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Elliptic Semi-complexes

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Introduction. The importance of elliptic complexes has been increasing in analytic topology and other branches of mathematics. (cf. [4], [5], [10]). But there are some geometric objects which can not be elliptic complexes. For example, if E is a (differentiable) vector bundle over a smooth manifold X with cotangent bundle T^*, then taking a covariant derivation D of E, we have the following series

$$\Gamma(E) \xrightarrow{D} \Gamma(E\otimes T^*) \xrightarrow{D} \Gamma(E\otimes \Lambda^2 T^*) \xrightarrow{D} \cdots \xrightarrow{D} \Gamma(E\otimes \Lambda^n T^*)$$

where $\Gamma(F)$ means the space of smooth sections of F, $\Lambda^p T^*$ is the p-th exterior product of T^*. ([6]). Since $DD=\Theta$, the curvature form of E, we know that this series can not be a differential complex unless E is induced from a representation of $\pi_1(X)$ in a linear group. ([2], [9]). But since we know the symbol $\sigma(D)$ of D is given by

$$\sigma(D)\{\pi^*(\xi)(x, \eta)\} = \pi^*(\xi)(x, \eta)_\wedge \pi^*(\eta),$$
$$x\in X, \quad \eta\in T^*_x, \textit{ the fibre of } T^* \textit{ at } x,$$

the symbol sequence

$$0 \to \pi^*(E) \xrightarrow{\sigma(D)} \pi^*(E\otimes T^*) \xrightarrow{\sigma(D)} \cdots\cdots \xrightarrow{\sigma(D)} \pi^*(E\otimes\Lambda^n T^*) \to 0,$$

of the above series is exact. From this, we can deduce some conclusions. For example, denoting D^* and Θ^* the formal adjoints of D and Θ, the operators

$$D+D^* : \sum_{p\geq 0}\Gamma(E\otimes\Lambda^{2p}T^*) \to \sum_{p\geq 0}\Gamma(E\otimes\Lambda^{2p+1}T^*),$$

$$\Box = DD^*+D^*D : \sum_p \Gamma(E\otimes\Lambda^p T^*) \to \sum_p \Gamma(E\otimes\Lambda^p T^*),$$

$$\bigcirc = (D+D^*)(D+D^*)$$
$$= \Box+\Theta+\Theta^* : \sum_p \Gamma(E\otimes\Lambda^p T^*) \to \sum_p\Gamma(E\otimes\Lambda^p T^*),$$

are all elliptic in the sense of Atiyah-Singer if E is a complex vector bundle. Moreover, if X is a compact oriented manifold, Atiyah-Singer's index theorem shows that

$$\kappa_n(\mathrm{ch}(E)) = \textit{the analytic index of } (D+D^*),$$

$$\kappa_n(\mathrm{ch}(E)) = \int_x \exp.\ (\Theta/2\pi\sqrt{-1}),\ \textit{if } E \textit{ is an Hermitian vector bundle and } D = 0,$$

where $\kappa_n(c)$, $c \in H^*(X)$ is the value of c on the fundamental class X. (cf. [3], [4], [6]).

Taking the above series as the guiding model, we define an elliptic semi-complex as follows:

Definition. *Let $E_1, \dots, E_m$ be (differentiable) vector bundles over a smooth manifold X, then the series*

$$\Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \cdots\cdots \xrightarrow{D_{m-1}} \Gamma(E_m),$$

D_i is a linear differental operator for each i,

is called an elliptic semi-complex if its symbol sequence

$$0 \to \pi^*(E_0) \xrightarrow{\sigma(D_0)} \pi^*(E_1) \xrightarrow{\sigma(D_1)} \cdots\cdots \xrightarrow{\sigma(D_{m-1})} \pi^*(E_m) \to 0,$$

is exact.

We denote an elliptic semi-complex by $\{\Gamma(E_i), D_i\}$. As usual, two elliptic semi-complexes $\{\Gamma(E_i), D_i\}$ and $\{\Gamma(E_i'), D_i'\}$ are called equivalent if there are bundle maps $h_i : E_i \to E_i'$, i = 0, ..., m, such that the following two diagrams are commutative.

$$\begin{array}{ccccccc}
\Gamma(E_0) & \xrightarrow{D_0} & \Gamma(E_1) & \xrightarrow{D_1} \cdots\cdots & \xrightarrow{D_{m-1}} & \Gamma(E_m) \\
\downarrow h_0^* & & \downarrow h_1^* & & & \downarrow h_m^* \\
\Gamma(E_0') & \xrightarrow{D_0'} & \Gamma(E_1') & \xrightarrow{D_1'} \cdots\cdots & \xrightarrow{D_{m-1}'} & \Gamma(E_m'),
\end{array}$$

$$\begin{array}{ccccccc}
0 \to \pi^*(E_0) & \xrightarrow{\sigma(D_0)} & \pi^*(E_1) & \xrightarrow{\sigma(D_1)} \cdots\cdots & \xrightarrow{\sigma(D_{m-1})} & \pi^*(E_m) \to 0 \\
\downarrow \pi^*(h_0) & & \downarrow \pi^*(h_1) & & & \downarrow \pi^*(h_m) \\
0 \to \pi^*(E_0') & \xrightarrow{\sigma(D_0')} & \pi^*(E_1') & \xrightarrow{\sigma(D_1')} \cdots\cdots & \xrightarrow{\sigma(D_{m-1}')} & \pi^*(E_m') \to 0.
\end{array}$$

Definition. *If $\{\Gamma(E_i), D_i,\ i = 0, 1, \cdots\cdots, m\}$ is an elliptic semi-complex, then m is called its length.*

We denote the chern character of the difference bundle $d(\pi^*(E_0), \cdots\cdots, \pi^*(E_m),\ \sigma(D_0), \cdots\cdots, \sigma(D_{m-1}))$ by ch $(d(\pi^*(E_0), \cdots\cdots, \pi^*(E_m),\ \sigma(D_0), \cdots\cdots, \sigma(D_{m-1}))$. We also denote the Thom isomorphism from $H^*(T(T^*))$ to $H^*(X)$, where $T(T^*)$ is the Thom complex of T^*, by ϕ. Then we define (cf. [3], [4]),

Definition. ϕ^{-1} $(\mathrm{ch}(d(\pi^*(E_0), \cdots\cdots, \pi^*(E_m), \sigma(D_0), \cdots\cdots, \sigma(D_{m-1}))$ *is called the chern character of* $\{\Gamma(E_i), D_i\}$ *and denoted by* ch $(\{\Gamma(E_i), D_i\})$.

