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Associate Vector Bundles of Microbundles

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Introduction. The main purpose of this paper is to introduce the notion of dual bundles of topological microbundles. For this purpose, we construct associate vector bundles of topological microbundles in §2. One of the interesting result is that a cross-section of the cotangent microbundle (i. e. the dual bundle of the tangent microbundle) of a topological manifold X is a 1-cochain of X in the sense of Alexander-Spanier. This justifies our definition of dual bundles. We also define the tensor product of associate vector bundles (or their dual bundles) of microbundles and show the correspondence between the cross-sections of the tensor products of cotangent microbundles and Alexander-Spanier cochains. In §4, we consider the relation between associate vector bundles of $H^d{}_*(n)_c$-bundles and jet bundles.

§1 Transition functions of microbundles.

1. Definition of the sheaf $H_*(n)_c$. We set

$$E_0(n) = \{ f \mid f \text{ is a homeomorphism from } R^n \text{ into } R^n \text{ and } f(0) = 0\}.$$

We regard $E_0(n)$ to be a topological semigroup by compact open topology. We denote by X a topological space with $\{(U_\alpha(x)\}$ the neighborhood basis of $x \in X$. The semigroup of all continuous maps from $U_\alpha(x)$ into $E_0(n)$ is denoted by $H(U_\alpha(x), E_0(n))$. For $f \varepsilon H(U_\alpha(x), E_0(n))$, we set

$$(1) \qquad \hat{f}(y, a) = (y, f(y)(a)).$$

By definition, $\hat{f}$ is a homeomorphism from $U_\alpha(x) \times \boldsymbol{R}^n$ into $U_\alpha(x) \times \boldsymbol{R}^n$.

Definition. *We call f and g are equivalent if $\hat{f}$ and $\hat{g}$ coincide on some neighborhood of $x \times 0$ in $U_\alpha(x) \times R^n$ and denote $f \sim g$.*

The set of equivalence classes of $H(U_\alpha(x), E_0(n))$ by this relation is denoted by $H_*(U_\alpha(x), E_0(n))$. It is also a semigroup.

If $U_\alpha(x)$ contains $U_\beta(x)$, then there is the restriction homomorphism $\bar{r}^\alpha_\beta : H(U_\alpha(x), E_0(n)) \to H(U_\beta(x), E_0(n))$, and it induces a homomorphism $r^\alpha_\beta : H_*(U_\alpha(x), E_0(n) \to H_*(U_\alpha x), E_0(n))$. Since we know $\bar{r}^\alpha_\beta \bar{r}^\beta_\gamma = \bar{r}^\alpha_\gamma$ if $U_\alpha(x) \supset U_\beta(x) \supset U_\gamma(x)$, we set

$$(2) \qquad H^*(n)_x = \lim\, [H_*(U_\alpha(x), E_0(n)), \bar{r}^\alpha_\beta].$$

Lemma 1. *For any* $f \in H(U_\alpha(x), E_0(n))$, *there exists a* $U_\beta(x)$ *such that* $\bar{r}^\alpha_\beta f$ *has the inverse in* $H_*(U_\alpha(x), E_0(n))$.

Proof. Since f is a homeomorphism, for suitable $U_\gamma(x)$, if $y \in U_\gamma(x)$ then $f(y)$ $(\boldsymbol{R}^n)$ contains Q', a neighborhood of the origin of $\boldsymbol{R}^n$. Moreover, we may assume that there exists a homeomorphism e from $\boldsymbol{R}^n$ onto Q' as follows :

$$e(a) = a, \text{ if } a \in Q'', \ Q'' \text{ is a neighborhood of the origin of } \boldsymbol{R}^n.$$

Then $f^{-1}e$ belongs in $H(U_\gamma(x), E_0(n))$ and on some suitable $U_\beta(x) \times Q$, $U_\beta(x) \subset U_\beta(x)$, $Q \subset Q''$ and Q is a neighborhood of the origin of R^n, we get

$$(\hat{f}(\hat{f}^{-1}e))(y, a) = ((\hat{f}^{-1}e)\hat{f})(y, a) = (y, a), \quad y \in U_\beta(x), \quad a \in Q.$$

Therefore we have the lemma.

Corollary. $H_*(n)_x$ *is a group.*

Definition. *If* $f \in H(U, E_0(n))$, *then its class in* $H_*(n)_x$ *is denoted by* f_x.

For a neighborhood $V(x)$ of x in X, we set

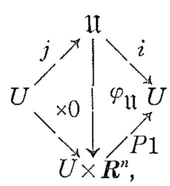
$$(3) \qquad U(f_x, V(x)) = \{f_y \mid y \in V(x)\}.$$

In $\bigcup_{x \in X} H_*(n)_x$, we take $\{U(f_x, V(x))\}$ to be the neighborhood basis of f_x. Then $\bigcup_{x \in X} H_*(n)_x$ becomes a sheaf of groups over X. This sheaf is denoted by $H_*(n)_c$.

Note. Similarly, starting from a pseudogroup Γ of continuous transformations of R^n, we can construct a sheaf Γ_{*c} (or $\Gamma^d{}_{*c}$) over X. (cf. [3] or § 4).

2. *The cohomology set* $H^1(X, H_*(n)_c)$. **Theorem 1.** *If* X *is a normal paracompact topological space, than there is a 1 to 1 correspondence between the set of all equivalence classes of n-dimensional topological microbundes over* X *and* $H^1(X, H_*(n)_c)$.

