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Multi-Terminal Graphs and Equilibrium Problem of Flows and Tensions

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Synopsis

The notion of graph was generalized to make the abstract network theory applicable to economic networks, where multi-commodities are traded among productive sectors, intermediate sectors, and final demand sectors. The generalized graph, called the multi-terminal graph, has contributed to developing a new network theory, deriving that flows of multi-commodities correspond to electric currents in electric circuits. A well-known Leontief model has been reformulated in this generalized system. A notion of tension is defined to correspond the price in economics to the voltage in electric circuits. The equilibrium problem of flows and tensions is discussed on the basis of new network, and solved under a general hypothesis that characteristics of branches are represented by complete increasing curves, which express demand and supply curves.

1. INTRODUCTION

The theory of network flows has many applications to various fields in different appearance, e.g., electrical networks, communication networks, the transportation theory, hydraulic networks, and PERT networks.

In this work, we generalize the notion of graph to make the network theory applicable to economic networks, where multi-commodities are traded among productive sectors, intermediate sectors, and final demand sectors. Each branch has a characteristic abstractly represented by the so-called complete increasing curve, which is also represented by the demand curve or the supply curve in economics. When characteristics are given to each branch, our first problem is to find feasible flows and feasible tensions, which are compatibly defined on the network. The second problem is to know whether quantities and prices of commodities on characteristic curves can be determined or not, which is a sort of equilibrium problem.

To solve the first problem, we present a method to examine the existence of solutions for a general system of linear inequalities, which is discussed in Chap. 4.

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To solve the second problem, we refer to Rockafeller's theorem in Chap. 5, which reduces the second problem to the first. We did not discuss about the method of getting concrete solutions of the first problem, while is not so difficult. It will rather be more difficult to get equilibrium solutions of the second problem, but we will not discuss it here.

2. MULTI-TERMINAL GRAPHS

Let N and B be finite sets, the elements of which are called *nodes* and *branches*, respectively, and the cardinarities of which are l and m, respectively. Let ∂^+ and ∂^- be multi-valued functions from B to N (i.e., $\partial^+(b) \subseteq N$ and $\partial^-(b) \subseteq N$), which are called *positive incidence mapping* and *negative incidence mapping*, respectively. A node n is *positively incident* to a branch b if $n \in \partial^+(b)$ and *negatively incident* to a branch b if $n \in \partial^-(b)$. For simplicity we assume

$$\partial^+(b) \cup \partial^-(b) \neq \phi \tag{2.1}$$

for all $b \in B$, and

$$\{b : n \in \partial^+(b) \cup \partial^-(b)\} \neq \phi \tag{2.2}$$

for all $n \in N$. The quadruple $\mathscr{G}=(N, B, \partial^+, \partial^-)$ is called *a multi-terminal graph*.

If the positive and negative incidence mappings ∂^+ and ∂^- are single-valued functions, $\mathscr{G}$ becomes a usual graph. We define other multi-valued functions δ^+ and δ^- from N to B as

$$\delta^+(n) = \{b : n \in \partial^+(b)\} \tag{2.3}$$

and

$$\delta^-(n) = \{b : n \in \partial^-(b)\}, \tag{2.4}$$

hence for all $n \in N$

$$\delta^+(n) \cup \delta^-(n) = \{b : n \in \partial^+(b) \cup \partial^-(b)\} \neq \phi \tag{2.5}$$

by (2.2), and for all $b \in B$ there exists a node n such that $n \in \partial^+(b) \cup \partial^-(b)$ by (2.1), i.e.,

$$b \in \{b : \partial^+(b) \cup \partial^-(b)\} = \delta^+(n) \cup \delta^-(n). \tag{2.6}$$

Hence $(B, N, \delta^+, \delta^-)$ is again a multi-terminal graph, which is called a *dual graph* of $\mathscr{G}$ and is denoted by $\mathscr{G}^*$. Thus, the branch and the node are completely dual concepts in the multi-terminal graph. The multi-

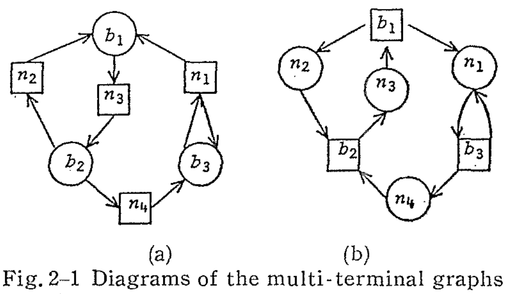

Fig. 2-1 Diagrams of the multi-terminal graphs (a) and (b) which are dual to each other.

terminal graph is represented by a diagram shown in Fig. 2–1, where the branches are denoted by circles and the nodes are by squares. An arrow is directed from branch to node if the node is positively incident to the branch, and is directed from node to branch if the node is negatively incident to the branch (which is a converse settlement from the electric circuit). Fig. 2–1 (b) represents the dual graph of (a), where the arrows are reversed and the circles are converted to squares.

The multi-terminal graph is regarded as a usual graph with colored nodes (say, red and green), the branches of which are positively incident to nodes of one color and negatively incident to nodes of another color.

