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Remarks on Loop Spaces

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For any space X, $\Omega X \xrightarrow{i} EX \xrightarrow{p} X$ is a Hurewicz-fibring, which is a powerful tool in algebraic topology. In this note, we shall define a fibre–map q: $\Omega X \to X$ and give some of its properties.

Throughout this note, all spaces are assumed to have base points, which are denoted by $*$; all maps (homotopies) are assumed to preserve (keep fixed) base points.

Let I be the closed unit interval. For any apace X, let $F(I;X)$ denote the mapping–space of I into X (base points free) with the compact–open topology. Put

$$EX = \{\alpha \in F(I;\ X);\ \ \alpha(0) = *\}\ ,$$

$$\Omega X = \{\alpha \in EX;\ \ \alpha(1) = *\}\ .$$

Then, as is well known, $\Omega X \xrightarrow{i} EX \xrightarrow{p} X$, $p(\alpha) = \alpha(1)$, is a Hurewicz–fibring, i. e., for any space Y, p has the lifting homotopy property (cf. [1], [2] and [4]).

Proposition 1. *Define* $q: \Omega X \to X$ *by* $q(\alpha) = \alpha(1/2)$, *then* q *is a fibre map in the sense of Serre,* i. e., *for any polyhedron* P, q *has the lifting homotopy property.*

Proof. Let $P = |K|$, K being a finite simplicial complex, $F: P\times I \to X$, $g: P\times(0) \to \Omega X$ and $q\circ g(x, 0) = F(x, 0)$ for all $x \in P$.

Let K^n denote the n–skeleton of K, and $P^n = |K^n|$. We shall define G_n : $P^n \times I \to \Omega X$, $G_n|P^n\times(0) = g|P^n\times(0)$, $q\circ G_n = F|P^n\times I$ by the induction on n.

Let v be a vertex and $g(v, 0) = \alpha_v \in \Omega X$. Define $\bar{G}_v'$: $(0)\times I \cup \dot{I}\times I \cup I\times(1/2) \to X$ by

$$\bar{G}_v'((0, s) = \alpha_v(s) \qquad ;$$

$$\bar{G}_v'(t, \varepsilon) = * \ , \qquad \varepsilon = 0, 1;$$

$$\bar{G}_v'(t, 1/2) = F(v, t) \qquad .$$

Since $(0)\times I \cup I\times\dot{I}$ is a strong deformation retract of $I\times I$, we may extend

$\bar{G}'_v$ to a map $\bar{G}_v: I \times I \to X$. Defining $G_0 : v \times I \to \Omega X$ by $G_0(v, t)(s) = \bar{G}_v(t, s)$, we have the required map.

Now, assume that we have already defined G_i for all $i < n$. Let σ be an n-simplex of K. By the inductive assumption, there exists a map $G'_{\dot\sigma} : |\dot\sigma| \times I \to \Omega X$, satisfying $q \cdot G'_{\dot\sigma} = F \mid |\dot\sigma| \times I$. Putting $\bar{G}'_{\dot\sigma}(y, t, s) = G'_{\dot\sigma}(y, t)(s)$, we have a map $\bar{G}'_{\dot\sigma}: |\dot\sigma| \times I \times I \to X$ satisfying $\bar{G}'_{\dot\sigma}(y, 0, s) = g(y)(s)$ and $\bar{G}'_{\dot\sigma}(y, t, 1/2) = F(y, t)$. Extend $\bar{G}'_{\dot\sigma}$ to $\bar{G}'_\sigma: |\dot\sigma| \times I \times I \cup |\sigma| \times (0) \times I \cup |\sigma| \times I \times \{0, 1/2, 1\} \to X$ by $\bar{G}'_\sigma(y, 0, s) = g(y, 0)(s)$, $\bar{G}'_\sigma(y, t, \varepsilon) = *$, $\varepsilon = 0, 1$, and $\bar{G}'_\sigma(y, t, 1/2) = F(y, t)$.