Definition. *Setting*

$$\Theta_i = D_{i+1}\ D_i,$$

we call Θ_i *the curvature form of* $\{\Gamma(E_i), D_i\}$.

Note. Since $\sigma(\Theta_i)$ is equal to 0, we get

$$\deg.\ (\Theta_i) \leq \deg.\ (D_i) + \deg.\ (D_{i-1}) - 1.$$

If $\{\Gamma(E_i), D_i\}$ is an elliptic semi-complex over a compact oriented manifold and each E_i is an Hermitian vector bundle, then we can define inner product on each $\Gamma(E_i)$ and by this inner product, each $\Gamma(E_i)$ becomes a pre-Hilbert space. The completion of this pre-Hilbert space is denoted by $\tilde{\Gamma}(E_i)$. We denote the weak extension of D_i to $\tilde{\Gamma}(E_i)$ by D_i, too. In this case, the adjoint operators of D_i and Θ_i are denoted by D_i^* and Θ_i^*. Then we get

Theorem 1. *Let* $\{\Gamma(E_i), D_i\}$ *be an elliptic semi-complex over a compact oriented manifold. We assume that* Θ_i *is a matrix and rank* $(\Theta_i(x))$ *does not depend on* x, $x \in X$, *the manifold on which* $\{\Gamma(E_i), D_i\}$ *is defined, for each* i. *Then the sequence*

$$\tilde{\Gamma}(E_0) \xrightarrow{D_0} \tilde{\Gamma}(E_1) \xrightarrow{p_2 D_1} (\Theta_0 \tilde{\Gamma}(E_0))^{\perp} \xrightarrow{p_3 D_2} \cdots\cdots$$

$$\cdots \xrightarrow{p_m D_{m-1}} (\Theta_{m-2} \tilde{\Gamma}(E_{m-2}))^{\perp},$$

where p_i *is the projection from* $\tilde{\Gamma}(E_i)$ *to* $(\Theta_{i-2}\tilde{\Gamma}(E_{i-2}))^{\perp}$, *becomes an elliptic complex if and only if*

$$\deg.(D_{i+1}\Theta_i^* - \Theta_i^* D_{i-1}) \leq \max.\ (\deg.(D_{i+1}),\ \deg.(D_{i-1})) - 1,$$

is hold for each i.

Next, define the map $D+D^*: \sum_{p\geq 0} \tilde{\Gamma}(E^{2p}) \to \sum_{p\geq 0} \tilde{\Gamma}(E^{2p+1})$ by

$$(D+D^*)(\sum_p u^{2p}) = \sum_p (D_{2p}u^{2p} + D_{2p}{}^* u^{2p}.)$$

Then the adjoint $(D+D^*)^*$ of $D+D^*$ is given by

$$(D+D^*)^*\ (\sum_p u^{2p+1}) = \sum_p (D_{2p+1}u^{2p+1} + D_{2p+1}{}^* u^{2p+1}).$$

Hence we may denote $D+D^*$ instead of $(D+D^*)^*$. By the exactness of the symbol sequence

$$0 \to \pi^*(E_0) \xrightarrow{\sigma(D_0)} \pi^*(E_1) \xrightarrow{\sigma(D_1)} \cdots\cdots \xrightarrow{\sigma(D_{m-1})} \pi^*(E_m) \to 0,$$

$D + D^*$ is elliptic in the sence of Atiyah-Singer. We also set

$$\Box = DD^* + D^*D, \ (\Box u^p = D_{p-1}D_p{}^*u^p + D_{p+1}{}^*D_p u^p),$$

$$\bigcirc = (D+D^*)\ (D + D^*) = \Box + \Theta + \Theta^*.$$

Then we get

$$\kappa_n(\mathrm{ch}(\{\Gamma(E_i,\ D_i)\}T_R(X)) = \textit{the analytic index of } D + D^*,$$

$$\tilde{\Gamma}(E) = \Box\tilde{\Gamma}(E) + \mathrm{ker}.\Box, \quad \mathrm{ker}.\Box = \{u \mid Du = D^*u = 0\}.$$

Moreover, although we can not obtain usual orthogonal decomposition theorem, we can prove the following orthogonal decomposition theorem.

Theorem 2. *For each p, we can set*

$$\tilde{\Gamma}(E_p) = D_{p-1}\mathrm{ker}.\,\Theta_{p-1} \oplus D_{p+1}{}^*\mathrm{ker}.\,\Theta_{p+1}{}^* \oplus$$
$$\oplus D_{p-1}\tilde{\Gamma}(E_{p-1}) \cap D_{p+1}{}^*\tilde{\Gamma}(E_{p+1}) \oplus \mathrm{ker}.\Box \cap \tilde{\Gamma}(E_p),$$

where $\oplus$ means the direct sum and the spaces in the right hand side are orthogonal each other.

In usual differential geometry, the relation

$$D\Theta = 0,$$

is called the Bianchi identity. (It is known that the Bianchi identity is true if the connection, by which D is determined, is torsionless. [6]). Therefore we define

Definition. *An elliptic semi-complex $\{\Gamma(E_i)D_i\}$, is called a Bianchi semi-complex if*

$$D_{i+2}\Theta_i = \Theta_{i+1}D_i = 0,$$

for all i.