3. FEASIBLE FLOWS AND FEASIBLE TENSIONS

Let $\mathscr{G}=(N, B, \partial^+, \partial^-)$ be a multi-terminal graph defined in Chap. 2., and let $P^+=(p^+_{nb})$ and $P^-=(p^-_{nb})$ be $l\times m$ matrices, rows and columns of which are indexed by the nodes in N and branches in B, respectively. Moreover, we assume that the elements are nonnegative and

$$n\in\partial^+(b) \quad \text{if and only if} \quad p^+_{nb}>0 \tag{3.1}$$

and

$$n\in\partial^-(b) \quad \text{if and only if} \quad p^-_{nb}>0. \tag{3.2}$$

The matrices P^+ and P^- are called a *positive incidence matrix* and a *negative incidence matrix*, respectively, and the matrix $P=P^+-P^-$ is called simply as an *incidence matrix*. The transpose matrices $(P^+)^T$ and, $(P^-)^T$ are positive and negative incidence matrices in $\mathscr{G}^*$, respectively, because $b\in\delta^+(n)$ if and only if $n\in\partial^+(b)$.

Now let $R(B)$ be a linear space of all real functions on B, the dimension of which is m, and let $R(N)$ be another linear space of all real functions on N, the dimension of which is l. We assume that a vector $\xi\in R(B)$ is a column vector, i.e., ξ is written by

$$\xi=\begin{pmatrix}\xi_{b_1}\\ \xi_{b_2}\\ \vdots\\ \xi_{b_m}\end{pmatrix}. \tag{3.3}$$

Accordingly, the positive (and negative) incidence matrix is considered as a linear transformation from $R(B)$ into $R(N)$ if we define

$$P^{\pm}\xi=\begin{pmatrix} p^{\pm}_{n_1b_1} & \cdots\cdots & p^{\pm}_{n_1b_m} \\ \cdots & \cdots & \cdots \\ p^{\pm}_{n_lb_1} & \cdots\cdots & p^{\pm}_{n_lb_m} \end{pmatrix}\begin{pmatrix} \xi_{b_1} \\ \xi_{b_2} \\ \vdots \\ \xi_{b_m} \end{pmatrix}. \tag{3.4}$$

The vector ξ of $R(B)$ is called a *compatible flow* if it belongs to the kernel of P, i.e., $\xi \in P^{-1}(\{0\})$, and the vector η of $R(B)$ is called a *compatible tension* if it belongs to the range of P^T (the transpose of P). The set of all compatible flows is $P^{-1}(\{0\})$ and the set of all compatible tensions is $P^T(R(N))$, and they are both subspaces of $R(B)$ and are orthogonal complements of each other.

Let I_{b_j} and $I^*_{b_j}$ be intervals (not necessarily closed) in the real line R^1 for $j=1, 2,\ldots, m$, which represent the intervals within which the flow and the tension can take the values. The product sets $I=\Pi^m_{j=1} I_{b_j}$ and $I^*=\Pi^m_{j=1} I^*_{b_j}$ are regarded as subsets (rectangles) of $R(B)$. A *feasibleflow* is a compatible flow in I and a *feasible tension* is a compatible tension in I^*.

If c_j and d_j are the left and right end points of the interval I_j, respectively, and c^*_j and d^*_j are defined similarly for I^*_j, feasible flow $\xi=(\xi_{b_j})$ satisfies

$$c_j \ (\leqq) \ \xi_{b_j} \ (\leqq) \ d_j \quad (j=1, 2,\ldots, m) \tag{3.5}$$

and

$$\textstyle\sum^m_{j=1} p_{n_ib_j}\,\xi_{b_j}=0 \ (i=1, 2,\ldots, l), \tag{3.6}$$

which is a system of linear inequalities and equalities (c_j's and d_j's may take $-\infty$ or $+\infty$). The feasible tension $\eta=(\eta_{b_j})$ satisfies

$$c^*_j \ (\leqq) \ \eta_{b_j} (\leqq) \ d^*_j \tag{3.7}$$

and

$$\textstyle\sum^l_{i=1} \zeta_{n_i} P_{n_ib_j}=\eta_{b_j} \ (j=1, 2,\ldots, m), \tag{3.8}$$

for some $\zeta=(\zeta_{n_i}) \in R(N)$.

EXAMPLE 1 (The Leontief Model) The well known Leontief model in economics (Leontief [2]) may be expressed by the words of our multi-terminal graph theory. The following is a three-dimensional case. Put $B=\{b_1, b_2, \ldots, b_6\}$ and $N=\{n_1, n_2, n_3\}$, and

$$\partial^+(b_i)=\{n_i\} \ \ i=1, 2, 3; \quad \partial^+(b_j)=\phi \quad j=4, 5, 6, \tag{3.9}$$

$$\partial^-(b_i)=\{n_1, n_2, n_3\} \ \ i=1, 2, 3; \quad \partial^-(b_j)=\{n_j\} \quad j=4, 5, 6. \tag{3.10}$$

The branch b_i represents the i th endogenous sector ($i=1, 2, 3$), the branch b_j represents j th exogenous sector ($j=4, 5, 6$), and the node n_i represents a market of the i th goods. Such a network is represented by Fig. 3–1. Let

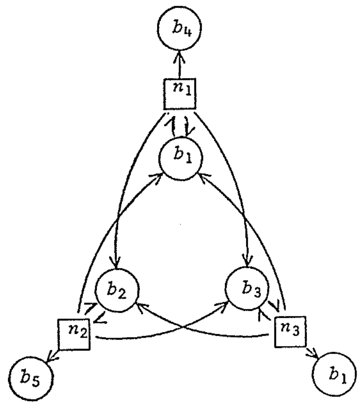

Fig. 3-1 A representation of the Leontief Model by the multi-commodity network.