Since $|\dot\sigma| \times I \times [0, 1/2] \cup |\sigma| \times (0) \times [0, 1/2] \cup |\sigma| \times I \times \{0, 1/2\}$ and $|\dot\sigma| \times I \times [1/2, 1] \cup |\sigma| \times (0) \times [1/2, 1] \cup |\sigma| \times I \times \{1/2, 1\}$ are strong deformation retracts of $|\sigma| \times I \times [0, 1/2]$ and $|\sigma| \times I \times [1/2, 1]$, respectively, we may extend $\bar{G}'_\sigma$ to a map $\bar{G}_\sigma: |\sigma| \times I \times I \to X$, such that $\bar{G}_\sigma(y, 0, s) = g(y, 0)(s)$, $\bar{G}_\sigma(y, t, 1/2) = F(y, t)$ for all $y \in |\sigma|$. Define $G_n : P^n \times I \to \Omega X$ by

$$G_n(y, t)(s) = \bar{G}_\sigma(y, t, s)$$

for all $(y, t) \in |\sigma| \times I$, then, by a usual discussion, we may conclude that G_n is a required lifting.

Corollary. *For any $s \in I$, defining $q_s : \Omega X \to X$ by $q_s(\alpha) = \alpha(s)$, we have a fibre map in the sense of Serre.*

Proposition 2. *Let F be the fibre of the fibre map q, i, e., $F = q^{-1}(*)$. Then, we have exact sequences*

$$0 \to \pi_{i+1}(X) \to \pi_i(F) \to \pi_{i+1}(X) \to 0$$

for all i.

Proof. By the above corollary, we have $q : \pi_i(\Omega X) \to \pi_i(X)$ are trivial[1] for all i. Considering the homotopy exact sequence

$$\to \pi_{i+1}(\Omega X) \xrightarrow{q_*} \pi_{i+1}(X) \to \pi_i(F) \to \pi_i(\Omega X) \to$$

of the fibre space $F \to \Omega X \to X$, we have short exact sequences

$$0 \to \pi_{i+1}(X) \to \pi_i(F) \to \pi_i(\Omega X) \to 0.$$

Since $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, we have the required ones.

The multiplication $\mu : \Omega X \times \Omega X \to \Omega X$ defined by

$$\mu(\alpha, \beta)(t) = \begin{cases} \alpha(2t), & 0 \leqq t \leqq 1/2, \\ \beta(2t-1), & 1/2 \leqq t \leqq 1, \end{cases}$$

makes X an H-space, which is homotopy-associative and has the homotopy-inversion ν defined by $\nu(\alpha)(t) = \alpha(1-t)$.

Since $\mu' = \mu \mid F \times F : F \times F \to F$, (F, μ') is an H-space, however, (F, μ') is not

1) Notice that q itself is not homotopic to $*$.

homotopy-associative and has not homotopy-inversion.[1]

If X is an H-space with a multiplication m, ΩX has the induced multiplication $\bar{m}$ defined by

$$\bar{m}(\alpha, \beta)(t) = m(\alpha(t), \beta(t)),$$

which is homotopic to μ, and $q(\bar{m}(\alpha, \beta)) = m(q(\alpha), q(\beta))$.
Therefore, $(F, \bar{m}| F \times F)$ is an H-space, moreover, if m is homotopy-associative or has homotopy-inversion, then $\bar{m}$ and $\bar{m}|F \times F$ has the same properties.

Proposition 3. *If X is an H-space with a multiplication m, $F \xrightarrow{i} \Omega X \xrightarrow{q} X$ is an H-fibration in the sense of [3].*

Proof. It sufficies to prove that there is a homotopy $H_s : F \vee E \to E$ satisfying the following conditions:

(i) $H_0 = \mu j$,

(ii) $H_1 = \nabla(i \vee 1)$,

(iii) $qH_s(F \vee F) = *$,

where $j: F \vee E \to F \times E$ is the inclusion map and $\nabla : E \vee E \to E$ is the folding map. Such a homotopy H is easily defined from the homotopies

$$m(*, x) \simeq x \simeq m(x, *).$$

References

[1] Fadell, E. On Fibre Spaces, Trans. Amer. Math. Soc. **90** (1959), 1-14.
[2] Hu, S. T. *Homotopy Theory*, Academic Press, 1959.
[3] Meyer, J.-P. Principal Fibrations, Trans. Amer. Math. Soc. **107** (1963), 177-185.
[4] Serre, J.-P. Homologie singulière des espaces fibrés, Ann. of Math. (2) **54** (1951), 425-505.

1) $\nu(F) \subset F$, but homotopies $\mu(\nu(\alpha), \alpha) \simeq * \simeq \mu(\alpha, \nu(\alpha))$ are not necessarily in F.