For a Bianchi semi-complex, we have the following theorems.

Theorem 3. *If $\{\Gamma(E_i),\ D_i\}$ is a Bianchi semi-complex over a compact oriented manifold and each E_i is an Hermitian vector bundle, then*

$$\tilde{\Gamma}(E_p) = \Theta_{p-2}\tilde{\Gamma}(E_{p-2}) \oplus D_{p-1}\mathrm{ker}.\,\Theta_{p-1}\tilde{\Gamma}\ \mathrm{ker}.\,\Theta_p{}^* \oplus$$
$$\oplus \Theta_{p+2}{}^*\tilde{\Gamma}(E_{p+2}) \oplus D_{p+1}{}^*\ \mathrm{ker}.\,\Theta_{p+1}{}^*\tilde{\Gamma}\ \mathrm{ker}.\,\Theta_p \oplus$$
$$\oplus D_{p-1}\tilde{\Gamma}(E_{p-1}) \cap D_{p+1}{}^*\tilde{\Gamma}(E_{p+1}) \oplus \mathrm{ker}.\ \Box \cap \tilde{\Gamma}(E_p)$$

holds for each p. Here the spaces in the right hand side are orthogonal each other.

Theorem 4. *If $\{\Gamma(E_i), D_i\}$ is a Bianchi semi-complex over a compact oriented manifold and each E_i is an Hermitian vector bundle, then*

(i). $\bigcirc u = 0$ *if and only if* $(D + D^*)u = 0$.
(ii). *If* u *is orthogonal to* ker. $\square$ *and* $\bigcirc u = 0$, *then*

$$u = \Theta v + \Theta^* w.$$

Note. (i) is true although $\{\Gamma(E_i), D_i\}$ is not a Bianchi semi-complex.

Note. (ii) is the global property of the operators $\square$ and $\bigcirc$. As the local property of operators $\square$ and $\bigcirc$, we have the following

Theorem 5′. *If each* Θ_i *is a matrix and either of* $\square$ *or* $\bigcirc$ *has the Green function* $g(x, \xi)$ *locally, and*

$$\int_{B(r,\ a)} |\, g(x, \xi)|\, d\xi \leq f(r),$$

$B(r, a) = \{\xi \mid |\, \xi - a \,| < r\}$, $f(r)$ *is continuous (positive increasing) function in* r *and* $\lim_{r \to 0} f(r) = 0$,

then there is a 1 *to* 1 *correspondence between the set of germs of solutions of* $\square u = 0$ *and the set of germs of solutions of* $\bigcirc u = 0$. *Moreover, if* $Gu = \int g(x, \xi)u(\xi)d\xi$ *is the Green operator of* $\square$, *then this correspondence is given by*

$$\{\hat{u} \mid \square u = 0\} \ni \hat{u} \to P(u) \in \{\hat{v} \mid \bigcirc v = 0\},$$

$\hat{u}$ *is the germ of* u,

$$P(u) = \sum_{n \geq 0} I_n(u), \quad I_0(u) = u, \quad I_n(u) = G(e(\Theta + \Theta^*)I_{n-1}(u)).$$

$e = 1$ *on* $B(r, a)$, $e = 0$ *outside of* $B(s, a)$, $(s > r)$, $o \leq e \leq 1$. s *depends on* u.

The purpose of this paper is to give the proofs these theorems. (Since the proof of theorem 5′ is quite similar to that of theorem 5 which is proved in § 7, we do not give the proof of theorem 5′). Although we do not know more refined announcements about elliptic semi-complexes, we treat the semi-complex

$$\Gamma(E) \xrightarrow{D} \Gamma(E \otimes T^*) \xrightarrow{D} \Gamma(E \otimes \Lambda^2 T^*),$$

over a compact oriented surface in the last paragraph. It suggests us that there would be exist some analogy between the solutions of $\bigcirc u = 0$ and holomorphic sections of some holomorphic bundle derived by E.

Added in proof. Prof. Kohara remarked to the authour that the solutions of the equation (19) of § 7 are called pseudoanalytic functions of the first kind and their properties have been studied by Bers. (Bers, L. : *Theory of pseudo-analytic functions.* 1953, New York Univ.. see also Courant, R. -Hilbert, D. : *Methods of Mathematical Physics*, II. New York, 1961. p. 374. Theorem 5 of § 7 is essentially proved at p. 378 of Courant-Hilbert's book).

§ 1. Proof of theorem 1.

We assume that each E_i is an Hermitian vector bundle and consider each $\pi^*(E_i)$ is also an Hermitian vector bundle. First we remark that by the assumption of the theorem, each Θ_i defines a bundle map from E_i into E_{i+2}. We denote this bundle map by Θ_i too. Similarly, the adjoint operator Θ_i^* of Θ_i also defines a bundle map from E_i into E_{i-2}. It is also denoted by Θ_i^*. Then on $T^*(X)$, the total space of the cotangent bundle of X, we have

$$(1)\qquad \pi^*(\Theta E)=\pi^*(\Theta)\pi^*(E),\quad \pi^*(\Theta^*)=(\pi^*(\Theta))^*,$$
$$\pi^*(E)=\text{ker.}\ ((\pi^*(\Theta))^*)\oplus\pi^*(\Theta)\pi^*(E),$$
$$\text{ker.}\ ((\pi^*(\Theta))^*)=\pi^*((\Theta E)^{\perp}),$$

where $\pi^*(E)=\sum\pi^*(E_i)$, $\pi^*(\Theta)(\sum_i u_i)=\sum_i\pi^*(\Theta_i)u_i$.

