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State Vector Renormalization in Weak Processes

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Abstract

Transition Processes induced by a weak interaction are formulated by using DEWITT's method of state vector normalization. It is shown that the transition matrix elements take a form of the product of the scattering matrix element of the final interaction and the matrix element of the weak interaction Hamiltonian between two $|\psi^{(1)}>$-type state vectors. The well-known phases of the transition matrix elements, when the theory is invariant under time reversal, are easily obtained from this form of the S-matrix.

1 Introduction

LIPPMANN and SCHWINGER's notations[1] are used throughout the present article. Let H be the total Hamiltonian and H_0, H_1 be the free and interaction parts of H, respectively. And let the mass renormalization have already been made. Then DEWITT[2] has proved the following relations.

The Z-factors defined as

$$\hat{Z}_a^{\frac{1}{2}} |\psi^{\pm}{}_a> = \pm i \varepsilon (E_a - H \pm i \varepsilon)^{-1} | \phi_a> , \tag{1}$$

with $<\psi^{\pm}{}_a | \psi^{\pm}{}_a> = 1$, are given by

$$\hat{Z}_a^{\frac{1}{2}} = <\phi_a | \psi^{\pm}{}_a> , \tag{2}$$

and the stationary states $|\psi^{\pm}{}_a>$ have the orthogonality properties,

$$<\psi^{\pm}{}_b | \psi^{\pm}{}_a> = \delta_{ba} ,$$

where $|\phi_a>, |\phi_b>, \ldots$ are the eigen vectors of the operator H_0 belonging to the eigen values E_a, E_b, ...

Now let

$$R^{\pm}(E) = H_1 \{1 + (E - H \pm i \varepsilon)^{-1} H_1\}, \tag{3}$$

and

$$R^{\pm}(E)_{ba} = <\phi_b \,|\, R^{\pm}(E) \,|\, \phi_a > .$$

Then $R^{\pm}(E)_{ba}$ can be separated into two parts, the singular part $\mathrm{Sing}R^{\pm}(E)_{ba}$ which is independent of the normalization volume V and the remainder $\mathrm{Rem}R^{\pm}(E)_{ba}$.

The singular parts are connected directly with the Z-factors such that

$$\mathrm{Sing}\ R^{\pm}(E_a)_{ba} = \pm\, i\varepsilon(\hat{Z}_a - 1)\,\delta_{ba}, \tag{4}$$

$$\mathrm{Sing} < \phi_b \,|\, \psi^{\pm}{}_a > = \hat{Z}_a^{\frac{1}{2}}\,\delta_{ba}\,.$$

From these formulae the scattering and reactance matrices become

$$S_{ba} = \delta_{ba} - 2\pi i\ \delta(E_b - E_a)\ (\hat{Z}_a\hat{Z}_b)^{-\frac{1}{2}}\ \mathrm{Rem}\ R^{+}(E_a)_{ba}, \tag{5}$$

and

$$\boldsymbol{K}_{ba} = 2\pi\,\delta(\,E_b - E_a)\,\hat{Z}_a^{\frac{1}{2}}\ \hat{Z}_b^{-\frac{1}{2}}\,K_{ba}\ , \tag{6}$$

where

$$K_{ba} = < \phi_b \,|\, (1 - H_1\,\mathrm{P}\frac{1}{E_a - H_0})^{-1}\ H_1 \,|\, \phi_a > .$$

Here the symbol P denotes Cauchy's principal value when appearing in an integral.

2 The Renormalization Constant

We now extend the above formalism to the case in which the interaction Hamiltonian H_1 is of the form

$$H_1 = H_s + H_w,$$

where H_s is the strong interaction Hamiltonian (including electromagnetic interactions) which is supposed to be renormalizable and H_w is the effective weak interaction Hamiltonian. If we neglect the second and higher order terms with respect to H_w, the propagators can be written as

$$1 + (E - H \pm i\varepsilon)^{-1}\,H_1 = F^{(\pm)}(E) + F^{(\pm)}(E)\,(E - H_0 \pm i\varepsilon)^{-1}\,H_w F^{(\pm)}(E)\,, \tag{7}$$

where $F^{(\pm)}(E)$ are defined as

$$\begin{aligned} F^{(\pm)}(E) &= 1 + (E - H_0 \pm i\varepsilon)^{-1}\,H_s + (E - H_0 \pm i\varepsilon)^{-1} H_s\,(E - H_0 \pm i\varepsilon)^{-1} H_s + \cdots \\ &= 1 + (E - H_0 - H_s \pm i\varepsilon)^{-1} H_s. \end{aligned} \tag{8}$$