$$P^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.11}$$

and

$$P^- = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}, \tag{3.12}$$

where a_{ij} 's are so-called input coefficients. We write a flow on this network as

$$\xi = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{3.13}$$

so that ξ is compatible if and only if $\xi \in P^{-1}(\{0\})$, i. e.,

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -a_{13} & -1 & 0 & 0 \\ -a_{21} & 1-a_{22} & -a_{23} & 0 & -1 & 0 \\ -a_{31} & -a_{32} & 1-a_{33} & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0, \tag{3.14}$$

which is equivalent to the well known Leontief fundamental equation. We can impose restrictions on ξ such as

$$0 \leqq x_i \ (i=1,\ 2,\ 3) \tag{3.15}$$

and

$$0 \leqq c_i \ (i=1,\ 2,\ 3), \tag{3.16}$$

i. e.,

$$I_i = [0,\ \infty) \quad (i=1,\ 2,\ \ldots,\ 6). \tag{3.17}$$

The flow is feasible if ξ satisfies (3.14), (3.15), and (3.16).

The Leontief model treats the case in which each industrial sector produces one kind of goods. In general, it is probable that plural goods are produced jointly by one sector, and our network is also able to represent such a general case.

4. LINEAR INEQUALITIES

In the preceding section, we have seen that the feasible flow and the feasible tension satisfy some linear inequalities and equalities. To know wheather the feasible flow or the feasible tension exist or not, let us study the existence problem of solutions of a given system of linear inequalities. For this, we need some roundabout preparations.

Let R^n be an n-dimensional Euclidean space, and R^{n*} be its conjugate space, i. e., a linear space of all real linear functionals on R^n. Then for any $f \in R^{n*}$ there exists a unique vector $a \in R^n$ such that $f(x) = \langle x, a \rangle = \sum_{i=1}^{n} x(i)a(i)$, by the Riesz representation theorem, where $\langle .,. \rangle$ is an inner product in R^n and $x(i)$ is the i th coordinate of $x \in R^n$. Let $\Xi = \{>, \geqq, <, \leqq, =\}$ be a set of great and small relations among real numbers. Single-valued maps $\bar{r}$, r^i, r^o and r^c of r in Ξ, the ranges of which are also in Ξ, are defined by Table 4–1. These operations $\bar{r}$, r^i, r^o, and r^c for r are called *closure*, *interior*, *opposite*, and *complement*, respectively. A relation r is *closed* if and only if $r = \bar{r}$ and *open* if and only if $r = r^i$.

Table 4–1 Definition of four kinds of maps from Γ into Γ. A symbol ϕ means "not defined".

r	$\bar{r}$	r^i	r^o	r^c
$>$	$\geqq$	$>$	$<$	$\leqq$
$\geqq$	$\geqq$	$>$	$\leqq$	$<$
$<$	$\leqq$	$<$	$>$	$\geqq$
$\leqq$	$\leqq$	$<$	$\geqq$	$>$
$=$	$=$	ϕ	$=$	ϕ

Let f be in R^{n*}, c be a real number, and r be in Ξ. We assume that a triplet (f, c, r) represents a linear inequality (or equality)

$$f(x) \; r \; c, \tag{4.1}$$

i. e., it represents one of the equations $f(x) > c$, $f(x) \geqq c$, $f(x) < c$, $f(x) \leqq c$, and $f(x) = c$.

A convex set $K(f, c, r)$ in R^n is defined as

$$K(f, c, r) = \{x \in R^n : f(x) \; r \; c\}, \tag{4.2}$$

which is a set of all solutions of the inequality (4.1). Clearly, $K(f, c, r)$ is closed if and only if r is closed or $f = 0 \in R^{n*}$, and open if and only if r is open or $f = 0 \in R^{n*}$. The validity of the following is clear:

PROPOSITION 1. i) *If* $f \neq 0$, *then* $K(f, c, r) \neq \phi$, ii) *if* $f = 0$ *and the inequality* $(0 \; r \; c)$ *is true, then* $K(f, c, r) = R^n$, *and* iii) *if* $f = 0$ *and the inequality*

$(0\ r\ c)$ is not true, then $K(f,\ c,\ r)=\phi$.

A *line* $L(f,\ c)$ is defined as

$$L(f,\ c)=K(f,\ c,\ =), \tag{4.3}$$

which is a closed convex set in R^n. A *system* Λ (of linear inequalities) is defined as $\Lambda=\{(f_i,\ c_i,\ r_i):\ i=1,\ 2,\ \ldots,\ m\}$, which represents m linear inequalities;

$$\begin{cases} f_1(x) & r_1 & c_1. \\ f_2(x) & r_2 & c_2 \\ & \vdots & \\ f_m(x) & r_m & c_m \end{cases} \tag{4.4}$$

$K_o(\Lambda)$ (or simply K_o) is a set of all solutions of (4.4), i.e.,

$$K_o=K_o(\Lambda)=\cap_{i=1}^{m} K(f_i,\ c_i,\ r_i). \tag{4.5}$$

Let B be a convex set in R^n. For a vector b in B, we put $B_b=B-b=\{b'-b:\ b'\in B\}$. The *dimension* of the convex set B is the dimension of the subspace spanned by B_b. The dimension is then independent of the choice of b. We denote the dimension of B by $\dim(B)$.