We denote the projections from $\pi^*(E)$ onto ker. $((\pi^*(\Theta))^*)$ and from $\pi^*(E)$ onto $\pi^*(\Theta)\pi^*(E)$ by p^1 and p^2. Then the symbol $\sigma(pD)$ of the operator pD given by

$$pD:\Gamma(E)\to(\Theta\Gamma(E))^{\perp},$$
$$\Gamma(E)=\sum\Gamma(E_i),\quad pD(\sum_i u_i)=\sum_i p_{i-1}D_iu_i,$$

is given by $p^1\sigma(D)$, because $(\Theta\Gamma(E))^{\perp}=\Gamma((\Theta E)^{\perp})$. Then the sequence

$$0\to p^1\sigma(D)\pi^*((\Theta E)^{\perp})\xrightarrow{\ i\ }\pi^*((\Theta E)^{\perp})\xrightarrow{p^1\sigma(D)}$$
$$\to p^1\sigma(D)\pi^*((\Theta E)^{\perp})\to 0,$$

where i is the inclusion, is exact if and only if the relation

$$(2)\qquad \sigma(D)\pi^*(\Theta^*)=\pi^*(\Theta^*)\sigma(D),$$

holds. Since we know (2) is true if and only if

$$\text{deg.}\ (D\Theta^*-\Theta^*D)\le\text{deg.}\ (D)-1,$$

we have the theorem.

Note 1. The condition of the theorem is rewritten as follows:

(i). $\text{deg.}\ (D^*\Theta-\Theta D^*)\le\text{deg.}\ (D^*)-1.$

(ii). $\text{deg.}\ (\Box D-D\Box)\le\text{deg.}\ (D)-1.$

(ii)'. $\text{deg.}\ (\Box D^*-D^*\Box)\le\text{deg.}\ (D^*)-1.$

(iii). $\text{deg.}\ (\bigcirc D-D\bigcirc)\le\text{deg.}\ (D)-1$ and

$$\deg.(\bigcirc D^* - D^*\bigcirc) \leq \deg.(D^*) - 1.$$

Note 2. This theorem is meaningless in most of the cases. For example, if $\{\Gamma(E_i), D_i\}$ is a Bianchi semi-complex, then each D_i is a 0-map on $\Theta_{i-2}(E_{i-2})$. On the other hand, if the theorem holds on this semi-complex, then the sequence

$$0 \to \pi^*(\Theta_0 E_0) \xrightarrow{\sigma(D_2)} \pi^*(\Theta_1 E_1) \xrightarrow{\sigma(D_3)} \cdots\cdots$$
$$\cdots\cdots \xrightarrow{\sigma(D_{m-1})} \pi^*(\Theta_{m-2} E_{m-2}) \to 0,$$

must be exact. Hence each Θ_i should be equal to 0 in this case. This shows $\{\Gamma(E_i), D_i\}$ is an elliptic complex.

§ 2. Proof of theorem 2.

In the rest of this paper, we often denote Du, etc., instead of $D_p u$, etc..

We also assume that $\{\Gamma(E_i), D_i\}$ is defined on a compact oriented manifold X and each E_i is am Hermitian vector bundle. Then the operator $\square$ is self adjoint by its definition. Hence we get

$$\tilde{\Gamma}(E_p) = \square\tilde{\Gamma}(E_p) + \ker.\square \cap \tilde{\Gamma}(E_p), \tag{3}$$

$$\ker.\square = \{u \mid Du = D^*u = 0\},$$

for each p. Moreover, we get

$$DD^*\tilde{\Gamma}(E_p) = D\tilde{\Gamma}(E_{p-1}), \quad D^*D\tilde{\Gamma}(E_p) = D^*\tilde{\Gamma}(E_{p+1}),$$

because we know

$$\tilde{\Gamma}(E_q) = D\tilde{\Gamma}(E_{q-1}) + \ker.D^* = D^*\tilde{\Gamma}(E_{q+1}) + \ker.D. \tag{4}$$

If $u = Dv \in D\tilde{\Gamma}(E_{p-1})$ is orthogonal to $D^*\tilde{\Gamma}(E_{p+1})$, then $\Theta v = 0$ because $(Dv, D^*w) = (\Theta v, w)$ and w is arbitrary. Therefore we have theorem 2.

As the corollary of this theorem, we have the following

Lemma 1. *If $Du = 0$ and u is orthogonal to* $\ker.\square$, *then we can write*

$$u = Dv$$

Proof. By (4), we get

$$DD^*u \neq 0 \text{ if } D^*u \neq 0. \tag{4$'$}$$

Therefore $Du=0$ implies $u \in D \ker.\ \Theta$. Hence we obtain the lemma.

Lemma 1$'$. *If $D^*u = 0$ and u is orthogonal to $\ker.\square$, then we can write*

$$u = D^*v.$$

Note. If $\{\Gamma(E_i), D_i\}$ is an elliptic complex, then theorem 2 is a corollary of the orthogonal decomposition theorem. (i. e. ker. Θ = ker. $\Theta^* = \tilde{\Gamma}(E)$ and $D\tilde{\Gamma}(E) \cup D^*\tilde{\Gamma}(E) = 0$ in this case.)

§ 3. Proof of theorem 3.

Since $\{\Gamma(E_i), D_i\}$ is a Bianchi semi-complex, $D\Theta$ is equal to 0. Hence we have

$$D \text{ ker. } \Theta \supset \Theta \tilde{\Gamma}(E).$$

Then as we know

$$(u, \Theta v) = (\Theta^* u, v),$$

we obtain

$$D\text{ker. } \Theta \supset (\Theta \tilde{\Gamma}(E))^{\perp} = D\text{ker. } \Theta \cap \text{ker. } \Theta^*.$$

This proves theorem 3.