Proof. Let $\mathfrak{X} : X \xrightarrow{i} E \xrightarrow{j} X$ be an n-dimensional microbundle over X defined by the diagram



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where $\mathfrak{U}$ is an open set in E, $\varphi_{\mathfrak{U}}$ is a homeomorphism, then we can set

$$\varphi_{\mathfrak{U}}\varphi_{\mathfrak{B}}^{-1}(x, a) = (x, \bar{\varphi}_{UV}(x)(a)), \quad (x, a) \in \varphi_{\mathfrak{B}}(\mathfrak{U} \cap \mathfrak{B}),$$

because $\varphi_{\mathfrak{U}}\varphi_{\mathfrak{B}}^{-1}(x, a)$, $(x \in U \cap V)$, does not change x. By definition, $\bar{\varphi}_{UV(x)}$ is a homeomorphism from some neighborhood of the origin of R^n to some neighborhood of the origin of $\mathbf{R}^n$ and $\bar{\varphi}_{UV}(x)(0) = 0$. Since $\varphi_{\mathfrak{U}}\varphi_{\mathfrak{B}}^{-1}$ is a homeomorphism from $\varphi_{\mathfrak{U}}(\mathfrak{U} \cap \mathfrak{U})$ onto $\varphi_{\mathfrak{B}}(\mathfrak{U} \cap \mathfrak{U})$, $\bar{\varphi}_{UV}(x)$ is continuous in x. We denote the class of $\bar{\varphi}_{UV}(x)$ in $H_*(n)_x$ by $\varphi_{UV}(x)$. (cf. the proof of lemma 1). Then by definition,

$$\{\varphi_{UV}(x)\} \in Z^1(X, H_*(n)_c).$$

If $\mathfrak{X}$ and $\mathfrak{X}'$ are equivalent microbundles by the bundle map $h : E \to E'$, then there is an $h_U \in H(U, E_0(n))$ for any U such that the diagram

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\quad h_U \quad} & \mathfrak{U}' \\ \varphi_{\mathfrak{U}} \downarrow & h_U & \downarrow \varphi'_{\mathfrak{U}'} \\ U \times R^n & \xrightarrow{\qquad\quad} & U \times \mathbf{R}^n \end{array} ,$$

is commutative. Therefore if the transition functions of $\mathfrak{X}$ and $\mathfrak{X}'$ are given by $\{\varphi_{UV}(x)\}$ and $\{\varphi_{UV}'(x)\}$, then we get

$$\bar{\varphi}_{UV}'(x) = h_U(x) \bar{\varphi}_{UV}(x) h_V(x)^{-1}.$$

Hence they are cohomologous and therefore the equivalence class of $\mathfrak{X}$ corresponds to an element of $H^1(X, H_*(n)_c)$.

To show the converse, we take an element $\{\varphi_{\alpha\beta}(x)\} \in Z^1(X, H_*(n)_c)$ and assume that its covering system $\{U_\alpha\}$ is locally finite. We denote the representation of $\varphi_{\beta\alpha}(x)$ by $\bar{\varphi}_{\alpha\beta}(x)$.

Since X is paracompact, for suitable locally finite covering system $\{U_\alpha\}$, we obtain

$$\bar{\varphi}_{\beta\gamma}(x) \bar{\varphi}_{\gamma\alpha}(x) \bar{\varphi}_{\alpha\beta}(x)(a) = a, \quad \text{if } x \in U_\alpha \cap U_\beta \cap U_\gamma, \quad a \in Q_{\alpha\beta},$$

where $Q_{\alpha\beta}$ is a neighborhood of the origin of $\mathbf{R}^n$.

In $U_\alpha \times \mathbf{R}^n \times \alpha$, we set

$$\mathfrak{U}_{\beta\alpha} = ((U_\alpha \cap U_\beta) \times Q_{\beta\alpha} \cap \bar{\varphi}_{\alpha\beta}((U_\alpha \cap U_\beta) \times Q_{\alpha\beta})) \times \alpha.$$

Then $\mathfrak{U}_{\beta\alpha}$ is an open set of $U_\alpha \times R^n \times \alpha$. We identify $\mathfrak{U}_{\alpha\beta} \ni x \times a \times \alpha$ and $x \times \bar{\varphi}_{\alpha\beta}(x)(a) \times \alpha \in \mathfrak{U}_{\beta\alpha}$. By the assumption about $\bar{\varphi}_{\alpha\beta}(x)$ and the definition of $\mathfrak{U}_{\alpha\beta}$, the equivalence relation is well defined. The quotient space of $\cup_\alpha U_\alpha \times \mathrm{R}^n \times \alpha$ by this relation is denoted by E. Then by the definition of the equialence, we can defne $i: X \to E$ and $j: E \to X$.

Since $\{U_\alpha\}$ is locally finite, for all $x \in X$, $\cap_{x \in U_\alpha \cap U_\beta} \mathfrak{U}_{\beta\alpha} \subset U_\alpha \times R^n \times \alpha$ is a neighborhood of $x \times 0 \times \alpha$ in $U_\alpha \times R^n \times \alpha$. Therefore there is a neighborhood $V(x)$ of x in X and Q of the origin of R^n such that

$$V(x) \times Q \times \alpha \subset \bigcap_{x \in U_\alpha \cap U_\beta} \mathfrak{U}_{\beta\alpha}.$$

Then by the definition of E, $V(x) \times Q \times \alpha$ becomes an open set of E. Therefore $X \xrightarrow{j} E \xrightarrow{i} X$ is a microbundle and by the above construction, the transition function of this bundle is $\{\varphi_{\alpha\beta}(x)\}$.

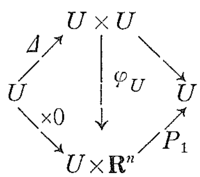
If $\{\bar{\varphi}_{\alpha\beta}'(x)\}$ is another rrpresentation of $\{\varphi_{\alpha\beta}(x)\}$, then we get

$$\bar{\varphi}_{\alpha\beta}'(x) \mid W(x) \times Q' = \bar{\varphi}_{\alpha\beta}(x) \mid W(x) \times Q',$$

on some $W(x) \subset X$ and $Q' \subset \mathbf{R}^n$. Hence they define same microbundles.