They are propagators when H_w is switched off, and satisfy the relations

$$F^{(\pm)}(E)\,(E - H_0 \pm i\varepsilon)^{-1} = \{F^{(\mp)}(E)\,(E - H_0 \mp i\varepsilon)^{-1}\}^*, \tag{9}$$

where * means Hermitian conjugation.

On the analogy of eq. (1), the Z-factors and the stationary states, which are obtained when H_w is switched off, are defined as

$$Z_a^{\frac{1}{2}} |\psi^{(\pm)}{}_a\rangle = F^{(\pm)}(E_a) |\phi_a\rangle , \qquad \langle\psi_a{}^{(\pm)} | \psi_a{}^{(\pm)}\rangle = 1. \tag{10}$$

Then we find from eqs. (7), (9) and (2) that

$$\hat{Z}_a = Z_a + Z_a \frac{\langle \psi_a{}^{(\mp)} | H_w | \psi_a{}^{(\pm)}\rangle}{\pm i \varepsilon} .$$

Since $\langle \psi_a{}^{(-)} | H_w | \psi_a{}^{(+)} \rangle$ has no singular part, we see

$$\hat{Z}_a = Z_a . \tag{11}$$

The Z-factors are not affected by the presence of the weak interaction.

To obtain the standing wave solutions, the propagator $F^{(1)}(E)$ is needed, which is defined as

$$F^{(1)}(E) = 1 + \mathrm{P}\frac{1}{E-H_0}H_s + \mathrm{P}\frac{1}{E-H_0}H_s\mathrm{P}\frac{1}{E-H_0}H_s + \cdots$$

$$= (1 - \mathrm{P}\frac{1}{E-H_0}H_s)^{-1}. \tag{12}$$

Corresponding to eq. (10), the standing waves are given by

$$|\psi_a{}^{(1)}\rangle = F^{(1)}(E_a) |\phi_a\rangle . \tag{13}$$

Since

$$\langle\phi_a | \mathrm{P}\frac{1}{E_a - H_0} = 0 ,$$

the solution $|\psi_a{}^{(1)}\rangle$ conserves its norm, i.e.

$$\langle \phi_a | \psi_a{}^{(1)}\rangle = 1.$$

But it is to be noticed that the energy shell limit of $F^{(1)}(E)_{aa}$ is not 1, but Z_a. Therefore we have

$$\lim_{E \to E_a} F^{(1)}(E) | \phi_a\rangle = Z_a |\psi_a{}^{(1)}\rangle . \tag{14}$$

3 The Scattering Matrix

To calculate the S-matrix elements, we substitute eqs. (7) and (9) into eq. (3), and find

$$R^{+}(E) = H_s F^{(+)}(E) + F^{(-)*}(E) H_w F^{(+)}(E). \tag{15}$$

Because the term which contains H_w in eq. (15) has no singular part, Rem $R^{+}(E_a)$ can be written as

$$\text{Rem } R^{+}(E_a) = \text{Rem } H_s F^{(+)}(E_a) + F^{(-)*}(E_a) H_w F^{(+)}(E_a). \tag{16}$$

Substituting eq. (16) into eq. (5), we obtain

$$S_{ba} = S^{(s)}{}_{ba} - 2\pi i\, \delta(E_b - E_a) < \psi_b{}^{(-)} | H_w | \psi_a{}^{(+)} > , \tag{17}$$

by virtue of eqs. (10) and (11). Here $S^{(s)}$ is the S–matrix when H_w is absent.