As the croollary of this theorem, we have the following

Theorem 3'. *If* $\Gamma(E_i)$, D_i *is a Bianchi semi-complex over a compact oriented manifold and each* E_i *is an Hermitian vector bundle, then*

$$\tilde{\Gamma}(E_p) = \Theta_{p-2}\tilde{\Gamma}(E_{p-2}) \oplus \Theta_{p+2}{}^* \tilde{\Gamma}(E_{p+2}) \oplus \oplus \text{ ker. } \Theta_p \cap \text{ ker. } \Theta_p{}^*. \tag{5}$$

Proof. Since we know

$$\tilde{\Gamma}(E_p) = \Theta_{p-2}\tilde{\Gamma}(E_{p-2}) \oplus \text{ker. } \Theta_p{}^* = \Theta_{p+2}{}^* \tilde{\Gamma}(E_{p+2}) \oplus \text{ker. } \Theta_p, \tag{6}$$

we obtain

$$\begin{aligned}\text{ker. } \Theta_p = {} & \Theta_{p-2}\tilde{\Gamma}(E_{p-2}) \oplus D_{p-1}\text{ker. } \Theta_{p-1} \cap \text{ker. } \Theta_p{}^* \oplus \\ & \oplus D_{p+1}{}^*\text{ker. } \Theta_{p+1}{}^* \cap \text{ker. } \Theta_p \oplus \\ & \oplus D_{p-1}\tilde{\Gamma}(E_{p-1}) \cap D_{p+1}{}^* \tilde{\Gamma}(E_{p+1}) \oplus \text{ker. } \square \cap \tilde{\Gamma}(E_p),\end{aligned}$$

$$\begin{aligned}\text{ker. } \Theta_p{}^* = {} & D_{p-1}\text{ker. } \Theta_{p-1} \cap \text{ker. } \Theta_p{}^* \oplus \Theta_{p+2}{}^* \tilde{\Gamma}(E_{p+2}) \oplus \\ & \oplus D_{p+1}{}^*\text{ker. } \Theta_{p+1}{}^* \cap \text{ker. } \Theta_p \oplus \\ & \oplus D_{p-1}\tilde{\Gamma}(E_{p-1}) \cap D_{p+1}{}^* \tilde{\Gamma}(E_{p+1}) \oplus \text{ker. } \square \cap \tilde{\Gamma}(E_p).\end{aligned}$$

Therefore we get

$$\begin{aligned}& \text{ker. } \Theta_p \cap \text{ker. } \Theta_p{}^* \\ & = D_{p-1}\text{ker. } \Theta_{p-1} \cap \text{ker. } \Theta_p{}^* \oplus \mathrm{D}_{p+1}{}^*\text{ker. } \Theta_{p+1}{}^* \cap \text{ker. } \Theta_p \oplus \\ & \quad \oplus D_{p-1}\tilde{\Gamma}(E_{p-1}) \cap D_{p+1}{}^* \tilde{\Gamma}(E_{p+}) \oplus \text{ker. } \square \cap \tilde{\Gamma}(E_p).\end{aligned}$$

This proves the theorem.

By (6), we also obtain

Lemma 2. *$\Theta^*\Theta u$ and $D^*\Theta u$ can not be equal to 0 unless Θu is equvl to 0.*

Lemma 2′. *$\Theta\Theta^* u$ and $D\Theta^* u$ can not be equal to 0 unless $\Theta^* u$ is equal to 0.*

Lemma 3. *Under the same assumptions of theorem 3, we have the following isomorphisms.*

$$(\Theta+\Theta^*)\Gamma(E) = \Theta\Gamma(E)\oplus\Theta^*\Gamma(\mathrm{E}),$$

$$\ker.(\Theta+\Theta^*) = \ker.\Theta\cap\ker.\Theta^*.$$

Proof. By definition, $(\Theta+\Theta^*)\Gamma(E)$ is contained in $\Theta\Gamma(E)\oplus\Theta^*\Gamma(E)$. On the other hand, we get $(\Theta+\Theta^*)\Gamma(E)\supset\Theta\Gamma(E)$ because we know $\Theta\Gamma(E)=\Theta\Theta^*\Gamma(E)$. Hence we have the first isomorphism. The second isomorphism follows from theorem 3′.

§ 4. Preparations of the proof of theorem 4.

Since $\square$ is an elliptic operator and X has no boundary under the assumptions of theorem 4, it has the Green operator G. ([1], [8], [10]). By Rellich's theorem, G is a compact Hermitian operator and maps $\square\Gamma(E)$ isomorphically onto itself. Moreover, since

$$\deg.\ (\Theta+\Theta^*) < \deg.(D) < \deg.(\square),$$

$G(\Theta+\Theta^*)$ is also a compact operator. We set

$$F = G(\Theta+\Theta^*).$$

Lemma 4. *$u\in\square\Gamma(E)$ becomes a proper vector of F belonging to a proper value λ if and only if*

$$-\lambda\square u + (\Theta+\Theta^*)u = 0.$$

Lemma 5. *$\ker.F$ is equal to $\ker.\Theta\cap\ker.\Theta^*$.*

By lemma 5, we have

(7) $$F\Gamma(E) = F\square\Gamma(E) = F(\Theta+\Theta^*)\Gamma(E).$$

Lemma 6. *$\square(\Theta+\Theta^*)\Gamma(E)$ is equal to $(\Theta+\Theta^*)\Gamma(E)$.*