In the above construction, if $\{\varphi_{UV}(x)\}$ is induced from $\mathfrak{X}$, then $\{\varphi_{UV}(x)\}$ induces $\mathfrak{X}$. Therefore we obtain the theorem.

Example. If X is a topological manifold with coordinate system $\{(U, h_\alpha,)\}$ then the tangent microbundle τ: $X \xrightarrow{\Delta} X \times X \rightarrow X$ is defined by the diagram



$\varphi_\alpha(x, y) = (x, h_U(y) - h_U(x))$,

where Δ is the diagonal map. ([7]). Therefore, setting

$$h_{U,x}(y) = h_U(y) - h_U(x), \quad y \in U,$$

the transition function $\{g_{UV}(x)\}$ of τ is given by

$$(4) \qquad g_{UV}(x)(a) = h_{U,x}h_{V,x}{}^{-1}(a) = h_U h_V^{-1}(a + h_V(x)) - h_U(x).$$

Note. If we start from Kister's theorem, we can construct an associate vector bundle of a microbundle as a $C(\mathbf{R}^n)$-bundle. In this case the dual bundle of this associate vector bundle must be a $C(\mathbf{R}^n)^*$-bundle. Here $C(\mathbf{R}^n)$ is the topological vector space of all real valued continuous functions on $\mathbf{R}^n$ with normally convergence topolgy and $C(\mathbf{R}^n)^*$ is its dual space. (i. e. the space of compact carrier measures of $\mathbf{R}^n$). But in this paper, we construct another type of associate vector bundle and its dual bundle starting from theorem 1.

§ 2 Associate vector bundles of microbundles.

3. *Construction of associate vector bundles.* For an open U of X, a normal paracompact topological space, we set

$C_0(U \times \mathbf{R}^n) = \{f \mid f$ is a (real valued) continuous functions on $U \times \mathbf{R}^n$ such that setting $f_y(a) = f(y, a)$, $y \in U$, the carrier of f_y is compact$\}$.

we regard $C_0(U \times \mathbf{R}^n)$ to be a topological vector space over $\mathbf{R}$ by normally convergence topology. Similarly, we denote by $C(U \times \mathbf{R}^n)$ the topological vector space of all continuous functions on $U \times \mathbf{R}^n$ with normally convergence topology. The submodule of $C(U \times \mathbf{R}^n)$ consisted by those functions that vanish on some neighborhood of $U \times 0$ is denoted by $C_e(U \times \mathbf{R}^n)$. Then, since X is normal, we have

$$(5) \quad C(U \times \mathbf{R}^n)/C_e(U \times \mathbf{R}^n) \simeq C_0(U \times \mathbf{R}^n)/C_e(U \times \mathbf{R}^n) \cap C_0(U \times \mathbf{R}^n).$$

We denote this module by $C_*(U \times \mathbf{R}^n)$. It is a vector space over $\mathbf{R}$, but not a topological vector space.

Since the restriction homomorphism r_V^U: $C(U \times \mathbf{R}^n) \rightarrow C(V \times \mathbf{R}^n)$, $U \supset V$, maps $C_e(U \times \mathbf{R}^n)$ into $C^n(V \times \mathbf{R}^n)$, we set

$$(6) \qquad C_*(n)_x = lim. \ [C_*(U_\alpha(x) \times \mathbf{R}^n) \mid r_\beta^\alpha].$$

Here $\bar{r}^{\alpha}_{\beta}$ is the induced homomorphism from r^{α}_{β}.

Lemma 2. *If φ_x belongs in $H_*(n)_x$, then φ_x induces an isomorphism $\varphi_x{}^*$ from $C_*(n)_x$ onto $C_*(n)_x$.*

Proof. We assume φ belongs in $H(U, E_0(n))$. Then φ induces an isomorphism φ^* from $C_0(\varphi(U \times \mathbf{R}^n))$ onto $C_0(U \times \mathbf{R}^n)$ and by this map, $C_e(U \times \mathbf{R}^n) \cap C_0(\varphi(U \times \mathbf{R}^n))$ is mapped on $C_0(U \times \mathbf{R}^n) \cap C_e(U \times \mathbf{R}^n)$. Here $C_0(\hat{\varphi}(U \times \mathbf{R}^n))$ is defined similarly as $C_0(U \times \mathbf{R}^n)$. Then we get the lemma because we know

$$C_*(U \times \mathbf{R}^n) \simeq C_0(\hat{\varphi}(U \times \mathbf{R}^n)) / C_0(\hat{\varphi}(U \times \mathbf{R}^n)) \cap C_e(U \times \mathbf{R}^n).$$

We denote the class of $f \in C(U \times \mathbf{R}^n)$ in $C_*(n)_x$ by f_x and for a neighborhood $V(x)$ of x, we set

$$U(f_x, V(x)) = \{f_y \mid y \in V(x)\}.$$

Then taking $U(f_x, V(x))$ to be the neighborhood basis of $\cup_{x \in U} x \times C_*(n)_x$, $\cup_{x \in U} x \times C_*(n)_x$ becomes a topological space. It is denoted by $(U, C_*(n))$. By the definitons of the topologies of $H_*(n)_c$ and $(U, C_*(n))$, we have

Lemma 3. *If φ belongs in $H^0(U, H_*(n)_c)$, then φ induces a homeomorphism φ^* from $(U, C_*(n))$ onto $(U, C_*(n))$ and $\varphi^*(x)$ is an isomorphism from $C_*(n)_x$ onto $C_*(n)_x$.*

Using this lemma and theorem 1, we construct the associate vector vector bundle $v\,\mathfrak{X}$ of $\mathfrak{X}$ by the following way. Let $\{\varphi_{UV}\}$ be the transition function of $\mathfrak{X}$, then we classify $\cup_U(U, C_*(n))$ by the relation

$$((U, C_*(n)) \ni (x, f_x) \sim (x, \varphi_{UV}(x)^* f_x) \in (V, C_*(n))$$

and set this quotient space by $v(E)$. By definition, $v(E)$ is a topological space and the projection p from $v(E)$ onto X is defined and continuous.