PROPOSITION 2. *If* $f\neq 0$, *then* $\dim(L(f,\ c))=n-1$. *If* $f\neq 0$ *and* $r\neq$"$=$", *then* $\dim(K(f,\ c,\ r))=n$.

PROOF. By Proposition 1, there exists an x_o in $L(f,\ c)$, thus,

$$L_{x_o}=L(f,\ c)-x_o=f^{-1}(\{0\}), \tag{4.6}$$

the dimension of which is $n-1$ if $f\neq 0$, as well known. If $r\neq$"$=$", then $L(f,\ c+e)\subsetneq K(f,\ c,\ r)$, where $e=+1$ if $r^i=$"$>$" and $e=-1$ if $r^i=$"$<$". Therefore $K(f,\ c,\ r)$ must be n-dimensional. Q. E. D.

The following two propositions are well known, but we give the proofs for completeness' sake.

PROPOSITION 3. *Let K be an $(n-1)$-dimensional convex set, and L be an $(n-1)$-dimensional linear manifold, i.e., $L-l$ $(l\in L)$ is an $(n-1)$-dimensional subspace. If $K\subseteq L$ and $x_o\notin L$, then*

$$A=\{tx_o+(1-t)y_o:\ y_o\in K,\ 0<t<1\} \tag{4.7}$$

is an n-dimensional convex set.

PROOF. We can assume $0\in K$. Linearly independent vectors $y_1,\ y_2,\ \ldots$, and y_{n-1} exist in K. Clearly $x_o,\ y_1,\ \ldots$, and y_{n-1} are also linearly independent. It is easy to prove that the vectors $z_o=\frac{2}{3}x_o+\frac{1}{3}y_1$ and $z_i=\frac{1}{2}x_o+\frac{1}{2}y_i$ $(i=1,\ 2,\ \ldots,\ n-1)$ are in A and linearly independent. Q. E. D.

PROPOSITION 4. *If K is an n-dimensional convex set in R^n, then $K^i \neq \phi$.*

PROOF. We assume $0 \in K$. Then, there exist linearly independent vectors $x_1, x_2, \ldots,$ and x_n in K. The convex hull co $\{0, x_1, \ldots, x_n\}$ is contained in K. A vector x in R^n is uniquely represented by a linear combination of x_i's with cofficients $\alpha_i (i = 1, 2, \ldots, n)$. If a mapping ψ from R^n onto R^n is defined by

$$\psi(x) = \sum_{i=1}^{n} \alpha_i e_i, \tag{4.8}$$

where $\{e_i\}$ is an orthogonal basis in R^n, then ψ is a one-to-one linear mapping from R^n onto R^n, hence it is a homeomorphism (topological isomorphism) from R^n onto R^n. Since co$\{0, x_1, \ldots, x_n\}$ corresponds to an n-dimensional simplex co $\{0, e_1, e_2, \ldots, e_n\}$ by ψ, it follows that $K^i \neq \phi$. Q. E. D.

A *subsystem* $\Lambda_i = \{(g_j, c_j, r_j) : j = 1, 2, \ldots, i-1, i+1, \ldots, m\}$ of a system $\Lambda = \{(f_k, c_k, r_k) : k = 1, 2, \ldots, m\}$ (assuming $f_k \neq 0$ for all k) is defined by

$$g_j(y) = f_j(\psi^{-1}(y)), \quad j = 1, 2, \ldots, i-1, i+1, \ldots, m \tag{4.9}$$

for y in R^{n-1}, where ψ is an isomorphism from $L(f_i, c_i)$ to R^{n-1} (the existence of which is assured by Prop. 2). A subsystem is also a system of linear inequalities of $n-1$ variables.

PROPOSITION 5. *If $r_i =$ "$=$", then the subsystem Λ_i has at least one solution if and only if Λ has at least one solution.*

PROOF. Λ has a solution x if and only $(f_j(x)\ r_j\ c_j)$ $(j \neq i)$ and $f_i(x) = c_i$, which is equivalent to $(g_j(\psi(x))\ r_j\ c_j)$ $(j \neq i)$ and $x \in L(f_i, c_i)$. Q. E. D.

Thus, we can remove the case $r_i =$"$=$" from the system when the existence problem is under cosideration. *If* $f_i = f_j$ and $c_i = c_j$ for i and j $(i \neq j)$, we can merge the two equations into one or we can see that the system is void, e. g., $f_i(x) \leqq c_i$ and $f_j(x) \geqq c_j$ and therefore they are merged into $f_i(x) = c_i$ and $f_i(x) < c_i$ and $f_j(x) > c_j$ and therefore it is a void case. By such a method, we can remove the case

$$L(f_i, c_i) = L(f_j, c_j) \quad \text{for } i \neq j. \tag{4.10}$$

Also, the case $K(f_i, c_i, r_i) \subseteq K(f_j, c_j, r_j)$ $(i \leqq j)$ may easily be removed from the system, even if $L(f_i, c_i) \neq L(f_j, c_j)$. Thus, we define that the system Λ is *primary* if $f_k \neq 0$ for all k, and $i \neq j$ implies $L(f_i, c_i) \neq L(f_j, c_j)$ and $K(f_i, c_i, r_i) \nsubseteq K(f_j, c_j, r_j)$. Therefore if the system is primary and $m \geqq 2$, then $K_o \subsetneqq K(f_i, c_i, r_i)$ for all i. The *i th closed side S_i of K_o* is defined by $S_i = L(f_i, c_i) \cap \bigcap_{j \neq i} K(f_j, c_j, r_j)$. The i th closed side is also a convex set in R^n. The following is easy

PROPOSITION 6. *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are $(n-1)$-dimensional subspaces of R^n and* *if* $y \perp \mathscr{M}_1$ *and* $y \perp \mathscr{M}_2$ *for some nonzero vector* y *in* R^n ($y \perp \mathscr{M}$ *means that* y *is orthogonal to each vector in* $\mathscr{M}$), *then* $\mathscr{M}_1 = \mathscr{M}_2$.

