Maximal rank of extremal marginal tracial states

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States on the coupled quantum system $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ whose restrictions to each subsystem are the normalized traces are called marginal tracial states. We investigate extremal marginal tracial states and compute their maximal rank. Diagonal marginal tracial states are also considered. © *2010 American Institute of Physics.* [doi:10.1063/1.3481567]

I. INTRODUCTION

States on coupled quantum systems have recently studied from many points of view. In this paper, we focus on states on the coupled quantum system $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ whose restrictions to each subsystem are the normalized traces. Such states are called marginal tracial states. In Ref. 2, Arveson showed a marginal tracial state of rank r is almost surely entangled when $r \leq n/2$.

In Ref. 5, Parthasarathy showed that every extremal marginal tracial state on $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is a pure state. After that, in Ref. 6, Price and Sakai gave necessary and sufficient conditions for a state to be extremal among marginal tracial states and conjectured that every such a state is pure if $n \geq 3$. Actually, in classical probability theory, the Birkhoff's theorem says that every extremal point of the set of doubly stochastic matrices of order n is a permutation matrix, so that the only extremal points are the trivial ones. In this paper, by using the one-to-one correspondence between marginal tracial states on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ and unital completely positive trace preserving (UCPT) maps on $M_n(\mathbb{C})$ (see, e.g., Refs. 1, 7, and 8), we give a negative answer to the conjecture with the help of the result of Landau and Streater (Ref. 4, Theorem 1) which shows the existence of a nonunitary extremal UCPT map on $M_n(\mathbb{C})$ for $n \geq 3$. To the author's knowledge, this result, first known by Arveson and told to the author by Price in private, has never been published, and so we will give it for the completeness of the paper. Moreover, we compute the maximal rank of extremal marginal tracial states for some special cases and consider diagonal marginal tracial states which correspond to diagonal UCPT maps introduced in Ref. 4.

In Sec. II, we recall the relation between marginal tracial states and UCPT maps and construct a nonpure extremal marginal tracial state. Furthermore, we compute the maximal rank of extremal marginal tracial states on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ for $n=3,4$ and give a lower bound on the maximal rank of extremal marginal tracial states for $n \geq 5$. In Sec. III, we investigate the maximal rank of extremal diagonal marginal tracial states and show that the set of such states of rank a with $a^2 \leq n$ is dense in the set of all diagonal marginal tracial states of rank a or less.

II. MARGINAL TRACIAL STATES AND UCPT MAPS

In this section we recall the relation between marginal tracial states and UCPT (unital completely positive trace preserving) maps and compute the maximal rank of extremal marginal tracial states.

Definition 2.1: A state ρ on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is a marginal tracial state if the restrictions of ρ to $M_n(\mathbb{C}) \otimes I$ and $I \otimes M_n(\mathbb{C})$ are the normalized traces. Denote by $\Gamma(n)$ the set of all marginal tracial states on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$.

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Extremal marginal tracial states are considered in Refs. 5 and 6, and they are all pure states, if $n=2$.[5] We will show that this is not the case when $n\geq 3$ by using the one-to-one correspondence between marginal tracial states and UCPT maps.

Denote by $UCPT(n)$ the set of all UCPT maps on $M_n(\mathbb{C})$. Every map $\varphi\in UCPT(n)$ can be written as

$$\varphi(A)=\sum_{i=1}^{k} v_i^* A v_i$$

for some $\{v_i\}\subset M_n(\mathbb{C})$ with $\Sigma_{i=1}^k v_i^* v_i=\Sigma_{i=1}^k v_i v_i^*=I$. The choice of the matrices $\{v_i\}$ is not unique, but they can be taken linearly independent. Thus, the number k of the terms is uniquely determined (see Ref. 3) and is denoted by $r(\varphi)=k$.