Definition. *A continuous maps from X into $v(E)$ such that*

$$p^* s(x) = x$$

is called a cross-section.

Since the class of a constant function is invariant under the operation of $\varphi_x{}^*$, $v(\mathfrak{X})$ always has a non-trivial cross-section. Or in other word, $v(\mathfrak{X})$ contains a 1-imensional trivial vector bundle $\mathbf{R}$ and we have a direct sum decomposition

$$v(\mathfrak{X}) = v_0(\mathfrak{X}) \oplus \boldsymbol{R}. \tag{7}$$

Note. If we use the vector space of complex valued continuous functions instead of the vector space of real valued continuous functions, then we can construct another associate vector bundle $v^c(\mathfrak{X})$ of $\mathfrak{X}$, then we get

$$v^c(\mathfrak{X}) = v(\mathfrak{X}) \otimes \mathbf{C}.$$

4. Tensor product, symmetric product and alternative product. If $\varphi \in H(U, E_0(n))$, $\phi \in H(U, E_0(m,))$ then we can define $\varphi \otimes \phi \in H(U, E_0(n+m))$ by

$$((\varphi\otimes\psi)(y))(a)=(\varphi(y)(a'),\ \psi(y)(a'')),\ a=(a',a''),\ a'\in \boldsymbol{R}^n,\ a''\in\boldsymbol{R}^m.$$

By definition, if $\{\varphi_{UV}(x)\}$ is a transition function of $\mathfrak{X}$, $\{\psi_{UV}(x)\}$ is a transition function of $\mathfrak{Y}$, then $\{\varphi_{UV}(x)\otimes\psi_{UV}(y)\}$ is a transition function of $\mathfrak{X}\oplus\mathfrak{Y}$, the whitney sum of $\mathfrak{X}$ and $\mathfrak{Y}$ in the sense of Milnor. ([7]). But since $(\varphi\otimes\psi)_x{}^*$ is an isomorphism of $C_*(n+m)_x$ which may be regarded to be the tensor product of $C_*(n)_x$ and $C_*(m)_x$, we call $v(\mathfrak{X}\oplus\mathfrak{Y})$ the tensor product of $v(\mathfrak{X})$ and $v(\mathfrak{Y})$ and denote

$$v(\mathfrak{X})\otimes v(\mathfrak{Y})=v(\mathfrak{X}\oplus\mathfrak{Y}).$$

For an element f of $C(\boldsymbol{R}^{nm})=C(\boldsymbol{R}^n\times\cdots\times\boldsymbol{R}^n)$, we call f is aymmetric (or alternative) (with respect to $\boldsymbol{R}^n$) if f satisfies

$$f(a_1,\cdots\cdots,a_m)=f(a_{\sigma(1)},\cdots\cdots,a_{(\sigma m)}),\quad a_i\in\boldsymbol{R}^n,\ \sigma\in\mathfrak{S}_m,$$

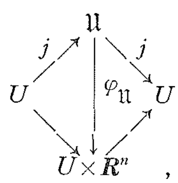
or

$$f(a_1,\cdots\cdots,a_m)=sgn(\sigma)f(a_{\sigma(1)};\cdots\cdots,a_{\sigma(m)}).$$

Then $(\varphi\otimes\cdots\otimes\varphi)^*f$ is symmetric or alternative if and only if f is symmetric or alternative. Here φ is an element of $E_0(n)$. Therefore we can define m-th symmetric product $S^m(v(\mathfrak{X}))$ and m-th alternative product $A^m(v(\mathfrak{X}))$ of $v(\mathfrak{X})$.

§ 3 Dual bundles of microbundles.

5. *Definition of dual budles.* If $\mathfrak{X}$: $X\overset{i}{\to}E\overset{j}{\to}X$ is a microbundle defined by the diagram



then we may assume $\varphi_{\mathfrak{U}}(\mathfrak{U})\supset U\times Q$, where Q is a neighborhhood of the origin of $\boldsymbol{R}^n$. Therefore we have an into isomorphism

$$\varphi^*_{\mathfrak{U}}:C_0(U\times Q)\to C_0(\mathfrak{U}).$$

Moreover, we may assume

$$\varphi_{\mathfrak{V}}\varphi_{\mathfrak{U}}{}^{-1}((U\cap V)\times Q)\supset(U\cup V)\times Q'\ ,$$

for some neighborhood of the origin Q'. Hence $(\varphi_{\mathfrak{V}}{}^{-1*}\varphi_{\mathfrak{U}}{}^*)\ (C_0(U\times Q))$ contains $C_0((U\cap V)\times Q')$. Therefore we get

Lemma 4. $(\varphi_{\mathfrak{U}}{}^{-1*}\varphi_{\mathfrak{V}}{}^*)\ (x)$ *induces an isomorphism of* $C_*(n)_x$.

Lemma 5. $\varphi_{\mathfrak{U}}{}^{-1*}\varphi_{\mathfrak{V}}{}^*$ *induces a homeomorphism from* $(U\cap V, C_*(n))$ *onto* $(U\cap V, C_*(n))$.