The following two propositions are key steps to the conclusion of this section.

PROPOSITION 7. *If the system* A *is primary, then* $S_i^i = \phi$ *for all* i *implies* $K_o^i = \phi$, *where* S_i^i *is an interior of* S_i, *topology of which is relative one on* $L(f_i, c_i)$.

PROOF. If $K_o^i \neq \phi$, there exists a vector x_o in K_o^i. As well known, we can find a unique vector y_i in $L(f_i, c_i)$, which gives the minimum distance between x_o and $L(f_i, c_i)$. Let i_o be a number such that

$$||x_o - y_{i_o}|| = \min_i ||x_o - y_i||. \tag{4.11}$$

The relation $x_o - y_{i_o} \perp L(f_{i_o}, c_{i_o}) - y_{i_o}$ is also well known. The vector y_{i_o} belongs to $K(f_j, c_j, r_j)$ for all j $(\neq i_o)$, because if $y_{i_o} \notin K(f_j, c_j, r_j)$, then, since $x_o \in K(f_j, c_j, r_j)$, $tx_o + (1-t)y_{i_o}$ must be in $L(f_j, c_j)$ for some t $(0 < t < 1)$, which contradicts to (4.11). Hence if $y_{i_o} \notin K(f_j, c_j, r_j)^i$ for some j $(j \neq i_o)$, then $y_{i_o} \in L(f_j, c_j)$, so

$$||x_o - y_{i_o}|| = \min \{||x_o - y||: y \in L(f_j, c_j)\} \tag{4.12}$$

and $x_o - y_{i_o} \perp L(f_j, c_j) - y_{i_o}$, then by Proposition 6,

$$L(f_{i_o}, c_{i_o}) - y_{i_o} = L(f_j, c_j) - y_{i_o}, \tag{4.13}$$

thus

$$L(f_{i_o}, c_{i_o}) = L(f_j, c_j), \tag{4.14}$$

which contradicts to the primary condition.

Hence $y_{i_o} \in K(f_j, c_j, r_j)^i$ for all j $(\neq i_o)$. Then there exists $\delta > 0$, and for all $j (\neq i_o)$

$$S_\delta(y_{i_o}) \subseteq K(f_j, c_j, r_j), \tag{4.15}$$

which implies

$$S_\delta(y_{i_o}) \cap L(f_{i_o}, c_{i_o}) \subseteq \bigcap_{j \neq i_o} K(f_j, c_j, r_j) \cap L(f_{i_o}, c_{i_o}) = S_{i_o}. \tag{4.16}$$

Therefore $(S_{i_o})^i \neq \phi$. Q. E. D.

PROPOSITION 8. *If the system* A *is primary and* $m \geqq 2$, *then* $K_o^i \neq \phi$ *if and only if* $S_{i_1}^i \neq \phi$ *and* $S_{i_2}^i \neq \phi$ *for at least two indices* i_1 *and* i_2 $(i_1 \neq i_2)$.

PROOF. $\Rightarrow$) If $K_o^i \neq \phi$, then there exists $x_o \in K_o^i$. By Proposition 7, we can assume that $S_{i_1}^i \neq \phi$ for some i_1. If ε (>0) is taken as $\overline{S_\varepsilon(x_o)} \subseteq K_o^i$ ($S_\varepsilon(x_o)$ is a sphere with center x_o and radius ε), we put

$$L_y = \{z_t:\ z_t = t(y - x_o) + x_o,\ t \geqq 1\} \tag{4.17}$$

for any y such that $||x_o - y|| = \varepsilon$. Accordingly, there exist y_o and k_o ($\neq i_1$) such that

$$L_{y_o} \cap L(f_{k_o},\ c_{k_o}) = \{z_{t_o}\} \neq \phi \tag{4.18}$$

and $t < t_o$ implies

$$z_t \notin \cup_{k=1}^{m} L(f_k,\ c_k). \tag{4.19}$$

If (4.19) is false ((4.18) is clear as $S_\varepsilon(x_o)$ spans R^n and the system is primary), then $y_o \in L(f_k,\ c_k)$ for some k, hence $y_o \notin K(f_k,\ c_k,\ r_k)^i$ and

$$y_o \notin K(f_k,\ c_k,\ r_k)^i \supseteq K_o^i\ , \tag{4.20}$$

which contradicts to the fact $\overline{S_\varepsilon(x_o)} \subseteq K_o^i$.

