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Decreasing Rearrangements of Non-Negative (c_0) *Sequences and Some Extensions of Hardy-Littlewood-Pólya's Theorems*

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Decreasing rearrangements of non-negative (c_0) sequences and three preorder relations which are extensions of Hardy-Littlewood-Pólya's one are defined. Some generalizations of Hardy-Littlewood-Pólya's inequalities for rearrangements and convex functions are given.

1 Introduction

In recent years a number of inequalities have appeared which involve rearrangements of vectors in R^n or sequences in (l^1) and of measurable functions on a finite measure space or non-negative L^1 functions on an infinite measure space [1; 6]. These inequalities are not only interesting themselves, but also have many applications in probability theory, information theory, mathematical economics, and so on [8]. But many times we are forced to consider sequences which belong to (c_0).

In this paper we define decreasing rearrangements of non-negative (c_0) sequences and we introduce three preorder relations in the positive cone $(c_0)_+$ of, (c_0), two of which are new and one is equivalent to that of Markus [7, p. 103]. Consequently, some generalizations of well-known results of Hardy-Littlewood-Pólya [5, Theorem 108, p. 89] and Pólya [9] are given. Moreover, two results of Chong [3, Theorem 2.7, p. 158; 4, Theorem 3.9, p. 434] are generalized.

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2 Notations and Preliminaries

Let R^n denote the set of all n-tuples of real numbers. For any n-tuple $x=(x_1,\cdots, x_n) \in R^n$, we denote by

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$$x^* = (x_1^*, \cdots, x_n^*)$$

the n–tuple whose components are those of x arranged in decreasing order of magnitude. If $a = (a_1, \cdots, a_n) \in R^n$ and $b = (b_1, \cdots, b_n) \in R^n$, then $a \ll b$ means that

$$\sum_{i=1}^{k} a_i^* \leqq \sum_{i=1}^{k} b_i^* \tag{2. 1}$$

for $1 \leqq k \leqq n$, and we write $a \prec b$ if, in addition to $a \ll b$, there is equality in (2. 1) for $k = n$. These two preorder relations in R^n were originally defined by Hardy–Littlewood–Pólya [5], and the following theorems give characterizations of $\prec$ and $\ll$ [5, Theorem 108, p. 89; 9].

Suppose $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$ are n–tuples in R^n, then the following hold.

(H_1) *$a \prec b$ is equivalent to* $\sum_{i=1}^{n} \phi(a_i) \leqq \sum_{i=1}^{n} \phi(b_i)$ (2. 2)

for all convex functions ϕ: $[b_n^*, b_1^*] \to R$.

(H_2) *$a \ll b$ is equivalent to* $\sum_{i=1}^{n} \phi(a_i) \leqq \sum_{i=1}^{n} \phi(b_i)$ (2. 3)

for all non–decreasing convex functions ϕ: $[b_n^*, b_1^*] \to R$, and this is equivalent to

$$\sum_{i=1}^{n} (a_i - u)^+ \leqq \sum_{i=1}^{n} (b_i - u)^+ \tag{2. 4}$$

for all real numbers u, where $x^+ = \max\{x, 0\}$ for any $x \in R$.

If $f(x)$ is non–increasing and right continuous on $[0, \infty)$,

$$f^*(x) = \sup\{\lambda : f(\lambda) > x\} \ (x \geqq 0) \tag{2. 5}$$

is called the right continuous inverse of f on $[0, \infty)$, and the following are well known [1, p. 24].

(R_1) *$f^*(x)$ is right continuous and decreasing.* (2. 6)

(R_2) *$f^*(x) > \lambda$ is equivalent to $f(\lambda) > x$.* (2. 7)

(R_3) $d_{f^*}(\lambda) = \mu\{x : f^*(x) > \lambda\} = f(\lambda)$, (2. 8)

where μ is the Lebesgue measure on $[0, \infty)$.

Throughout this paper, we write N in place of the set of all positive integers, and Z_+ denotes the set of all non–negative integers. Also R_+ stands for the positive cone of R, and $\bar{R}_+$ for the set of all non-negative extended real numbers, while $(l^p)_+$ denotes the positive cone of (l^p) $(1 \leqq p)$. Moreover, if $a=(a_1, a_2, \cdots) \in (c_0)_+$ we write $a_i = a(i)$ for any integer $i \in N$, and S_+ stands for a set $\{a:\ a \in (c_0)_+$, there exists an $m \in N$ so that $i > m$ implies $a(i)=0\}$. $d_a(\lambda) = \text{Card}\ \{i:\ a(i) > \lambda\}$ is called the distribution of a. Then $(c_0)_+$ is characterized by a set such that $\{a: a=(a_1, a_2, \cdots) \geqq 0,\ d_a(\lambda) < \infty$ for any $\lambda > 0\}$.* In the sequel, we use the term "convex" in a narrow meaning: a convex function is a function ϕ such that $\lambda_1, \lambda_2 \geqq 0$ and $\lambda_1 + \lambda_2 = 1$ imply $\phi(\lambda_1 x + \lambda_2 y) \leqq \lambda_1 \phi(x) + \lambda_2 \phi(y)$ for any x and y in the domain of ϕ.