We denote the homeomorphism from $(U \cap V,\ C_*(n))$ onto $(U \cap V,\ C_*(n))$ induced from $\varphi_{\mathfrak{U}}{}^{-1*}\varphi_{\mathfrak{V}}{}^*$ by φ_{UV}'. Its value at x is denoted by $\varphi_{UV}'(x)$. Then by lemma 5, we can classify $\cup_U(U,\ C_*(n))$ by the relation

$$(U,\ C_*(n)) \ni (x, f_x) \sim (x, \varphi_{UV}'(x)f_x) \in (V,\ C_*(n)).$$

The quotient space of $\cup_U(U,\ C_*(N))$ by this relation is denoted by $x^*(E)$. The projection from $v^*(E)$ onto X is denoted by p^*.

Definition. *We set* $v^*(\mathfrak{X}) = \{X,\ v^*(E),\ p^*\}$ *and call* $v^*(\mathfrak{X})$ *the dual bundle of* $v(\mathfrak{X})$.

Since we know that if φ and ϕ are two isomorphisms of $C_0(\boldsymbol{R}^n)$ and $C_0(\boldsymbol{R}^m)$ then we can define the isomorphism $\varphi\otimes\phi$ of $C_0(\boldsymbol{R}^{n+m})$ by

$$(\varphi\otimes\phi)\ (f(a, b)) = \varphi((g_b(a)),\ g_b(a) = \phi(f_a(b)),\ a \in \boldsymbol{R}^n,\ \ b \in \boldsymbol{R}^m,$$

we can define $(v^*(\mathfrak{X}) \otimes v(\mathfrak{Y}))$, $v(\mathfrak{X})\otimes v^*(\mathfrak{Y})$ and $v^*(\mathfrak{Y})\otimes v^*(\mathfrak{X})$. Moreover, we obtain

$$(8) \qquad v^*(\mathfrak{X})\otimes v^*(\mathfrak{Y}) = v^*(\mathfrak{X}\oplus\mathfrak{Y}).$$

Definition. $v^*(\tau)$ *is called the cotangent microbundle of* X. *Here* X *is a topological manifold and* τ *is the tangent microbundle of* X.

As $v(\mathfrak{X})$, we can define the cross-sections of $v^*(\mathfrak{X})$ and there is the following direct sum decomposition.

$$(7)' \qquad v^*(\mathfrak{X}) = v_0{}^*(\mathfrak{X}) \oplus \boldsymbol{R}.$$

6. *Cross-sections of cotangent microbundles.* **Theorem 2.** *If* s *is a cross-section of* p*-tensor prouct of* $v^*(\tau)$, *then* s *is a* p*-cochain of* X *in the sense of Alexander-Spanier. The converse is also true.*

Proof. We assume that the manifold structure of X is given by $\{(U, h_U)\}$. Then we may set

$$(9) \qquad g_{UV}'(x) = h_{U,x}{}^{-1*}h_{V,x}{}^*,\quad h_{U,x}(y) = h_U(y) - h_U(x),$$

and a cross-section s of $v^*(\tau)\otimes\cdots\otimes v^*(\tau)$ is given by

$$s = \{s_U(x,\ a_1, \cdots\cdots, a_p)\},\quad x \in U,\quad a_i \in \boldsymbol{R}^n,\quad (n = \dim. X),$$
$$s_U(x, g_{UV}'\ (x)\ (a_1), \cdots\cdots, g_{VU}'(x)(a_p)) = s_V(x, a_1, \cdots\cdots, a_p).$$

Hence by (9), we obtain

$$(10) \qquad s_U(x, h_{U,x}{}^{-1}(a_1), \cdots\cdots, h_{U,x}{}^{-1}(a_p))$$
$$= s_V(x, h_{V,x}{}^{-1}(a_1), \cdots\cdots, h_{V,x}{}^{-1}(a_p)),\quad x \in U \cap V.$$

This shows our assertion.

Similarly, we have

Theorem 3. *A Cross-section of* $v\ (\mathfrak{X}) \otimes \overbrace{v^*(\tau)\otimes\cdots\otimes v^*(\tau)}^{s}$ *is an* s*-cross-section of* $\mathfrak{X}$ *and the converse is also tsrue. (For the definition of* s*-cross-sections, see [2]).*

Note 1. In [2], we use another definition of Alexander-Spanier cochain and s-cross-sections, but they correspond to cross-sections of a subbundle of

$$v_0^*(\tau)\otimes\cdots\otimes v_0^*(\tau) \text{ and } v(\mathfrak{X})\otimes v_0^*(\tau)\otimes\cdots\otimes v_0^*(\tau).$$

Note 2. In general, if X is a paracompact topological manifold and ξ is a vector bundle over X, then we obtain

$$(11)\quad C^s(X,\xi)\subset H^0(X,\xi\otimes\overbrace{v_0^*(\tau)\otimes\cdots\cdots\otimes v_0^*(\tau)}^{s}).$$

Note 3. If X is a normal paracompact manifold with locally finite covering system $\{U\}$, $\mathfrak{X}$ is a microbundle over X with transition function $\{\varphi_{UV}(x)$, then setting

$$\theta_U(x_0,\ x_1)=\sum_{U\cap W\neq\phi}\sqrt{e_W(x_0)e_W(x_1)}\ \varphi_{UW}(x_0)\,(\varphi_{UW}(x_1)-\varphi_{UW}(x_0)),$$

$$(D(\varphi))_U=d\varphi_U+\theta_U\varphi_U,\ \ \varphi=(\varphi_U)\in C^s\,(X,\mathfrak{X}),\ \ s\geq 0,$$

we get the following sequence.

$$(12)\qquad C^0(X,\ \mathfrak{X})\xrightarrow{D}C^1(X,\mathfrak{X})\xrightarrow{D}\cdots\xrightarrow{D}C^s\,(X,\mathfrak{X})\xrightarrow{D}C^{s+1}((X,\mathfrak{X})\xrightarrow{D}\cdots.$$

Here $\{e_W\}$ is the partition of unity of X with car. $e_W\subset W$ and d is the derivation in Alexander-Spanier cohomolgy. (cf. [2]).