Proof. We denote the projections from $\square\Gamma(E)$ to $(\Theta+\Theta^*)\Gamma(E)$ and to $\ker.\Theta\cap\ker.\Theta^*\cap\square\Gamma(E)$ by P_1 and P_2. Then since

$$(P_i\square P_j u,\ P_j u) = (\square P_j u, P_i P_j u) = 0,\quad i\neq j,$$

if $\square P_j u = P_i\square P_j u$, then $P_j u$ is equal to 0 because

$$(\square u,\ u) = ||Du||^2 + ||D^*u||^2$$

and ker. $\square \cap \square \Gamma(E)$ is equal to 0. Here $||v||$ means the norm of v. Moreover, since $\square$ is an isomorphism of $\square\Gamma(E)$, we get

$$(Gu,\ u) = (v,\ \square v) \neq 0,\ \ if\ u \neq 0\ (u = \square v),$$

for any $u \in \square\Gamma(E)$. Therefore if $GP_j u = P_i G P_j u$, $i \neq j$, then $P_j u$ is equal to 0. Hence we obtain

$$(8)' \qquad F^n u = 0 \text{ if and only if } u \in \ker.\Theta \cap \ker.\Theta^*,$$

because $\Theta + \Theta^*$ is an automorphism of $(\Theta+\Theta^*)\Gamma(E)$. By (8)′, we get

$$(8) \qquad F^n u = 0 \text{ if and only if } Fu = 0 \text{ for any } n.$$

Similarly, we get

$$(P_1F-\lambda)^n v = 0 \text{ if and only if } v = P_1 u \text{ and } (F-\lambda)^n u = 0,$$

$$\text{for any } n. \text{ In this case, } v \neq 0 \text{ if } u \neq 0.$$

For any proper value $\lambda \neq 0$ of F, we set

$$I_{F,\lambda} = \frac{1}{2\pi\sqrt{-1}} \int_{|\alpha-\lambda|=\varepsilon_\lambda} \alpha^{-1}(I-\alpha^{-1}F)^{-1}d\alpha,$$

where ε_λ is taken as follows :$\{\mu \mid |\alpha-\mu| \leq \varepsilon_\lambda\}$ contains no proper value of F which is different from λ. (I is the identity map). Then by (8) and the theorem of residue, we get

$$\sum_{\lambda\neq 0} I_{F,\lambda} = I - P_2 = P_1.$$

Then since F is a compact operator and

$$I_{F,\lambda}{}^* = \frac{1}{2\pi\sqrt{-1}} \int_{|\alpha-\bar\lambda|=\varepsilon^\lambda} \alpha^{-1}(I-\alpha^{-1}F^*)^{-1}d\alpha$$

$$= I_{F^*,\bar\lambda},$$

we have (cf. [7], p. 577 and p. 1119),

$$(9) \qquad \sum_{\lambda\neq 0} I_{F^*,\lambda} = P_1.$$

This shows F^*u is equal to 0 if and only if $u \in \ker.\Theta \cap \ker.\Theta^* \square\Gamma(E)$. But we know

$$(10)' \qquad F^*u = 0 \text{ if and only if } u \in G\ (\ker.\Theta \cap \ker.\Theta^* \cap \square\Gamma(E)),$$

because F^* is given by

$$F^* = (\Theta + \Theta^*)G.$$

Hence we have

(10) $$P_1GP_2 = 0.$$

Then since $P_2GP_1 = (P_1GP_2)^*$, P_2GP_1 is also equal to 0. Therefore G maps $(\Theta+\Theta^*)\tilde{\Gamma}(E)$ onto itself. Hence we obtain the lemma.

§ 5. Proof of theorem 4.

Theorem 4′. *If u is a non-zero proper vector belonging to a non-zero proper value of F, then u belongs in $(\Theta + \Theta^*)\tilde{\Gamma}(E)$.*

Proof. Since we know

$$Fu = \lambda u \text{ if and only if } F^*\square u = \lambda\square u,$$

and if $F^*v=\lambda v$, then v must belong in $(\Theta + \Theta^*)\tilde{\Gamma}(E)$ because $F^* = (\Theta + \Theta^*)G$, we have the theorem by lemma 6.

Lemma 7. *$\bigcirc(\Theta + \Theta^*)u$ is equal to 0 if vnd only if $(D + D^*)(\Theta + \Theta^*)u$ is equal to 0.*

Proof. Since $\bigcirc=(D+D^*)(D+D^*)$, $(D + D^*)(\Theta + \Theta^*)u = 0$ implies $\bigcirc(\Theta + \Theta^*)u=0$.

On the other hand, since $D + D^*$ is a self adjoint elliptic operator, $(D + D^*)(D + D^*)v = 0$ implies $(D + D^*)\, v = 0$. Hence we have the lemma.

Note. As was remarked at introduction, this lemma is true for arbitrary $v \in \tilde{\Gamma}(E)$ and arbitrary elliptic semi-complex $\{\Gamma(E_i),\ D_i\}$, if it is defined over a compact oriented manifold and each E_i is an Hermitian vector bundle.

Proof of theorem 4. (i) follows from lemma 7. (ii) follows from lemma 4 and theorem 4′.

Lemma 8. *If $\{\Gamma(E_i),\ D_i\}$ is a Bianchi semi-complex over a compact oriented manifold and each E_i is an Hermitian vector bundle, then Θ^*v (resp. Θu) is determined uniquely by Θu (resp. Θ^*v) if $(D + D^*)(\Theta u + \Theta^*v)$ is equal to 0.*

Proof. By assumption, We get

$$D^*\Theta u = -D\Theta^*v.$$

Hence $D\Theta^*v$ (resp. $D^*\Theta v$) is determined uniquely by $D^*\Theta u$ (resp. $D\Theta^*v$). Then since $D^*\Theta u$ and $D\Theta^*v$ are determined uniquely by Θu and $\Theta^* v$ by lemmas 2 and 2′, we have the lemma.

§ 6. Graduation of ker. $\bigcirc$.