For any $\varphi\in UCPT(n)$, the map $\varphi\otimes \mathrm{id}_{M_n(\mathbb{C})}$ is a unital positive map, so that the composition of $\varphi\otimes \mathrm{id}_{M_n(\mathbb{C})}$ with any state on $M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ is also a state. Let, in particular, choose the pure marginal tracial state given by the vector $\xi=\frac{1}{\sqrt{n}}\Sigma_{i=1}^n e_i\otimes e_i\in\mathbb{C}^n\otimes\mathbb{C}^n$ with a fixed orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$. Then the composition is a marginal tracial state. Indeed, it easily follows from the equation

$$\mathrm{tr}(e_{ij}{}^t e_{kl})=\langle (e_{ij}\otimes e_{kl})\xi,\xi\rangle=\frac{1}{n}\delta_{ik}\delta_{jl},$$

that $\langle(\varphi(A)\otimes B)\xi,\xi\rangle=\mathrm{tr}(\varphi(A)^t B)$ for any $A,B\in M_n(\mathbb{C})$, where $\{e_{ij}\}$ is the set of matrix units of $M_n(\mathbb{C})$, tr is the normalized trace, and ${}^tB$ is the transpose of B. Therefore, we have

$$\langle(\varphi(A)\otimes I)\xi,\xi\rangle=\mathrm{tr}(\varphi(A)^t I)=\mathrm{tr}(\varphi(A))=\mathrm{tr}(A),$$

$$\langle(\varphi(I)\otimes B)\xi,\xi\rangle=\mathrm{tr}(\varphi(I)^t B)=\mathrm{tr}({}^tB)=\mathrm{tr}(B).$$

We define the map π from $UCPT(n)$ to $\Gamma(n)$ by

$$\pi(\varphi)(A\otimes B)=\langle(\varphi(A)\otimes B)\xi,\xi\rangle.$$

The map π is known to be bijective (see Refs. 1, 7, and 8), but we give its proof for the completeness of the paper. In what follows, we identify a state ρ with its density matrix D_ρ and write $\mathrm{rank}(\rho):=\mathrm{rank}(D_\rho)$.

Theorem 2.2: *The map π from $UCPT(n)$ to $\Gamma(n)$ defined above is bijective and preserves the convex structure. In particular, $\pi(\varphi)$ is an extremal point of $\Gamma(n)$ if and only if φ is an extremal point of $UCPT(n)$. Moreover,* $\mathrm{rank}(\pi(\varphi))$ *is equal to $r(\varphi)$.*

Proof: It is easy to see that π is injective, and so we show that π is surjective. Take a marginal tracial state ρ and consider the decomposition of the corresponding density matrix D_ρ into rank one positive operators which has the form $D_\rho=\Sigma_{i=1}^k|\zeta_i\rangle\langle\zeta_i|$ for some $\zeta_i\in\mathbb{C}^{n^2}=\mathbb{C}^n\otimes\mathbb{C}^n$. We write

$$\zeta_i=\sum_{j=1}^{n}\zeta_{ij}\otimes e_j\in\mathbb{C}^n\otimes\mathbb{C}^n$$

and define $v_i\in M_n(\mathbb{C})$ by $v_i(e_j)=\sqrt{n}\zeta_{ij}$ and put $\varphi=\Sigma_{i=1}^k v_i^*\cdot v_i$. Since φ is completely positive and the domain of π can be extended to the set of all completely positive maps, one can define a positive linear functional $\pi(\varphi)$ on $M_n(\mathbb{C})\otimes M_n(\mathbb{C})$. Then we have

$$D_{\pi(\varphi)} = \sum_{i=1}^{k} |(v_i \otimes I)\xi\rangle\langle(v_i \otimes I)\xi| = \sum_{i=1}^{k} \left| \sum_{j=1}^{n} (\zeta_{ij} \otimes e_j) \right\rangle \left\langle \sum_{j=1}^{n} (\zeta_{ij} \otimes e_j) \right|$$
$$= \sum_{i=1}^{k} |\zeta_i\rangle\langle\zeta_i| = D_\rho,$$

and hence $\pi(\varphi)=\rho$. Moreover, we have

$$\mathrm{tr}(\varphi(A)) = \mathrm{tr}(\varphi(A)^t I) = \rho(A \otimes I) = \mathrm{tr}(A),$$

$$\mathrm{tr}(\varphi(I)^t B) = \rho(I \otimes B) = \mathrm{tr}(B) = \mathrm{tr}({}^t B)$$

for all $A,B \in M_n(\mathbb{C})$, which imply that φ is a UCPT map, and hence π is surjective.

