 \otimes \widetilde{W}_2 \otimes \hat{\widetilde{W}}_3 \oplus \widetilde{W}_1 \otimes \hat{\widetilde{W}}_2 \otimes \widetilde{W}_3$$
$$\oplus \hat{\widetilde{W}}_1 \otimes \hat{\widetilde{W}}_2 \otimes \widetilde{W}_3 \oplus \widetilde{W}_1 \otimes \widetilde{W}_2 \otimes \hat{\widetilde{W}}_3$$
$$= (\widetilde{W}_1 \oplus \hat{\widetilde{W}}_1) \otimes (\widetilde{W}_2 \oplus \hat{\widetilde{W}}_2) \otimes (\widetilde{W}_3 \oplus \hat{\widetilde{W}}_3),$$

where $\widetilde{W}_i$ is a 1-dimensional $\widetilde{T}'$-**C**-module and $\hat{\widetilde{W}}_i$ is its dual $\widetilde{T}'$-**C**-module for $i = 1, 2, 3$.

In fact, take an additive **C**-base $u_i = e_{2i} - e_{2i+1}\sqrt{-1}$, $\hat{u}_i = e_{2i} + e_{2i+1}\sqrt{-1}$ for $i = 0, 1, 2, 3$ in $\varDelta^C$. Then, for $\tilde{t} = \tilde{\tau}(\theta_1, \theta_2, \theta_3) \in \widetilde{T}'$, using (9.3)—(9.5), we have

$$\begin{cases} \tilde{t}u_1 = u_1 \exp(\sqrt{-1}(\theta_1+\theta_2+\theta_3)/2), & \tilde{t}\hat{u}_1 = \hat{u}_1\exp(\sqrt{-1}(-\theta_1-\theta_2-\theta_3)/2), \\ \tilde{t}u_2 = u_2 \exp(\sqrt{-1}(\theta_1-\theta_2-\theta_3)/2), & \tilde{t}\hat{u}_2 = \hat{u}_2\exp(\sqrt{-1}(-\theta_1+\theta_2+\theta_3)/2), \\ \tilde{t}u_3 = u_3 \exp(\sqrt{-1}(-\theta_1+\theta_2-\theta_3)/2), & \tilde{t}\hat{u}_3 = \hat{u}_3\exp(\sqrt{-1}(\theta_1-\theta_2+\theta_3)/2), \\ \tilde{t}u_4 = u_4 \exp(\sqrt{-1}(-\theta_1-\theta_2+\theta_3)/2), & \tilde{t}\hat{u}_4 = \hat{u}_4\exp(\sqrt{-1}(\theta_1+\theta_2-\theta_3)/2). \end{cases}$$

These prove (10.2).

Now, put $\phi_{\widetilde{T}'}(\widetilde{W}_1) = a^{\frac{1}{2}}$, $\phi_{\widetilde{T}'}(\widetilde{W}_2) = b^{\frac{1}{2}}$ and $\phi_{\widetilde{T}'}(\widetilde{W}_3) = c^{\frac{1}{2}}$, then we have

$$R(\widetilde{T}') = \mathbf{Z}\,[a,\ a^{-1},\ b,\ b^{-1},\ c,\ c^{-1},\ (a\ b\ c)^{\frac{1}{2}}]$$

Denote $\phi_{\mathrm{Spin}(7)}(\varDelta^C) = \varDelta^C$, then we have by (10.2)

$$\tilde{j}^*(\varDelta^C) = (a^{\frac{1}{2}} + a^{-\frac{1}{2}})(b^{\frac{1}{2}} + b^{-\frac{1}{2}})(c^{\frac{1}{2}} + c^{-\frac{1}{2}}).$$