We assume that $\{\Gamma(E_i),\ D_i\}$ is a Bianchi semi-complex over a compact oriented manifold and each E_i is an Hermitian vector bundle. Then by theorem 4, we can set

(11) $$\ker. \bigcirc = \ker. \Box \oplus (\ker. \bigcirc \cap (\Theta + \Theta^*) \tilde{\Gamma}(E)).$$

Since $\Box$ maps $\tilde{\Gamma}(E_p)$ into $\tilde{\Gamma}(E_p)$, we can set

(12)′ $$\ker. \Box = \sum_{p \geq 0} \ker. \Box \cap \tilde{\Gamma}(E_p).$$

On the other hand, $\ker. \bigcirc \cap (\Theta + \Theta^*) \tilde{\Gamma}(E)$ does not allow such graduation. But setting

$$\Theta u + \Theta^* v = \sum_{p \geq 0} {}_p u^p + \sum_{p \geq 0} \Theta_p{}^* v^p, \quad u^p, \quad v^p \in \tilde{\Gamma}(E_p),$$

we have

$$(D + D^*)(\Theta u + \Theta^* v) \cap \tilde{\Gamma}(E_q)$$
$$= D_{q+1}{}^* \Theta_{q-1} u^{q-1} + D_{q-1} \Theta_{q+1}{}^* u^{q+1}.$$

Therefore we obtain

(13) $$(D + D^*)(\Theta u + \Theta^* v) = 0 \text{ if and only if}$$
$$D_{q+1}{}^* \Theta_{q-1} u^{q-1} + D_{q-1} \Theta_{q+1}{}^* u^{q+1} = 0 \text{ for each q.}$$

We set

(14) $$\mathscr{H}^q = \{u \mid u \in \tilde{\Gamma}(E_q), \ Du = D^* u = 0\},$$
$$\mathscr{F}^q = \{u + v \mid u \in \Theta_{q+2}{}^* \tilde{\Gamma}(E_{q+2}), \ v \in \Theta_q \tilde{\Gamma}(E_q), \ Du + D^* v = 0\}.$$

We note that $\mathscr{F}^q$ is contained in $\ker. \bigcirc$ because $D^* u = Dv = 0$.

By (12)′ and (13), we have

Theorem 6. *If $\{\Gamma(E_i), D_i\}$ is a Bianchi semi-cnmplex over a compact oriented manifold and each E_i is an Hermitian vector bundle, then*

(12) $$\ker. \bigcirc = \sum_{p=0}^{m} \mathscr{H}^p \oplus \sum_{p=0}^{m-2} \mathscr{F}^p,$$

where $\sum$ means direct sum and m is the length of $\{\Gamma(E_i), D_i\}$.

Corollary 1. *Under the same assumptions, we have*

(15) $$\kappa_n(\mathrm{ch}(\{\Gamma(E_i), D_i\}) T_R(X)) = \sum_{p=0}^{m} (-1)^p (\dim. \mathscr{H}^p + \dim. \mathscr{F}^p),$$

where $T_R(X)$ means the Todd class of the complexification of the tangent bundle of X. (*cf.* [4]).

Corollary 2. $\sum_{p=0}^{m} (-1)^p (\dim. \mathscr{H}^p + \dim. \mathscr{F}^p)$ *is equaj to* 0 *if X is an odd dimensional manifold.*

Note. By lemma 8, the projection from $\mathscr{F}^q$ to $\Theta_{q+2}{}^*\tilde{\Gamma}(E_{q+2})$ (resp. to $\Theta_q\tilde{\Gamma}(E_q)$) is an isomorphism. Hence we get

$$\mathscr{F}^q \simeq \pi^q \mathscr{F}^q, \quad (\mathscr{F}^q \simeq \pi^{q+2}\mathscr{F}^q), \tag{16}$$

where π^q is the projection from $\tilde{\Gamma}(E)$ to $\tilde{\Gamma}(E_q)$. Moreover, we have

$$\begin{aligned} &\pi^{q+2}\tilde{\Gamma}^q \perp \pi^{q+2}\tilde{\Gamma}^{q+2}, \\ &\pi^q\tilde{\Gamma}^q = 0, \quad p \neq q, \quad q+2. \end{aligned} \tag{16$'$}$$

§ 7. An example.

Let E be an oriented vector bundle over an oriented surface, then taking a connection θ of E, we have the following sequence.

$$\Gamma(E) \xrightarrow{D} \Gamma(E\otimes T^*) \xrightarrow{D} \Gamma(E\otimes\Lambda^2T^*), \tag{17}$$

where $D = d + \theta$ is the covariant derivative of E derived from θ.

Since $D\Theta = D^3$ maps $\Gamma(E)$ into $\Gamma(E\otimes\Lambda^3T^*) = 0$, (17) is a Bianchi semi-complex.

As we know

$$(D + D^*)u = 0 \text{ and } u \in \Gamma(E\otimes T^*) \text{ then } \mathrm{D}u = D^*u = 0,$$

we need only to consider the equation

$$\begin{aligned} &(D + D^*)(u^0 + u^2) = 0, \\ &u^0 \in \Gamma(E), \quad u^2 \in \Gamma(E\otimes\Lambda^2T^*). \end{aligned} \tag{18$'$}$$

Using local coordinates, we set

$$u^0 = f, \quad u^2 = g dx_1 \wedge dx_2, \quad \theta = \theta_1 dx_1 + \theta_2 dx_2,$$

$$Df = (\frac{\partial f}{\partial x_1} + \theta_1 f)\, dx_1 + (\frac{\partial f}{\partial x_2} + \theta_2 f)\, dx_2,$$

$$D^*(g dx_1 \wedge dx_2) = -(\frac{\partial g}{\partial x_2} + \theta_2)\, g dx_1 + (\frac{\partial g}{\partial x_1} + \theta_1) g dx_2,$$

Then (18)$'$ is rewritten as follows :

$$\frac{\partial f}{\partial x_1} + \theta_1 f = \frac{\partial g}{\partial x_2} + \theta_2 g, \quad \frac{\partial f}{\partial x_2} + \theta_2 f = -\frac{\partial g}{\partial x_1} - \theta_1 g, \tag{18}$$

or in another form,

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1}, & -\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2}, & \dfrac{\partial}{\partial x_1} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \theta_1, & -\theta_2 \\ \theta_2, & \theta_1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

We note that, if E is a real vector bundle and u^0, u^2 are real cross-sections, then setting $h=f+\sqrt{-1}g$, $z=x_1+\sqrt{-1}x_2$, (18) is rewritten as

$$\frac{\partial h}{\partial z}=-2(\theta_1-\sqrt{-1}\theta_2)\bar{h}, \tag{19}$$

where $\partial h/\partial z=1/2(\partial h/\partial x_1-1/\sqrt{-1}\partial h/\partial x_2)$, $\bar{h}=f-\sqrt{-1}g$. But the equation (19) can not be defined on X unless X is open or a complex torus and E is locally flat.