3 Decreasing Rearrangements of Non-Negative (c_0) Sequences and Some Extensions of H-L-P's Theorems

Our results are based on the next existence theorem for rearrangements of sequences in $(c_0)_+$.

THEOREM 1. *If a belongs to $(c_0)_+$, then we can rearrange all the components $a_i > 0$ of a in a non-increasing order of magnitude so that $a_1^* \geqq a_2^* \geqq \cdots$ holds.*

Proof. If a $\in S_+$, then the statement in our theorem is evident, therefore we may assume $a \notin S_+$ and $a \in (c_0)_+$. Then there exists a component $a_j > 0$ of a. Put $A_1 = N$, and then $d_a\left(\frac{a_j}{2}\right) = \text{Card}\ \{i : a(i) > \frac{a_j}{2}\}$ is finite, which insures the existence of $i_1 \in N$ so that $a_{i_1} = \max\ \{a_i:\ i \in A_1\}$. Define A_n and a_{i_n} $(n=2, 3, \cdots)$ by induction as follows :

$$a_{i_n} = \max\ \{a_i:\ i \in A_n\},\ A_{n+1} = A_n - \{i_n\}. \tag{3.1}$$

If we set $A_+ = \{i:\ a_i > 0\}$, we can define a single valued mapping

$$\phi : N \to A_+,\ \phi(j) = i_j \tag{3.2}$$

by means of (3.1). It is easy to see that ϕ is one-to-one; for any $a_k \in A_+$ there exists one coordinate i_j such that $a_k = a_{i_j}$, since $d_a\left(\frac{a_k}{2}\right) = \text{Card}\ \{i:\ a_i > \frac{a_k}{2}\}$ is finite. That is, $\phi: N \to A_+$ is a one-to-one and onto mapping, and if we put

$$a^*(j) = a(i_j) = a(\phi(j)), \tag{3.3}$$

a_i^* $(i \in N)$ is the desired one.

DEFINITION 1. If $a \in (c_0)_+$ and a $\notin S_+$, then we define $a^* = (a(\phi(1)), a(\phi(2)), \cdots)$, where ϕ is the mapping defined by (3.2). If $a \in S_+$, assume Card $\{i:\ a_i > 0\} = m$, and denote by a_i^* $(i=1, \cdots, m)$ the positive components of a rearranged in non-increasing order of magnitude. In this case, we define $a^* = (a_1^*, \cdots, a_m^*, 0, \cdots, 0)$.

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* This easy but important fact is suggested by Mr. Yukio Takeuchi.

we call a^* the *decreasing rearrangement of* $a \in (c_0)_+$.

It is easy to see that $d_a(\lambda) = d_{a*}(\lambda)$ for any $\lambda \in R$. Therefore $a^* \in (c_0)_+$ if a $\in (c_0)_+$.

DEFINITION 2. If $a,\ b \in (c_0)^+$, we write

$$a \sim b \text{ if and only if } d_a(\lambda) = d_b(\lambda) \tag{3.4}$$

for any $\lambda \in R$, and we say that a and b are *equidistributed* if $a \sim b$.

It is easy to see that $\sim$ is a preorder relation in $(c_0)_+$ and that $a \sim a^*$.

PROPOSITION 1. *If* $a,\ b \in (c_0)_+$, *then*

$$a \sim b \textit{ if and only if } a^* = b^*. \tag{3.5}$$

Proof. Both $a \sim a^*$ and $b \sim b^*$ with $a \sim b$ imply $a^* \sim b^*$; hence $a^* = b^*$ is clear. The proof of the converse implication is clear from Definition 1.

In the sequel, we regard $\aleph_0$ as ∞, an element of $\overline{R}_+$, and we consider $d_a(\cdot)$ as a mapping from R to $\overline{R}_+$.

THEOREM 2. *A mapping* $f(\cdot)$ *from* R *to* $\overline{R}_+$ *is a distribution* $d_a(\cdot)$ *for some* $a \in (c_0)_+$, *if and only if,* $f(\cdot)$ *satisfies the following three conditions* D_1, D_2, *and* D_3.

(D_1) $f(\lambda) \in Z_+$ *for any* $\lambda > 0$ *and* $f(\lambda) = \infty$ *for any* $\lambda < 0$.