It is obvious that π preserves the convex structure.

Since $\{v_i\}$ is a linearly independent set, so is $\{(v_i \otimes I)\xi\}$. Thus, the density matrix,

$$D_{\pi(\varphi)} = \sum_{i=1}^{k} |(v_i \otimes I)\xi\rangle\langle(v_i \otimes I)\xi|,$$

has rank k, which implies $\mathrm{rank}(\pi(\varphi))=r(\varphi)$. ■

By Theorem 1 in Ref. 4, if $n \geq 3$, there exists an extremal point φ of $UCPT(n)$ with $r(\varphi) \geq 2$ and hence φ is a nonunitary map. Therefore, by Theorem 2.2 there exists an extremal marginal tracial state in $\Gamma(n)$ which is not pure.

Next we compute the maximal rank, denoted by $MR(n)$, of extremal marginal tracial states in $\Gamma(n)$. To this end, we collect some results on the extremal points of $UCPT(n)$ and $\Gamma(n)$.

Definition 2.3: The pairs $(u_1,v_1),\ldots(u_k,v_k)$ of elements of a linear space are said to be bi-independent if

$$\sum_{j=1}^{k} c_j u_j = 0 \quad \text{and} \quad \sum_{j=1}^{k} c_j v_j = 0$$

imply $c_j=0$ for all $1 \leq j \leq k$. The k pairs of elements are denoted by $u_1,\ldots,u_k;v_1,\ldots,v_k$ (the relative order of the u 's and v 's being fixed).

Theorem 2.4: *(Reference 4) Let $\varphi=\sum_{i=1}^{k} v_i^* \cdot v_i$, where $\{v_i\}$ is a linearly independent set with $\sum_{i=1}^{k} v_i^* v_i = \sum_{i=1}^{k} v_i v_i^* = I$. Then φ is an extremal point of $UCPT(n)$ if and only if $\{v_i v_j^*\}_{i,j=1}^{k};\{v_j^* v_i\}_{i,j=1}^{k}$ is a bi-independent set.*

Theorem 2.5: *(Reference 6) Let ρ be a marginal tracial state on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ and let P_ρ be the support projection of ρ . Then ρ is an extremal point of $\Gamma(n)$ if and only if $(P_\rho(M_n(\mathbb{C}) \otimes M_n(\mathbb{C}))P_\rho) \cap ((M_n(\mathbb{C}) \ominus \mathbb{C}I) \otimes (M_n(\mathbb{C}) \ominus \mathbb{C}I)) = \{0\}$.*

Since the dimensions of $P_\rho(M_n(\mathbb{C}) \otimes M_n(\mathbb{C}))P_\rho$, $(M_n(\mathbb{C}) \ominus \mathbb{C}I) \otimes (M_n(\mathbb{C}) \ominus \mathbb{C}I)$ and $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ are $\mathrm{rank}(\rho)^2$, $(n^2-1)^2$ and n^4, respectively, it follows from Theorem 2.5 that an upper bound on $MR(n)$ is given by

$$MR(n) \leq \sqrt{2n^2 - 1}. \tag{1}$$

Every extremal point of $\Gamma(2)$ is pure, so that $MR(2)=1$. In the following, we compute $MR(3)$ and $MR(4)$.

Theorem 2.6: *The maximal rank of extremal marginal tracial states in* $\Gamma(3)$ *is* 4.