We can assume $z_{t_o} \in K(f_i, c_i, r_i)^i$. Hence, for some $\delta > 0$, $S_\delta(z_{t_o}) \subseteq K(f_{i_1},\ c_{i_1},\ r_{i_1})^i$. Putting $x_o' = \delta' x_o + (1 - \delta') z_{t_o}$ where $\delta' = \delta/(2||x_o - z_{t_o}||)$, we get

$$\min\{||x'_o - y|| : y \in L(f_{k_o},\ c_{k_o},\ r_{k_o})\} \leqq ||x'_o - z_{t_o}|| < \frac{\delta}{2}. \tag{4.21}$$

On the other hand,

$$\begin{aligned} &\min\{||x'_o - y|| : y \in L(f_{i_1},\ c_{i_1},\ r_{i_1})\} \\ &\qquad \geqq \min\{||z_{t_o} - y|| - ||x_o - z_{t_o}||\} > \delta - \frac{\delta}{2} = \frac{\delta}{2}. \end{aligned} \tag{4.22}$$

Hence, if we choose x'_o for x_o in Prop. 7, then i_o, which gives the minimum distance, can not be i_1. Thus we get another side S_{i_2}.

$\Leftarrow$) Put $A = \{z_t:\ z_t = tx_o + (1 - t)y_o,\ 0 < t < 1,\ \ x_o \in S_{i_1}^i\}$, where $y_o \in S_{i_2}^i$, and $A \subseteq K_o$ (which is easy as $z_t \in K(f_{i_1},\ c_{i_1},\ r_{i_1})$, $z_t \in K(f_{i_2},\ c_{i_2},\ r_{i_2})$, and $z_t \in \bigcap_{j \neq i_1, i_2} K(f_j,\ c_j,\ r_j)$) and A is n-dimensional by Proposition 3. Hence K is also n-dimensional and $K_o^i \neq \phi$ by Proposition 4. Q.E.D.

We can determine by the above proposition whether the interior of K is empty or not. The following proposition gives a light to the case when the interior of K_o is empty.

PROPOSITION 9. *If the system is primary and* $K_o^i = \phi$, *then,* $K_o = \phi$ *if and only if for any* i $S_i = \phi$ *or the relation* r_i *is open.*

PROOF. $\Rightarrow$) If $S_i \neq \phi$ and $L(f_i,\ c_i) \subseteq K(f_i,\ c_i,\ r_i)$, then $\phi \neq S_i = S_i \cap L(f_i,\ c_i) \subseteq S_i \cap K(f_i,\ c_i,\ r_i) = K_o \cap L(f_i,\ c_i) \subseteq K_o$.

$\Leftarrow$) In general, $E^b \cap E = \phi$ and $E^i = \phi$ implies $E = \phi$ (E^b is a boundary of E). Hence, with $K_j = K(f_j,\ c_j,\ r_j)$ and $L_j = L(f_j,\ c_j)$,

$$K_o^b \cap K_o = \overline{K_o} \cap \overline{K_o^c} \cap K_o = K_o \cap \overline{K_o^c} = \bigcap_j K_j \cap \overline{(\bigcap_j K_j)^c} = \bigcap_j K_j \cap \overline{\bigcup_k K_k^c}$$

$$= \bigcap_j K_j \cap \bigcup_k \overline{K_k^c} = \bigcup_k (\bigcap_j K_j) \cap K(f_k,\ c_k,\ \overline{r_k^c}) \tag{4.23}$$

$$= \bigcup_k (\bigcap_j K_j) \cap L_k = \bigcup_k (S_k \cap K_k \cap L_k) = \phi, \tag{4.24}$$

since $S_k \cap K_k = \bigcap_{j \neq k} K_j \cap L_k \cap K_k = \phi$ if $S_k = \phi$ or r_k is open. Thus $K_o = \phi$. Q. E. D.

Let us introduce a notion of degree of system. If A is a primary system, the *degree* of a system A, denoted by $\deg(A)$, is defined by

$$\deg(A) = \begin{cases} 0 & \text{if } K_o = \phi, \\ 1 & \text{if } K_o^i = \phi \text{ and } K_o \neq \phi, \\ 2 & \text{if } K_o^i \neq \phi. \end{cases} \tag{4.25}$$

Summing up the above propositions, we can determine the degree of system by reducing the problem to its subsystems.

THEOREM 4.1 *Let A be a primary system and $m > 2$. Then,*

(1) *if* $\deg(A_{i_1}) = 2$ *and* $\deg(A_{i_2}) = 2$ *for some* i_1, $i_2 (i_1 \neq i_2)$, *then* $\deg(A) = 2$,

(2) *otherwise, if* $\deg(A_{i_1}) \geqq 1$ *and* r_{i_1} *is closed for some* i_1, *then* $\deg(A) = 1$,

(3) *otherwise,* $\deg(A) = 0$.

EXAMPLE 4.1 Let us again consider the Leontief model, and let us treat some concrete values such as

$$P^- = \begin{pmatrix} \frac{1}{8} & \frac{5}{16} & \frac{5}{8} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{5}{8} & 0 & 1 & 0 \\ \frac{3}{16} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 1 \end{pmatrix}. \tag{4.26}$$

Restrictions on x_i's and c_i's are $6 < c_1$, $8 < c_2$, $2 < c_3$, $0 < x_1 < 36$, $0 < x_2 < 36$, and $0 < x_3 < 28$. Therefore, to know the existence of feasible flows, we must examine the following linear inequalities and equalities:

$$\frac{7}{8}x_1 - \frac{5}{16}x_2 - \frac{5}{8}x_3 - c_1 = 0 \qquad \text{(A1)},$$

$$-\frac{1}{4}x_1 + \frac{7}{8}x_2 - \frac{5}{8}x_3 - c_2 = 0 \qquad \text{(A2)},$$

$$-\frac{3}{16}x_1 - \frac{1}{8}x_2 + \frac{7}{8}x_3 - c_3 = 0 \qquad \text{(A3)},$$