Hence, $\varDelta^C$ coincides with an element induced by a unique irreducible representation $\varDelta_7$ which is containd in the Cliffiord algebra $C_7^C$ (for notations $\varDelta_7$ and $C_7^C$ we refer to [4] and [1] respectively). And we have [4]

$$R(\mathrm{Spin}(7)) = \mathbf{Z}\,[\tilde{\mu}_1^C,\ \tilde{\mu}_2^C,\ \varDelta^C],$$

where $\tilde{\mu}_1^C = p^*(\mu_1^C)$, $\tilde{\mu}_2^C = p^*(\mu_2^C)$ and there exists a relation

$$(\varDelta^C)^2 = p^*(1 + \mu_1^C + \mu_2^C + \mu_3^C).$$

As for $SU(3)$, let $N_1$ and $N_2$ denote **C**-submodules of $\mathfrak{C}\otimes_{\mathbf{R}}\mathbf{C}$ with respectively additive bases $u_i = e_{2i} - e_{2i+1}\sqrt{-1}$ and $\hat{u}_i = e_{2i} + e_{2i+1}\sqrt{-1}$ for $i = 1, 2, 3$. We shall show that these are invariant by $SU(3)$. In fact, for $x \in SU(3)$, $xu_i = x(e_{2i} - e_{2i+1}\sqrt{-1}) = x((e_0 - e_1\sqrt{-1})e_{2i}) = x(e_0 - e_1\sqrt{-1})x(e_{2i}) = (e_0 - e_1\sqrt{-1})x(e_{2i})$. Note that $x(e_{2i})$ is a linear combination of $e_2, \cdots\cdots, e_7$; $x(e_{2i}) = \sum_{i=2}^{7} e_i a_i = \sum_{i=1}^{3} e_{2i}a_{2i} + \sum_{i=1}^{3} e_{2i+1}a_{2i+1}$. Obviously, $(e_0 - e_1\sqrt{-1})(\sum_{i=1}^{3} e_{2i}a_{2i}) = \sum_{i=1}^{3} u_i a_{2i} \in N_1$. On the other hand,

$$(e_0 - e_1\sqrt{-1})(\sum_{i=1}^{3} e_{2i+1}a_{2i+1}) = \sum_{i=1}^{3} (e_0 - e_1\sqrt{-1})\, e_1 e_{2i} a_{2i+1} = \sum_{i=1}^{3} (e_1 + e_0\sqrt{-1})e_{2i}a_{2i+1}$$

$= \sum_{i=1}^{3}(e_0 - e_1\sqrt{-1})\, e_{2i}\, \sqrt{-1}\, a_{2i+1} = \sum_{i=1}^{3} u_i\sqrt{-1}\, a_{2i+1} \in N_1$, whence follows that $xu_i \in N_1$ for $i = 1, 2, 3$. Similarly $xu_i \in N_2$ for $x \in SU(3)$, $i = 1, 2, 3$.

Let $j_1 : T \to SU(3)$ denote the inclusion. Then, analogously to the case of $G_2$, we have in $M(T)$

$$(10.3)\qquad \begin{cases} j_1^{\#}(N_1) = W_1 \oplus W_2 \oplus W_3 \\ j_1^{\#}(N_2) = \hat{W}_1 \oplus \hat{W}_2 \oplus \hat{W}_3, \end{cases}$$

where $W_i$ is a 1-dimensional $T$-$\mathbf{C}$-module and $\hat{W}_i$ is its dual $T$-$\mathbf{C}$-module for $i = 1, 2, 3$, and there exists a relation $W_1 \otimes W_2 \otimes W_3 = \mathbf{C}$.