§4 Associate vectorbundles of $\mathbf{H}_*{}^d(\mathbf{n})_c$ -bundles.

7. *Associate vector bundles of Γ-bundles.* We denote by Γ a pseudogroup of continuous transformations of $\mathbf{R}^n$ and set

$$\Gamma_0=\{f\mid f\in\Gamma,\ f \text{ is defined on some neighborhood of the origin and } f(0)=0\},$$

$$\Gamma_0{}^\#=\{f\mid f\in E_0(n),\ \text{for some } Q, \text{ a neighborhood of the origin, } f\mid Q\in\Gamma_0\}.$$

By definition, $\Gamma_0{}^\#$ is a subsemigroup of $E_0(n)$ and we have

Lemma 6, *For any $f\in\Gamma_0$, there exists a $\bar f\in\Gamma_0{}^\#$ and a neighborhood of the origin Q such that*

$$(13)\qquad \bar f\mid Q=f\mid Q.$$

By lemma 6, we can define the group Γ_{*x} by

$$(14)\qquad \Gamma_{*x}=lim.\,[H_*\,(U_\alpha(x),\ \Gamma_0{}^\#)\mid \bar r^\alpha_\beta].$$

Here $H_*\,(U_\alpha(x),\Gamma_0{}^\#)$ is defined similarly as $H_*(U_\alpha\,(x),\ E_0(n))$.

Let $e_i,\ i=0,\cdots\cdots,n$ be carrier compact continuous functions on $\mathbf{R}^n$ such that

$$e_0(a)=1,\ \ e_i(a)=a_i,\ \ i=1,\cdots\cdots,n,$$
$$a=(a_1,\cdots\cdots,\ a_n)\in Q,\ \text{ a neighborhood of the origin,}$$

then we denote by $V(\{e\})$ the subspace of $C_0(\boldsymbol{R}^n)$ spanned by $e_0, \cdots\cdots, e_n$. The subspace of $C_0(\mathbf{R}^n)$ generated by $V(\{e\})$ and the operations of $\Gamma_0{}^{\#}$ is denoted by $C_0{}^{\Gamma\#}\{e\}$. Then we set

$$C_0{}^{\Gamma\#}{}_{(e)}(U\times \boldsymbol{R}^n) = \{f \mid f \in C_0(U\times R^n),\quad f_y \in C_0{}^{\Gamma\#}{}_{(e)},\quad f = f(y, a),\quad y \in U, \}$$

$$C_*^{\Gamma}(U\times \mathbf{R}^n) = C_0{}^{\Gamma\#}{}_{(e)}(U\times \mathbf{R}^n)/C_0{}^{\Gamma\#}{}_{(e)}(U\times \mathbf{R}^n) \cup C_e(U \times \mathbf{R}^n),$$

(15) $C^{\Gamma}{}_{*,\,x} = lim.\ [C_*^{\Gamma}(U_\alpha(x) \times \mathbf{R}^n) \mid \bar{r}^{\alpha}_{\beta}]$.

We note that in this second formula, $C^{\Gamma}{}_*(U\times \mathbf{R}^n)$ does not depend on the choice of $e_0, \cdots\cdots, e_n$ if we fix the coordinate of $\mathbf{R}^n$. By the definitions of Γ_{*x} and $C^{\Gamma}{}_{*,\,x}$, we have

Lemma 7. *If $f_x \in \Gamma_{*x}$, then f_x induces an isomorphism of $C^{\Gamma}{}_{*,\,x}$, and this representation of Γ_{*x} as an operator group of $C^{\Gamma}{}_{*,\,x}$ is isomorphism.*

We can define $(U, C^{\Gamma}{}_*)$ similarly as $(U, C^*(n))$. Then by lemma 7, we can construct the associate vector bundle $v^{\Gamma}(\gamma)$ of a Γ–bundle γ by classifying $\cup_U(U, C^{\Gamma}{}_*)$.

8. *Associate vector bundles of $H^d{}_*(n)_c$ –bundles and jet bundles.* We set

$$E_0(n) = \{ f \mid f \text{ is a } C^r\text{–class diffeomorphism from } \mathbf{R}^n \text{ into } \mathbf{R}^n \text{ and } f(0)=0\},$$
$$1 \leq r \leq \infty.$$

We regard $E_0{}^r(n)$ to be a topological semigroup by C^r–topology. (cf. [1]). $E_0{}^r(n)$ is also denoted by $E_0{}^d(n)$ if we need not to clarify the order of differentiability. By the same way as $H_*(n)_x$ and the sheaf $H_*(n)_c$, we can define the group $H_*{}^r(n)_x$ and the sheaf $H_*{}^r(n)_c$. They are also denoted by $H_*{}^d(n)_x$ and $H_*{}^d(n)_c$. Similarly, we set

$$C_0{}^r(\mathbf{R}^n) = \{f \mid f \in C_0(\mathbf{R}^n),\ f \text{ is a } C^r \text{ –class function}\}.$$

We consider $C_0{}^r(\mathbf{R}^n)$ to be a topological vector space by C^r–topology. We also set

$$C_0{}^r(U \times \mathbf{R}^n) = f \mid f \in C_0(U\times \mathbf{R}^n),\quad f_y \in C_0{}^r(\mathbf{R}^n),\quad f = f(y, a),\quad y \in U, \},$$

$$C_*{}^r(U \times \mathbf{R}^n) = C_0{}^r(U \times \mathbf{R}^n)/C_0{}^r(U \times \mathbf{R}^n) \cap C_e(U\times \mathbf{R}^n),$$

$$C_*{}^r(n)_x = lim.\ [C_*{}^r(U_\alpha(x) \times \mathbf{R}^n) \mid \bar{r}^{\alpha}_{\beta}].$$

We also denote $C_*{}^r(n)_x$, etc. by $C_*{}^d(n)_x$, etc.. Then we can define $(U, C_*{}^d(n))$ similarly as $(U, C_*(n))$ and we can construct the associate vector bundle $v^d(\xi)$ (or $v^r(\xi)$ of an $H_*{}^d(n)_c$ –bundle ξ by classifying $\cap_U(U, C_*{}^d(n))$.