$c_1 > 6$ (A4), $c_2 > 8$ (A5), $c_3 > 2$ (A6), $x_1 > 0$ (A7), $x_1 < 36$ (A8), $x_2 > 0$ (A9), $x_2 < 36$ (A10), $x_3 > 0$ (A11), and $x_3 < 28$ (A12).
To make the system primary, we remove (A1), (A2), and (A3), and we call the resulting system Λ:

$$\Lambda: \quad \frac{7}{8}x_1 - \frac{5}{16}x_2 - \frac{5}{8}x_3 > 6 \qquad \text{(B1)},$$

$$-\frac{1}{4}x_1 + \frac{7}{8}x_2 - \frac{5}{8}x_3 > 8 \qquad \text{(B2)},$$

$$-\frac{3}{16}x_1 - \frac{1}{8}x_2 + \frac{7}{8}x_3 > 2 \qquad \text{(B3)},$$

$x_1 > 0$ (B4), $x_1 < 36$ (B5), $x_2 > 0$ (B6), $x_2 < 36$ (B7),
$x_3 > 0$ (B8), and $x_3 < 28$ (B9).

The first subsystem Λ_1 is written as (the formula in brackets means the i th formula giving the i th subsystem)

$$\Lambda_1: \quad [x_1 = \frac{8}{7}(6 + \frac{5}{16}x_2 + \frac{5}{8}x_3)]$$

$$44x_2 - 45x_3 > 544,$$
$$-43x_2 + 166x_3 > 736,$$
$$480 > 5x_2 + 10x_3 > -96,$$
$$36 > x_2 > 0,$$

which is a two-dimensional case, and we can easily derive that $\deg(\Lambda_1) = 2$.
For the second subsystem Λ_2:

$$\Lambda_2: \quad [x_1 = -4(8 - \frac{7}{8}x_2 + \frac{5}{8}x_3)]$$

$$44x_2 - 45x_3 > 544,$$
$$-25x_2 + 43x_3 > -128,$$
$$136 > 7x_2 - 5x_3 > 64,$$
$$36 > x_2 > 0,$$
$$128 > x_3 > 0,$$

the degree is also 2. Hence $\deg(\Lambda) = 2$ by the theorem, and feasible flows exist. One of the solutions is a middle point between a point in $S^i_{j_1}$ $(\neq \phi)$ and a point in $S^i_{i_2}(\neq \phi)$. In this case $\xi = (32 \;\; 32 \;\; 16 \;\; 8 \;\; 10 \;\; 4)^T$ is one of the solutions. In a higher dimensional case, we can also use the theorem to determine the degree of system by induction, but the number of subsystems increases in a geometric progression at each step of induction.

5. BRANCH CHARACTERISTICS AND FEASIBLE FLOWS

In this section, we formulate additional characteristics of branches and nodes, which correspond to the notion of demand curve and supply curve in economics. The 2-dimensional Euclidean space R^2 is ordered by

$$(x_1,\ y_1)<(x_2,\ y_2) \text{ if and only if } x_1 \leqq x_2 \text{ and } y_1 \leqq y_2. \tag{5.1}$$

The subset Γ of R^2 is called a *complete increasing curve* if it is a maximal totally-ordered subset of $(R^2, <)$. Then Γ is a continuous increasing curve, which crosses each of the lines with slope -1 exactly once. The *inverse* Γ^* is defined by

$$\Gamma^* = \{(x^*,\ x):\ (x,\ x^*) \in \Gamma\}, \tag{5.2}$$

which is also a complete increasing curve. Let Γ_j ($j=1,\ 2,\ \ldots,\ m$) be complete increasing curves, and K and K^* be subspaces of R^m orthogonally complementary to each other, and let I_j and I_j^* be domains of Γ_j and Γ_j^*, i e.,

$$I_j = \{x:\ (x,\ y) \in \Gamma_j \text{ for some } y\} \tag{5.3}$$

and

$$I_j^* = \{x^*:\ (x^*,\ y^*) \in \Gamma_j^* \text{ for some } y^*\}. \tag{5.4}$$

Let I and I^* be rectangles similarly defined from I_j and I_j^* (instead of I_{b_j} and $I_{b_j}^*$) as in chap. 3. The following theorem by Rockafeller [3] is well known, which has been proved by Iri [1] partially when I_j's are closed.

THEOREM (Rockafeller) *If there are vectors $x \in I \cap K$ and $x^* \in I^* \cap K^*$, there are $x^{*o} = (x_j^{*o}) \in K^*$ and $x_o = (x_j^o) \in K$ such that*

$$(x_j^o,\ x_j^{*o}) \in \Gamma_j \textit{ for } j = 1,\ 2,\ \ldots,\ m. \tag{5.5}$$

The pair $(x^o,\ x^{*o})$ which satisfies the theorem is called an equilibrium solution. This theorem provides an interpretation to our network of multi-commodity flows. Each branch of a multi-terminal graph is assigned such a Γ_j as its characteristic curve. Choose $R^{-1}(\{0\})$ as the subspace K and $P^T(R(N))$ as the subspace K^*, and the theorem is rewritten into the following form:

THEOREM *If there are a feasible flow and a feasible tension on a multi-terminal graph, there exist a compatible flow $(\xi_{b_j}^o)$ and a compatible tension $(\eta_{b_j}^o)$*

such that $(\xi^o_{b_j},\ \eta^o_{b_j})$*'s are on the complete increaing curves* Γ_j*'s* $(j=1,\ 2,\ \dots,\ m)$.