Hence we have

$$R(T) = \mathbf{Z}[\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}]/(\alpha\beta\gamma - 1),$$

where $\alpha = \phi_T(W_1)$, $\beta = \phi_T(W_2)$ and $\gamma = \phi_T(W_3)$. Put $\phi_{SU(3)}(N_1) = \nu_1$ and $\phi_{SU(3)}(N_2) = \nu_2$, then we have by (10.3)

$$j_1^*(\nu_1) = \alpha + \beta + \gamma,$$
$$j_1^*(\nu_2) = \alpha^{-1} + \beta^{-1} + \gamma^{-1} = \beta\gamma + \gamma\alpha + \alpha\beta = j_1^*(\phi_{SU(3)}(\Lambda^2(N_1))).$$

So that we have [4],

$$R(SU(3)) = \mathbf{Z}[\nu_1, \nu_2].$$

Thus, in the following diagram

$$\begin{array}{ccccc} R(SO(7)) & & & & \\ \downarrow p^* & \searrow l^* & & & \\ R(\mathrm{Spin}(7)) & \xrightarrow{k^*} & R(G_2) & \xrightarrow{i^*} & R(SU(3)), \end{array}$$

that is,

$$\begin{array}{ccccc} \mathbf{Z}[\mu_1^C, \mu_2^C, \mu_3^C] & & & & \\ \downarrow p^* & \searrow l^* & & & \\ \mathbf{Z}[\tilde{\mu}_1^C, \tilde{\mu}_2^C, \Delta^C] & \xrightarrow{k^*} & \mathbf{Z}[\lambda_1^C, \lambda_2^C] & \xrightarrow{i^*} & \mathbf{Z}[\nu_1, \nu_2], \end{array}$$

we have the following relations

$$\begin{cases} p^*(\mu_1^C) = \tilde{\mu}_1^C, \ \ p^*(\mu_1^C) = \tilde{\mu}_2^C, \ \ p^*(\mu_2^C) = (\Delta^C)^2 - 1 - \tilde{\mu}_1^C - \tilde{\mu}_2^C \\ k^*(\tilde{\mu}_1^C) = \lambda_1^C, \ \ k^*(\tilde{\mu}_2^C) = \lambda_2^C, \ \ k^*(\Delta^C) = 1 + \lambda_1^C \\ l^*(\mu_1^C) = \lambda_1^C, \ \ l^*(\mu_2^C) = \lambda_2^C, \ \ l^*(\mu_3^C) = (\lambda_1^C)^2 + \lambda_1^C - \lambda_2^C \\ i^*(\lambda_1^C) = 1 + \nu_1 + \nu_2, \ \ i^*(\lambda_2^C) = 2(\nu_1 + \nu_2) + \nu_1\nu_2. \end{cases}$$

As for $RO(SO(7))$ and $RO(\mathrm{Spin}(7))$, we can discuss in the real range. Using the fact that the complexification $c$ is an isomorphism, we have $RO(SO(7))=\mathbf{Z}[\mu_1, \mu_2, \mu_3]$ and $RO(\mathrm{Spin}(7))=\mathbf{Z}[\tilde{\mu}_1, \tilde{\mu}_2, \Delta]$ where $\mu_i$ is the class of $\Lambda^i(L_1)$ for $i=1, 2, 3$, $\tilde{\mu}_i = p^*(\mu_i)$ for $i=1, 2$ and $\Delta$ is the class of $\boldsymbol{\Delta}$. And in the diagram

$$\begin{array}{ccc} RO(SO(7)) & & \\ \downarrow p^* & \searrow l^* & \\ RO(\mathrm{Spin}(7)) & \xrightarrow{k^*} & RO(G_2) \end{array}$$

we have the same relations as in the complex case, i. e.

$$\begin{cases} p^*(\mu_1)=\tilde{\mu}_1, \quad p^*(\mu_2)=\tilde{\mu}_2, \quad p^*(\mu_3)=\Delta^2-1-\mu_1-\mu_2 \\ k^*(\tilde{\mu}_1)=\lambda_1, \quad k^*(\tilde{\mu}_2)=\lambda_2, \ k^*(\Delta)=1+\lambda_1 \\ l^*(\mu_1)=\lambda_1, \quad l^*(\mu_2)=\lambda_2, \ l^*(\mu_3)=\lambda_1{}^2+\lambda_1-\lambda_2. \end{cases}$$

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