Definition. *If $s \leq r$, then we set*

$$C_0{}^{r,\,s}(\mathbf{R}^n) = \{f \mid f \in C_0{}^r(R^n),\ (D^p f)(0) = 0,\ |p| \leq s\},$$
$$p = (p_1, \cdots\cdots, p_n),\quad |p| = p_1 + \cdots\cdots + p_n,\quad D^0 f = f,$$

$$D^p f = \frac{\partial^{|p|} f}{\partial x_1^{p_1} \cdots\cdots \partial x_n^{p_n}},$$

$$C_0{}^r,{}_s(U \times \mathbf{R}^n) = \{f \mid f \in C_0(U \times \mathbf{R}^n),\ f_y \in C_0{}^r,{}_s(\mathbf{R}^n)\},$$

$$C^r{}_{s,*}(U \times \mathbf{R}^n) = C_0{}^r,{}_s(U \times \mathbf{R}^n)/C_0{}^r,{}_s(U \times \mathbf{R}^n) \cap C_c(U \times \mathbf{R}^n),$$

$$C^r{}_{s,*}(n)_x = \lim.\ [C^r{}_{s,*}(U^\alpha(x) \times \mathbf{R}^n) \mid \bar{r}^\alpha_\beta].$$

Note. In this definition, if $s = r = \infty$, then

$$C_0{}^\infty,{}_\infty(\mathbf{R}^n) = \{f \mid f \in C_0{}^\infty(\mathbf{R}^n),\ (D^p f)(0) = 0, \text{ for all } p\}.$$

Lemma 8. *If $\varphi_x \in H_*{}^r(n)_x$, then φ_x induces an isomorphism of $C^r{}_{s,*}(n)_x$ for any $s\,(\leq r)$.*

Since we can define $(U, C^r{}_{s,*}(n))$ similarly as $(U, C_*(n))$, we can construct an associate vector bundle of an $H_*{}^r(n)_c$ -bundle ξ by classifying $\cup_U (U, C^r{}_{s,*}(n))$ by lemma 8. This vector bundle is denoted by $v^r{}_s(\xi)$. By definition, $v^r{}_s(\xi)$ is a subbundle of $v^r(\xi)$.

As we can define the quotient bundle of $v^r(\xi)$ for any subbundle of $v^r(\xi)$, we get the following

Definition. *Setting*

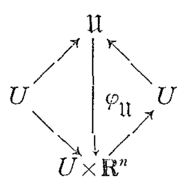
(16) $\quad j_s(\xi) = v^r(\xi)/v^r{}_s(\xi),$

we call $j_s(\xi)$ to be the associate s-jet bundle of ξ. Similarly, we set

(16)′ $\quad j^s{}_t(\xi) = v^r{}_s(\xi)/v^r{}_t(\xi), \quad 0 \leq s < t \leq r.$

The map from ξ to $j_s(\xi)$ (resp. to $j^s{}_t(\xi)$) is denoted by j_s. (resp. by $j^s{}_t$).

Note 1. If a Γ-bundle (or an $H_*{}^r(n)_c$-bundle) γ: $X \to E \to X$ is defined by the diagram



then $(\varphi_{\mathfrak{U}}{}^{-1*}\ \varphi_{\mathfrak{B}}{}^*)\,(x)$ induces an isomorphism of $C^{\Gamma *},{}_x$ (or $C^r{}_*(n)_x$). Therefore we can define the dual bundle $v_\Gamma{}^*(\gamma)$. (or $v^{r*}(\xi)$). Moreover, if ξ is an $H_*{}^r(n)_c$ -bundle, then $(\varphi_{\mathfrak{U}}{}^{-1*}\varphi_{\mathfrak{B}}{}^*)\,(x)$ induces a map from $C^r{}_{s,*}(n)_x$ onto $C^r{}_{s,*}(n)_x$. Hence we can define $v^r{}_s{}^*(\xi)$ and the associate s-jet bundle $j_s{}^*(\xi)$ by

$$j_s{}^*(\xi) = v^{r*}(\xi)/v^r{}_s{}^*(\xi).$$

Note 2. By definition, there is a canonical bundle map $p^s{}_t$: $j^r{}_s(\xi) \to j^r{}_t(\xi)$ if $s > t$ and we get

$$p^s{}_t p^t{}_u = p^s{}_u, \text{ if } s>t>u\geq 0.$$

Hence we can define the inverse limit of $\{j^t{}_s(\xi)\}$ if ξ is an $H_*{}^\infty(n)_c$ –bundle and we have (cf. [4]),

$$(17) \qquad lim. \ [j^t{}_s(\xi) \mid p^s{}_r] = j^t{}_\infty(\xi).$$

Definition. *The structure group of* $j_s(\xi)$ *(resp.* $j^s{}_t(\xi)$*)* $(s, t\neq\infty)$ *is denoted by* $G_s(n, \boldsymbol{R})$. *(resp.* $G^s{}_t(n, \boldsymbol{R})$*). (cf. [5], [6]). It is a linear group and there is a canonical (onto) homomorphism* J_s *(resp.* $J^s{}_t$*) from* $H_*{}^d(n)_x$ *to* $G_s(n, \boldsymbol{R})$. *(resp. to* $G^s{}_t(n, \boldsymbol{R})$*).*

9. *Associate jet bundles of the tangent microbundle.* If X is a smooth manifold, then its tangent microbundle τ is regarded to be an $H_*{}^d(n)_c$–bundle, ([3]), and we have

Theorem 4. $j^0{}_r(\tau)$ *is the (tangent) r–jet bundle of X tn the sense of Ehresmann, ([5]). Especially, we obtain*

$$(18) \quad j^0{}_1(\tau) = T(X).$$

Here $T(X)$ *is the tangent bundle of X.*

Definition. *We assume that X is a smooth manifold and* ξ *is a vector bundle over X, then a cross–section of* $\xi\otimes j^0{}_r(A^p(v_0{}^*(\tau)))$ *is called an r–jet p–form (over X) with coefficient in* ξ.