As we have shown in chap. 4., we have a method to determine whether the conditions of the theorem are satisfied or not. Thus we have a method to know the existence of equilibrium solutions.

At the conclusion, let us give some examples which are instructive to show the relation between a multi-terminal graph with characteristic curves and economic networks.

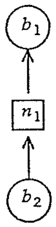

Fig. 5-1 A simple network, where b_1 is consumers and b_2 is suppliers.

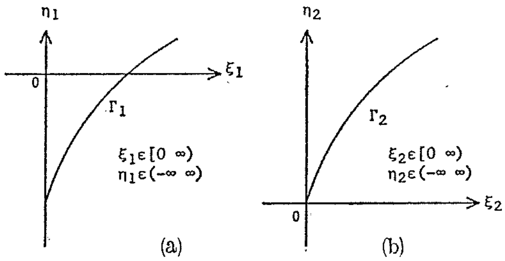

Fig. 5-2 Characteristic curves of b_1 and b_2.

EXAMPLE 5.1 Let us consider the simple case shown in Fig. 5-1, where $B=\{b_1,\ b_2\}$, $N=\{n_1\}$, $\partial^+(b_1)=\partial^-(b_2)=\phi$, and $\partial^-(b_1)=\partial^+(b_2)=\{n_1\}$. We have $P^+=(0\ 1)$ and $P^-=(1\ 0)$, so $P=(-1\ 1)$. The flow ξ is compatible if and only if $-\xi_1+\xi_2=0$ and the tension η is compatible if and only if $\eta_1+\eta_2=0$. The characteristic curves are shown in (a) and (b) of Fig. 5-2, and thus $I_1=(0,\ \infty)$, $I_1^*=(-\infty,\ \infty)$, $I_2=[0,\ \infty)$, and $I_2^*=(-\infty,\ \infty)$. The flow $\xi=(\xi_1\ \xi_2)^T$ is feasible if and only if $\xi_1=\xi_2$ and $0\leqq\xi_1$. The tension $\eta=(\eta_1\ \eta_2)^T$ is feaible if and only if $\eta_1=-\eta_2$. Clearly such a ξ and such an η exist and by the corollary, the equilibrium solution exists. Indeed, with $\Gamma_1'=\{(\xi,\ -\eta):\ (\xi,\ \eta)\in\Gamma\}$, the intersecting point $(\xi_o,\ \eta_o)$ of Γ_1' and Γ_2 gives an equilibrium solution, which is written as $\xi=(\xi_o\ \xi_o)^T$ and $\eta=(-\eta_o\ \eta_o)^T$. If we interpret the branch b_1 as consumers, the branch b_2 as suppliers, and the node n_1 as a market, the flow ξ_i $(i=1,\ 2)$ as the quantity of a commodity, and the tension η_i $(i=1,\ 2)$ as the difference between the prices of input and output commodities, the vector $\zeta\in R(N)$ is such that $P^T(\zeta)=\eta$ represents prices on each market. Fig. 5-3 is a well-known figure appearing in textbooks of economics.

EXAMPLE 5.2 The network shown in Fig. 5-4 represents the case that intermediate sectors exist. It is an important point of our theory that the character of an intermediator is also represented by the characteristic curve of a branch. Such a characteristic curve plays twofold roles in the network : the first is that

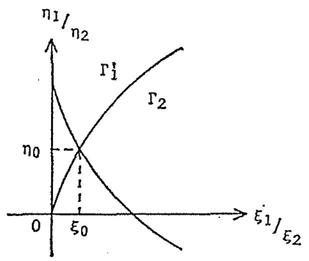

Fig. 5–3 Reversal of axis η results in a demand curve.

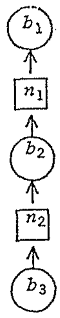

Fig. 5–4 The case that the intermediate sector b_2 exists.

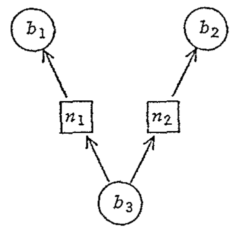

Fig. 5–5 A supplier b_3 produces two kinds of goods jointly.

it works as a sort of supply curve when considered from the side of consumers, and the second is that it works as a sort of demand curve, when considered from the side of suppliers.

EXAMPLE 5. 3 The preceding examples are single-commodity cases. On the other hand, a simple network shown in Fig. 5–5 is a multi-commodity case. The supplier b_3 produces two kinds of goods jointly. The supply curve at a market n_1 is dependent on the characteristic curve Γ_2 of another consumer b_2 which is a peculiar phenomenon in the multi-commodity case.

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REFFERENCES

1. M. Iri, "Network flow, transportation and scheduling," Academic Press, New York and London, 1969.
2. W. W. Leontief, "The structure of American economy 1919–29, an emperical application of equilibrium analysis," Oxford University Press, New York, 1941.
3. R. T. Rockafellar, J. of Math. Analysis and Applications, **19**, 543–564, 1967.