Proof: From (1), we have $MR(3) \leq 4$. Hence all we need is to construct a map φ which is extremal in $UCPT(3)$ and has $r(\varphi)=4$. Let

$$w_1 = e_{11}, \quad w_2 = e_{12} + \sqrt{2}e_{23}, \quad w_3 = \sqrt{2}e_{21} + \sqrt{3}e_{32}, \quad w_4 = e_{31} + \sqrt{2}e_{13}.$$

Then we have

$$\sum_{i=1}^{4} w_i w_i^* = \sum_{i=1}^{4} w_i^* w_i = 4I,$$

and hence $v_i = \frac{1}{2} w_i$ satisfies $\Sigma_{i=1}^4 v_i v_i^* = \Sigma_{i=1}^4 v_i^* v_i = I$. By using the equations

$$\begin{array}{llll}
w_1 w_1^* = e_{11} & w_2 w_1^* = 0 & w_3 w_1^* = \sqrt{2} e_{21} & w_4 w_1^* = e_{31} \\
w_1 w_2^* = 0 & w_2 w_2^* = e_{11} + 2e_{22} & w_3 w_2^* = \sqrt{3} e_{31} & w_4 w_2^* = 2e_{12} \\
w_1 w_3^* = \sqrt{2} e_{12} & w_2 w_3^* = \sqrt{3} e_{13} & w_3 w_3^* = 2e_{22} + 3e_{33} & w_4 w_3^* = \sqrt{2} e_{32} \\
w_1 w_4^* = e_{13} & w_2 w_4^* = 2e_{21} & w_3 w_4^* = \sqrt{2} e_{23} & w_4 w_4^* = e_{33} + 2e_{11}
\end{array}$$

and

$$\begin{array}{llll}
w_1^* w_1 = e_{11} & w_2^* w_1 = e_{21} & w_3^* w_1 = 0 & w_4^* w_1 = \sqrt{2} e_{31} \\
w_1^* w_2 = e_{12} & w_2^* w_2 = e_{22} + 2e_{33} & w_3^* w_2 = 2e_{13} & w_4^* w_2 = \sqrt{2} e_{32} \\
w_1^* w_3 = 0 & w_2^* w_3 = 2e_{31} & w_3^* w_3 = 2e_{11} + 3e_{22} & w_4^* w_3 = \sqrt{3} e_{12} \\
w_1^* w_4 = \sqrt{2} e_{13} & w_2^* w_4 = \sqrt{2} e_{23} & w_3^* w_4 = \sqrt{3} e_{21} & w_4^* w_4 = e_{11} + 2e_{33},
\end{array}$$

a simple calculation yields that $\Sigma_{i,j=1}^4 a_{ij} w_i w_j^* = 0$ and $\Sigma_{i,j=1}^4 a_{ij} w_j^* w_i = 0$ imply $a_{ij} = 0$ for all $1 \le i, j \le 4$. Therefore, $\{w_i w_j^*\}; \{w_j^* w_i\}$ is a bi-independent set, and so is $\{v_i v_j^*\}; \{v_j^* v_i\}$. Put $\varphi = \Sigma_{i=1}^4 v_i^* \cdot v_i$. Then φ is an extremal point of $UCPT(3)$ with $r(\varphi) = 4$ by Theorem 2.4, and $MR(3) = 4$ follows.■

Theorem 2.7: *The maximal rank of extremal marginal tracial states in* $\Gamma(4)$ *is* 5.

Proof: From (1), we have $MR(4) \le 5$. Hence all we need is to construct a map φ which is extremal in $UCPT(4)$ and has $r(\varphi) = 5$. Let

$$w_1 = e_{13} + e_{32}, \quad w_2 = \sqrt{2} e_{24} + \sqrt{2} e_{43}, \quad w_3 = \sqrt{2} e_{14} + \sqrt{3} e_{31},$$

$$w_4 = e_{21} + \sqrt{2} e_{42}, \quad w_5 = e_{12} + e_{23}.$$

Then it holds that

$$\sum_{i=1}^{5} w_i w_i^* = \sum_{i=1}^{5} w_i^* w_i = 4I,$$

and hence $v_i = \frac{1}{2} w_i$ satisfy $\Sigma_{i=1}^5 v_i v_i^* = \Sigma_{i=1}^5 v_i^* v_i = I$. By the equations