From eq. (8) we can show

$$F^{(\pm)}(E) = (1 - \text{P}\frac{1}{E-H_0} H_s)^{-1} \{1 \mp i\pi\, \delta(E - H_0) H_s F^{(\pm)}(E)\}, \tag{18}$$

hence $| \psi_a{}^{(+)} >$ can be written as

$$| \psi_a{}^{(+)} > = \sum_c Z_a{}^{-\frac{1}{2}} | \psi^{(1)}{}_c > \{\delta_{ca} - i\pi\, \delta(E_c - E_a) (H_s F^{(+)}(E_a))_{ca}\}. \tag{19}$$

On the other hand, when the weak interaction is switched off, eq. (4) shows

$$(H_s F^{(+)}(E_a))_{ca} = + i\varepsilon (Z_a - 1)\, \delta_{ca} + \text{Rem } (H_s F^{(+)}(E_a))_{ca}.$$

Thus we find

$$| \psi_a{}^{(+)} > = \frac{1}{2} \sum_c Z_c{}^{\frac{1}{2}} (1 + S^{(s)})_{ca} | \psi_c{}^{(1)} > . \tag{20}$$

In the same way, we get also

$$< \psi_b{}^{(-)} | = \frac{1}{2} \sum_c Z_c{}^{\frac{1}{2}} (1 + S^{(s)})_{bc} < \psi_c{}^{(1)} | . \tag{21}$$

If we substitute these results into eq. (17), the S–matrix elements can be written as

$$S_{ba} = S^{(s)}{}_{ba} - \frac{i\pi}{2} \delta(E_b - E_a) \sum_{c,d} (Z_c Z_d)^{\frac{1}{2}} (1 + S^{(s)})_{bc} \cdot$$
$$< \psi_c{}^{(1)} | H_w | \psi_d{}^{(1)} > (1 + S^{(s)})_{da}. \tag{22}$$

The same discussion can be made in terms of the K–matrix, too. In this case, however, we must use the equation

$$(1 - H_1 \text{P}\frac{1}{E-H_0})^{-1} H_1 = H_s F^{(1)}(E) + F^{(1)*}(E) H_w F^{(1)}(E),$$

instead of eq. (7). As in the previous case, we find

$$K_{ba} = K^{(s)}{}_{ba} + < \phi_b | F^{(1)*}(E_a) H_w F^{(1)}(E_a) | \phi_a >$$
$$= K^{(s)}{}_{ba} + < \phi_b | F^{(1)*}(E_a) H_w | \psi_a{}^{(1)} > . \tag{23}$$

The energy shell operation on K results in

$$\boldsymbol{K}_{ba} = 2\pi\ \delta(E_b - E_a)\ Z_a^{\frac{1}{2}} Z_b^{-\frac{1}{2}} K_{ba}$$
$$= \boldsymbol{K}^{(s)}{}_{ba} + 2\pi\ \delta(E_b - E_a)\,(Z_a\, Z_b)^{\frac{1}{2}} < \psi_b{}^{(1)}\,|H_w\,|\,\psi_a{}^{(1)} >, \tag{24}$$

where we have used eq. (14). After the Cayley transformation of $\boldsymbol{K}$, again we find eq. (22).

If we use the representation in which $S^{(s)}$ is diagonal, i. e.

$$S^{(s)} = e^{2i\delta},$$

then eq. (22) can be rewritten as

$$S_{ba} = S_{ba}{}^{(s)} - 2\pi i\ \delta(E_b - E_a)\ (Z_a\, Z_b)^{\frac{1}{2}} \cdot$$
$$\cos\delta_b\ \cos\delta_a\ e^{i(\delta_b+\delta_a)} < \psi_b{}^{(1)}\,|\,H_w\,|\,\psi_a{}^{(1)} >. \tag{25}$$

In ordinary weak processes the initial state is the one-particle state, hence $\delta_a = 0$. And if the theory is invariant under time reversal, $< \psi_b{}^{(1)}\,|\,H_w\,|\,\psi_a{}^{(1)} >$ is taken to be real.[3] Therefore, the phases of the transition matrix elements are completely determined by the phase shifts due to the final interaction.

References

1) Lippmann, B. A. and Schwinger, J., *Phys. Rev.* **79**, 469 (1950).
2) DeWitt, B. S., *Phys. Rev.* **100**, 905 (1955).
3) Coester, F., *Phys. Rev.* **89**, 619 (1953).