Example. A p–form with coefficient in ξ is a 1–jet p–form with coefficient in ξ. (The converse is also true).

This example and (11) shows that the notion of s–cross–section is a natural generalization of differential forms with coefficients in a vector bundle.

10. *Smooth* $H_*{}^d(n)$*–bundle and its* j_s*–image.* If X is a smooth manifold (of class r) then we can define the differentiability (up to r) of the functions on $U\times\mathbf{R}^n$, where U is an open set of X. The vector space of all C^r–class functions on $U\times\mathbf{R}^n$ is denoted by $S^r(U\times\mathbf{R}^n)$. We set

$$S_0{}^r(U\times\mathbf{R}^n) = S^r(U\times\mathbf{R}^n)\cap C_0{}^r(U\times\mathbf{R}^n).$$

Definition. $f \in H(U, E_0{}^r(n))$ *is called a* C^r*–class map if* f^* *maps* $S_0{}^r(\hat{f}(U\times\boldsymbol{R}^n))$ *onto* $S_0{}^r(U\times\boldsymbol{R}^n)$.

Definition. *The subsemigroup of* $H(U, E_0{}^r(n))$ *consisted by all* C^r*–class maps is denoted by* $H_r(U, E_0{}^r(n))$.

Starting from $H_r(U, E_0{}^r(n))$, we can construct a sheaf $H_*{}^r(n)_r$ by the same way as $H_*{}^r(n)_c$. We also denote this sheaf by $H_*{}^d(n)_d$. It is a subsheaf of $H_*{}^d(n)_c$. Therefore we can construct an $H_*{}^d(n)_d$–bundle.

Definition. *An* $H_*{}^r(n)_r$*–bundle is called a smooth (*C^r*–class)* $H_*{}^d(n)$*–bundle.*

Example. If X is a smooth manifold, then its tangent microbundle and cotangent microbundle are both $H_*{}^d(n)_d$–bundles.

If X is an $H_*{}^r(n)_r$–bundle, then we can define its C^r–class cross–section. Especially if X is a smooth manifold and ξ is a smooth vector bundle or an $H_*{}^r(n)_r$–bundle, then we can define the C^r–class cross–section of $\xi\otimes\overbrace{v_0{}^*(\tau)\otimes\cdots\otimes v_0{}^*(\tau)}^{s}$

or $\xi\otimes A^s\,({}_0{}^*(\tau))$. We set

(19) $C^s{}_r(X,\xi)=\{$the set of all C^r–class corss–sections of $\xi\otimes\overbrace{v_0{}^*(\tau)\otimes\cdots\otimes v_0{}^*(\tau)}^{s}\}$,

(19)′ $A^s{}_r(X,\xi)=\{$the set of all C^r–class cross–sections of $\xi\otimes A^s(v_0{}^*(\tau))\}$.

$C^s{}_r(X,\xi)$ and $A^s{}_r(X,\xi)$ are also denoted by $C^s{}_d(X,\xi)$ and $A^s{}_d(X,\xi)$. Since there is a projection from $C^s{}_d(X,\xi)$ to $A^s{}_d(X,\xi)$, if $\{\theta_U\}$ in the definition of D is a collection of C^r–class maps, (*e.g.* to construct $\{\theta_U\}$ by using C^r–class partition of unity), then we have the following sequence by (12),

(12)′ $A^0{}_d(X,\ \xi)\xrightarrow{D}A^1{}_d(X,\xi)\xrightarrow{D}\cdots\xrightarrow{D}A^s{}_d(X,\xi)\xrightarrow{D}A^{s+1}{}_d(X,\ \xi)\xrightarrow{D}\cdots$,

and since we know

$$j^0{}_r{}'d=dj^0{}_r{}'\ ,$$

we have the following commutative diagram.

$$\begin{array}{ccc} A^s{}_d(X,\xi) & \longrightarrow & A^{s+1}{}_d(X,\xi) \\ \downarrow j^0{}_r{}' & & \downarrow j^0{}_r{}' \\ H^0{}_d(X,\xi\otimes j^0{}_r(A^s(v_0{}^*(\tau)))) & \xrightarrow{D^0r} & H^0{}_d(X,\xi\otimes j^0{}_r(A^{s+1}(v_0{}^*(\tau))). \end{array}$$

Here $H^0{}_d(X,\eta)$ means the set of all smooth cross–sections of η, $j^0{}_r{}'$ is given by

$$j^0{}_r{}'=\text{identity}\otimes j^0{}_r\ ,$$

and $D^0{}_r$ is the map defined by

(20) $D^0{}_r=\{(D_U)^0{}_r\},\ (D_U)^0{}_r=j^0{}_r{}'d+J^0{}_r\theta_U$.

As we know

(21) $j^0{}_1d$ is the exterior differential,

we have by the commutativity of the above diagram,

Theorem 5. (*cf.* [*6*], [*8*]). *If ξ is a $GL(n,\boldsymbol{R})$-bundle, then we obtain*

(22) $D^0{}_1$ is the covariant derivation of ξ,

or in other word,

(22)′ $\{J^0{}_1\theta_U\}$ is a connection form of ξ in the usual sense.

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