$$\begin{array}{lll}
w_1 w_1^* = e_{11} + e_{33} & w_2 w_1^* = \sqrt{2} e_{41} & w_3 w_1^* = 0 \\
w_1 w_2^* = \sqrt{2} e_{14} & w_2 w_2^* = 2e_{22} + 2e_{44} & w_3 w_2^* = 2e_{12} \\
w_1 w_3^* = 0 & w_2 w_3^* = 2e_{21} & w_3 w_3^* = 2e_{11} + 3e_{33} \\
w_1 w_4^* = \sqrt{2} e_{34} & w_2 w_4^* = 0 & w_3 w_4^* = \sqrt{3} e_{32} \\
w_1 w_5^* = e_{31} + e_{12} & w_2 w_5^* = \sqrt{2} e_{42} & w_3 w_5^* = 0 \\
w_4 w_1^* = \sqrt{2} e_{43} & w_5 w_1^* = e_{13} + e_{21} & \\
w_4 w_2^* = 0 & w_5 w_2^* = \sqrt{2} e_{24} & \\
w_4 w_3^* = \sqrt{3} e_{23} & w_5 w_3^* = 0 & \\
w_4 w_4^* = e_{22} + 2e_{44} & w_5 w_4^* = \sqrt{2} e_{14} & \\
w_4 w_5^* = \sqrt{2} e_{41} & w_5 w_5^* = e_{11} + e_{22} &
\end{array}$$

and

$$
\begin{array}{lll}
w_1^* w_1 = e_{33} + e_{22} & w_2^* w_1 = 0 & w_3^* w_1 = \sqrt{2} e_{43} + \sqrt{3} e_{12} \\
w_1^* w_2 = 0 & w_2^* w_2 = 2e_{44} + 2e_{33} & w_3^* w_2 = 0 \\
w_1^* w_3 = \sqrt{2} e_{34} + \sqrt{3} e_{21} & w_2^* w_3 = 0 & w_3^* w_3 = 2e_{44} + 3e_{11} \\
w_1^* w_4 = 0 & w_2^* w_4 = \sqrt{2} e_{41} + 2e_{32} & w_3^* w_4 = 0 \\
w_1^* w_5 = e_{32} & w_2^* w_5 = \sqrt{2} e_{43} & w_3^* w_5 = \sqrt{2} e_{42} \\
w_4^* w_1 = 0 & w_5^* w_1 = e_{23} & \\
w_4^* w_2 = \sqrt{2} e_{14} + 2e_{23} & w_5^* w_2 = \sqrt{2} e_{34} & \\
w_4^* w_3 = 0 & w_5^* w_3 = \sqrt{2} e_{24} & \\
w_4^* w_4 = e_{11} + 2e_{22} & w_5^* w_4 = e_{31} & \\
w_4^* w_5 = e_{13} & w_5^* w_5 = e_{22} + e_{33}, &
\end{array}
$$

a simple calculation yields that $\Sigma_{i,j=1}^5 a_{ij} w_i w_j^* = 0$ and $\Sigma_{i,j=1}^5 a_{ij} w_j^* w_i = 0$ imply $a_{ij} = 0$ for all $1 \leq i, j \leq 5$. Therefore, $\{w_i w_j^*\};\{w_j^* w_i\}$ is a bi-independent set, and so is $\{v_i v_j^*\};\{v_j^* v_i\}$. Put $\varphi = \Sigma_{i=1}^5 v_i^* \cdot v_i$. Then φ is an extremal point of $UCPT(4)$ with $r(\varphi) = 5$ by Theorem 2.4, and $MR(4) = 5$ follows. ■

As to a lower bound on $MR(n)$ for $n > 2$, we have the following estimate.

Theorem 2.8: *The maximal rank of extremal marginal tracial states in $\Gamma(n)$ is at least n for $n > 2$.*

Proof: Fix $n > 2$ and we construct a map φ which is extremal in $UCPT(n)$ and has $r(\varphi) = n$. Let

$$
v_1 = \sqrt{\frac{n-2}{n-1}} \sum_{j=2}^{n} e_{jj},
$$

$$
v_i = \frac{1}{\sqrt{n-1}} (e_{1i} + e_{i1})
$$

for $2 \leq i \leq n$. Then we have

$$
v_1^* v_1 = \frac{n-2}{n-1} \sum_{j=2}^{n} e_{jj},
$$

$$
v_1^* v_j = \frac{\sqrt{n-2}}{n-1} e_{j1} \quad \text{for} \quad j \geq 2,
$$

$$
v_j^* v_1 = \frac{\sqrt{n-2}}{n-1} e_{1j} \quad \text{for} \quad j \geq 2,
$$

$$
v_j^* v_j = \frac{1}{n-1} (e_{11} + e_{jj}) \quad \text{for} \quad j \geq 2,
$$