In general, as for the the equation (18), we have the following

Theorem 5. *There is a* 1 *to* 1 *correspondence between the germ of the solutions of* (18) *and the germ of the following type functions.*

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{matrix} = \\ = \end{matrix} \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial x_1}+\dfrac{\partial\varphi_2}{\partial x_2} \\ -\dfrac{\partial\varphi_1}{\partial x_2}+\dfrac{\partial\varphi_2}{\partial x_1} \end{pmatrix}, \tag{20}$$

φ_1, φ_2 are the (vector valued) functions whose components are harmonic functions.

Before proving the theorem, we note that setting

$$T=\begin{pmatrix} \dfrac{\partial}{\partial x_1}, & -\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2}, & \dfrac{\partial}{\partial x_1} \end{pmatrix}, \qquad T^*=\begin{pmatrix} \dfrac{\partial}{\partial x_1}, & \dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_2}, & \dfrac{\partial}{\partial x_1} \end{pmatrix},$$

we get

$$TT^*=\begin{pmatrix} \dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}, & 0 \\ 0, & \dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2} \end{pmatrix}, \quad \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}=T^*\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

Hence $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ is the solution of $T\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}=0$. We also prepare the following

Lemma 9. *A solution of the equation*

$$T\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

is given by

$$f_1=\frac{1}{2\pi}\int_{R^2}\frac{-(x_1-\xi_1)g_1(\xi)+(x_2-\xi_2)g_2(\xi)}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}d\xi_1 d\xi_2 \tag{21}$$

$$f_2=\frac{1}{2\pi}\int_{R^2}\frac{-(x_1-\xi_1)g_2(\xi)-(x_2-\xi_2)g_1(\xi)}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}d\xi_1 d\xi_2,$$

if g_1, g_2 are C^1-class compact carrier (vector valued) functions.

This lemma follows from Poisson's formula. We set

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = T^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ is given by (21).}$$

Then using polar coordinates, we have

Lemma 10. *If* car. g_1 *and* car. g_2 *are contained in*

$$B(r, a)=\{\xi \mid |\xi-a|<r\},$$

and

$$\max_{\xi\in B(r, a)}. (|g_1(\xi)|, |g_2(\xi)|)\leq M,$$

then we have

$$\max_{x\in R^2}. (|f_1(x)|, |f_2(x)|)\leq Mr. \tag{22}$$

Proof of theorem 5. We set

$$\theta=\begin{pmatrix} \theta_1, & -\theta_2 \\ \theta_2, & \theta_1 \end{pmatrix}, \qquad I_0=\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where h_1, h_2 are given by (20). Since the problem is local, we may assume

$$\text{car.}\,\theta_1 \cup \text{car.}\,\theta_2 \subset B(r, a), \qquad \max.(|\theta_1|, |\theta_2|)\leq M$$

and

$$rM<\frac{1}{4}.$$

Then, setting $I_n(\theta, I_0)=T^{-1}(\theta I_{n-1}(\theta, I_0))$, $n>1$, $I_0(\theta, I_0)=I_0$, the Peano series

$$P(\theta, I_0)=\sum_{n\geq 0} I_n(\theta, I_0)$$

converges absolutely and uniformly on R^2. Then by its definition, $P(\theta, I_0)$ satisfies (18). Therefore we can define the map

$$P: \{\text{the set of germs of the solutions of } \mathrm{Tf}=0\}$$

$\rightarrow$ {the set of germs of the solutions of (18)},

by $P(I_0) = P(\theta, I_0)$, where f means the germ of f. Moreover, if we take

$$P(\theta, T^{-1}(g)) = \sum_{n\geq 0} I_n(\theta, T^{-1}(g)),$$

then we have

$$T(P(\theta, T^{-1}(g))) + \theta P(\theta, T^{-1}(g)) = g,$$

$$|P(\theta, T^{-1}(g))| < rN/(1-4rM), \text{ if car. g} \subset \mathrm{B}(r, a) \text{ and } |g| < N,$$

where $|f|$ means $\max.(|f_1|, |f_2|)$. Hence if $J_0 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ satisfies (18), then setting $T_\theta^{-1}(g) = P(\theta, T^{-1}(g))$, the Peano series

$$P_\theta(-\theta, J_0) = \sum_{n\geq 0} I_{\theta,n}(-\theta, J_0),$$

$$I_{\theta,0}(-\theta, J_0) = J_0, \quad I_{\theta,n}(-\theta, J_0) = T_\theta^{-1}(-\theta I_{\theta,n-1}(\theta, J_0)),$$

satisfies

$$P_\theta(-\theta, P(\theta, I_0)) = I_0,$$

$$T(P_\theta(-\theta, J_0)) = 0,$$

if r is sufficiently small. (i. e. if $rN/(1-4rM) < 1/4$). Hence we have the theorem.

Note. As was remarked at introduction, the proof of theorem 5′ is quite similar to the above proof, because the essential part of the above proof is the estimate (22) which was assumed in theorem 5′.

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