$$
v_j^* v_k = \frac{1}{n-1} e_{jk} \quad \text{for} \quad j,k \geq 2 \quad \text{and} \quad j \neq k,
$$

and hence $\Sigma_{i=1}^n v_i v_i^* = \Sigma_{i=1}^n v_i^* v_i = I$. It follows from a simple calculation that $\{v_j^* v_i\}$ is a linearly independent set. Put $\varphi = \Sigma_{i=1}^n v_i^* \cdot v_i$. Then φ is an extremal point of $UCPT(n)$ with $r(\varphi) = n$ by Theorem 2.4, and $MR(n) \geq n$ follows. ■

III. DIAGONAL UCPT MAPS

In this section, we give some properties of diagonal UCPT maps.

Definition 3.1: (Reference 4) A linear map φ on $M_n(\mathbb{C})$ is diagonal if it has the form

$$\varphi(A) = C \circ A$$

for some $C \in M_n(\mathbb{C})$, where $C \circ A$ is the Schur product of C and A.

Completely positive diagonal maps are characterized by the next proposition.

Proposition 3.2: (Reference 4) φ is a completely positive diagonal map if and only if, in any representation,

$$\varphi = \sum_{i=1}^{k} v_i^* \cdot v_i,$$

the matrices v_i are diagonal.

In Ref. 4, it is shown that the maximal rank of extremal diagonal UCPT maps on $M_n(\mathbb{C})$ is at most $\sqrt{n}$. The next theorem shows that for any $a \in \mathbb{N}$ with $a^2 \le n$, we can construct an extremal diagonal UCPT map φ with $r(\varphi) = a$. A marginal tracial state is called a diagonal marginal tracial state if the corresponding UCPT map is diagonal.

Theorem 3.3: *Let $a \in \mathbb{N}$ be such that $a^2 \le n$. Then there are diagonal matrices $v_1, \ldots, v_a \in M_n(\mathbb{C})$, such that the map*

$$\varphi(A) = \sum_{i=1}^{a} v_i^* A v_i, \quad A \in M_n(\mathbb{C})$$

is an extremal UCPT map. Thus, φ corresponds to an extremal diagonal marginal tracial state on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ of rank a.

Proof: If $a=1$, let v_1 be any diagonal unitary matrix in $M_n(\mathbb{C})$. So we suppose $a \ge 2$. Due to space limitation, in the following we write our diagonal matrices as row vectors with n entries.

We first prove the theorem in the case that $a^2 = n$. Choose $\theta_1, \ldots, \theta_a \in [0, 2\pi]$, such that the elements of the set $\{\theta_i - \theta_j : 1 \le i \ne j \le a\}$ are all distinct mod 2π. Let

$$\begin{array}{cccc}
b_{11} = \frac{1}{\sqrt{a}} e^{i\theta_1} & b_{12} = \frac{1}{\sqrt{a}} e^{2i\theta_1} & \ldots & b_{1l} = \frac{1}{\sqrt{a}} e^{li\theta_1} \\
b_{21} = \frac{1}{\sqrt{a}} e^{i\theta_2} & b_{22} = \frac{1}{\sqrt{a}} e^{2i\theta_2} & \ldots & b_{2l} = \frac{1}{\sqrt{a}} e^{li\theta_2} \\
\vdots & \vdots & \vdots & \vdots \\
b_{a1} = \frac{1}{\sqrt{a}} e^{i\theta_a} & b_{a2} = \frac{1}{\sqrt{a}} e^{2i\theta_a} & \ldots & b_{al} = \frac{1}{\sqrt{a}} e^{li\theta_a},
\end{array}$$

where $l = n - a = a^2 - a$. Consider the following vectors of length $n = a^2$:

$$v_1 = (\overbrace{1,0,0,\ldots,0,0}^{\text{length } a}, b_{11}, b_{12}, \ldots b_{1l}),$$

$$v_2 = (0,1,0,\ldots,0,0, b_{21}, b_{22}, \ldots b_{2l}),$$

$$\vdots$$

$$v_a = (0,0,0,\ldots,0,1, b_{a1}, b_{a2}, \ldots b_{al}).$$

If we write the set $\{v_i v_j^*\}_{1 \le i,j \le a}$ as a list of a^2 row vectors, we have

$$v_1v_1^* = \left(\overbrace{1,0,0,\ldots,0,0}^{\text{length } a},\overbrace{\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}}^{\text{length } l}\right),$$

$$v_1v_2^* = \left(0,0,0,\ldots,0,0,\frac{1}{a}e^{i(\theta_1-\theta_2)},\frac{1}{a}e^{2i(\theta_1-\theta_2)},\ldots,\frac{1}{a}e^{li(\theta_1-\theta_2)}\right),$$

$$v_1v_3^* = \left(0,0,0,\ldots,0,0,\frac{1}{a}e^{i(\theta_1-\theta_3)},\frac{1}{a}e^{2i(\theta_1-\theta_3)},\ldots,\frac{1}{a}e^{li(\theta_1-\theta_3)}\right),$$

$$\vdots$$

$$v_av_a^* = \left(0,0,0,\ldots,0,1,\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}\right).$$

Rearrange the rows in such a way that $v_1v_1^*, v_2v_2^*, \ldots, v_av_a^*$ appear first, then we have the following $n\times n$ matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ 0 & 1 & 0 & \cdots & 0 & 0 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{a}e^{i(\theta_1-\theta_2)} & \frac{1}{a}e^{2i(\theta_1-\theta_2)} & \frac{1}{a}e^{3i(\theta_1-\theta_2)} & \cdots & \frac{1}{a}e^{li(\theta_1-\theta_2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{a}e^{i(\theta_a-\theta_{a-1})} & \frac{1}{a}e^{2i(\theta_a-\theta_{a-1})} & \frac{1}{a}e^{3i(\theta_a-\theta_{a-1})} & \cdots & \frac{1}{a}e^{li(\theta_a-\theta_{a-1})} \end{bmatrix}.$$

Note that this matrix has rank n if and only if the lower corner $l\times l$ matrix has rank l. If one factors out $1/a$ from each row of this $l\times l$ matrix, then the remaining matrix is

$$\begin{bmatrix} e^{i(\theta_1-\theta_2)} & e^{2i(\theta_1-\theta_2)} & \cdots & e^{li(\theta_1-\theta_2)} \\ e^{i(\theta_1-\theta_3)} & e^{2i(\theta_1-\theta_3)} & \cdots & e^{li(\theta_1-\theta_3)} \\ \vdots & \vdots & \vdots & \vdots \\ e^{i(\theta_a-\theta_{a-1})} & e^{2i(\theta_a-\theta_{a-1})} & \cdots & e^{li(\theta_a-\theta_{a-1})} \end{bmatrix},$$

which is a Vandermonde matrix with nonzero determinant because of the choice of $\theta_1,\ldots,\theta_a$. Consequently, $\{v_iv_j^*\}_{1\le i,j\le a}$ is a linearly independent set and $\Sigma_{i=1}^a v_i^*v_i=\Sigma_{i=1}^a v_iv_i^*=I$. Thus, the diagonal map $\varphi=\Sigma_{i=1}^a v_i^*\cdot v_i$ is extremal in $UCPT(n)$.

If $a^2<n$, then add $n-a^2$ entries to the end of each of the vectors $v_1,\ldots,v_a$ to form

$$v_1' = v_1 \oplus (c_{11}, c_{12}, \ldots, c_{1,n-a^2}),$$

$$v_2' = v_2 \oplus (c_{21}, c_{22}, \ldots, c_{2,n-a^2}),$$

$$\vdots$$

$$v_a' = v_a \oplus (c_{a1}, c_{a2}, \ldots, c_{a,n-a^2}),$$

where $\Sigma_{i=1}^{a}|c_{ij}|^2=1$ for all $1 \leq j \leq n-a^2$. Then we have

$$\sum_{i=1}^{a} v_i'^* v_i' = \sum_{i=1}^{a} v_i' v_i'^* = I.$$

Furthermore, since $\{v_i v_j^*\}_{1\leq i,j\leq a}$ is a linearly independent set, so is $\{v_i' v_j'^*\}_{1\leq i,j\leq a}$. Consequently, the mapping

$$\varphi'(A) = \sum_{i=1}^{a} v_i'^* A v_i', \quad A \in M_n(\mathbb{C})$$

is an extremal UCPT map, and it corresponds to an extremal diagonal marginal tracial state on $M_n(\mathbb{C})$ of rank a. ■

Theorem 3.4: *Extremal diagonal marginal tracial states of rank a with $a^2 \leq n$ are dense in the set of all diagonal marginal tracial states of rank a or less.*

Proof: Let φ be a diagonal UCPT map on $M_n(\mathbb{C})$ with

$$\varphi(A) = \sum_{i=1}^{a} u_i^* A u_i$$

for any $A \in M_n(\mathbb{C})$, where u_i is possibly zero. Let ψ be a diagonal UCPT map on $M_n(\mathbb{C})$ with

$$\psi(A) = \sum_{i=1}^{a} v_i^* A v_i$$

for any $A \in M_n(\mathbb{C})$. By Theorem 3.3, we may assume that $\{v_i v_j^*\}$ is a linearly independent set, so that ψ is an extremal point of the set of all UCPT maps.

Fix $\varepsilon > 0$ and put $w_i = u_i + \varepsilon v_i$. Since $\Sigma_{i=1}^{a} u_i^* u_i = I$, $\Sigma_{i=1}^{a} w_i^* w_i$ is invertible for sufficiently small $\varepsilon > 0$, and hence one can define a diagonal UCPT map by

$$\varphi_\varepsilon(A) \coloneqq \left(\sum_{i=1}^{a} w_i^* w_i\right)^{-1} \sum_{i=1}^{a} w_i^* A w_i$$

for any $A \in M_n(\mathbb{C})$. Since φ_ε goes to φ as $\varepsilon \to 0$, all we need is to verify that φ_ε is an extremal point of the set of all UCPT maps. To this end, it suffices to prove that $\{w_i w_j^*\}$ is a linearly independent set for sufficiently small $\varepsilon > 0$.

Since $w_i w_j^*$ is a diagonal matrix in $M_n(\mathbb{C})$, it can be considered a column vector in $\mathbb{C}^n$ and then $W = [w_i w_j^*]_{i,j=1}^{a}$ is an $n \times a^2$ matrix. Similarly, $U = [u_i u_j^*]_{i,j=1}^{a}$ and $V = [v_i v_j^*]_{i,j=1}^{a}$ can be considered $n \times a^2$ matrices.

Since $w_i w_j^* = (u_i + \varepsilon v_i)(u_j + \varepsilon v_j)^* = u_i u_j^* + \varepsilon(v_i u_j^* + u_i v_j^*) + \varepsilon^2 v_i v_j^*$, if we put $X = [v_i u_j^* + u_i v_j^*]_{i,j=1}^{a}$, then we have

$$W = U + \varepsilon X + \varepsilon^2 V,$$

and hence

$$W^*W = U^*U + \varepsilon(U^*X + X^*U) + \varepsilon^2(U^*V + V^*U + X^*X) + \varepsilon^3(X^*V + V^*X) + \varepsilon^4 V^*V.$$

The determinant of W^*W is a polynomial of ε of degree $4a^2$ and the coefficient of ε^{4a^2} is the determinant of V^*V which is not zero because $\{v_i v_j^*\}$ is a linearly independent set. Therefore, the equation $|W^*W|=0$ has at most $4a^2$ solutions and $|W^*W|$ is not zero for sufficiently small $\varepsilon > 0$. Since $\{w_i w_j^*\}$ is a linearly independent set if and only if W^*W is invertible in $M_{a^2}(\mathbb{C})$, the proof is complete. ∎

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