(D_2) *There exists a* $\lambda_0 \in R_+$ *such that* $f(\lambda) = 0$ *for any* $\lambda > \lambda_0$.

(D_3) $f(\lambda)$ *is a non-increasing and right continuous function on* R.

Proof. If $f(\cdot) = d_a(\cdot)$ for some $a \in (c_0)_+$, then we have an alternative expression $f(\cdot) = \mathrm{d}_{a*}(\cdot)$; hence D_1 and D_2 are clear. D_3 is a consequence of the continuity of a measure Card $\{\cdot\}$.

To prove the converse implication, consider the right continuous inverse f^* of f. Then, as mentioned already, (2.6), (2.7), and (2.8) hold. Moreover, if $f^*(0) = \infty$, then $f^*(0) > K$ for any $K > 0$, which is equivalent to $f(K) > 0$ for any $K > 0$ by (2.7), contradictory to D_2 ; hence $f^*(0) < \infty$. Now, define $\bar{a}(s) = f^*(s-1)$ for any $s \in N$. Then,

$$\bar{a}(s) \geqq 0 \text{ is non-incrasing for any } s \in N, \text{ and } \bar{a}(1) = f^*(0) < \infty. \tag{3.6}$$

It is clear that $d_{\bar{a}}(\lambda) = f(\lambda) = \infty$ holds for any $\lambda < 0$. Assume $0 \leqq \lambda < \bar{a}(1) = f^*(0)$, then

$$\begin{aligned} d_{\bar{a}}(\lambda) &= \max\{s\colon s \in N,\ \bar{a}(s) > \lambda\} \\ &= \max\{s\colon s \in N,\ f^*(s-1) > \lambda\} \\ &= \max\{s\colon s \in N,\ f(\lambda) > s-1\} \\ &= f(\lambda). \end{aligned}$$

Next, assume $\lambda \geqq \bar{a}(1) = f^*(0)$, then (3.6) implies $d_{\bar{a}}(\lambda) = 0$, while $f(\lambda) \leqq 0$ follows from (2.7); hence $f(\lambda) = 0$. Thus we have proved that $d_{\bar{a}}(\lambda) = f(\lambda)$ for any $\lambda \in R$, and $d_{\bar{a}}(\lambda) = f(\lambda) < \infty$ for any $\lambda > 0$. Therefore $\bar{a}$ belongs to $(c_0)_+$, and $f(\cdot) = d_{\bar{a}}(\cdot)$: the proof is completed.

COROLLARY 1. *If a belongs to* $(c_0)_+$, *then*

$$a^*(s) > \lambda \textit{ if and only if } d_a(\lambda) > s-1 \tag{3.7}$$

for any $s \in N$.

Proof. Suppose a $\in (c_0)_+$, and put $f = d_a = d_{a^*}$. Then $f = d_{\bar{a}}$, where $\bar{a}$ is the element in $(c_0)_+$ defined in the proof of Theorem 2. Hence,

$$d_a(\lambda) > s-1 \Longleftrightarrow f(\lambda) > s-1 \Longleftrightarrow f^*(s-1) > \lambda \Longleftrightarrow a^*(\mathrm{s}) > \lambda$$

is clear from (2.7).

COROLLARY 2. *If* $a \in (c_0)_+$, *then*

$$\begin{aligned} a^*(s) &= \sup\{\lambda : d_a(\lambda) > s-1\} \\ &= \inf\{\lambda :\ d_a(\lambda) \leqq s-1\} \end{aligned}$$

necessarily holds for any $s \in N$.

Proof. Both $a^*(s) = \sup\ \{\lambda:\ a^*(s) > \lambda\} = \sup\ \{\lambda:\ d_a(\lambda) > s-1\}$ and $a^*(s) = \inf\ \{\lambda:\ a^*(s) \leqq \lambda\} = \inf\ \{\lambda:\ d_a(\lambda) \leqq s-1\}$ are immediate consequences of (3.7).

By virtue of Corollary 1 and Corollary 2, we can easily obtain the next convergence theorem for rearrangement.

THEOREM 3. *If* a_n *and* $a \in (c_0)_+$, *then*

$$a_n \uparrow a \quad \textit{implies both} \quad d_{a_n} \uparrow d_a \quad \textit{and} \quad a_n^* \uparrow a^*. \tag{3.8}$$

Proof. It is easy to see that $a_n \uparrow a$ implies $d_{a_n}(\lambda) \leqq d_{a_{n+1}}(\lambda) \leqq d_a(\lambda)$ for any $\lambda \in R$. Then $a_n^* \leqq a_{n+1}^* \leqq a^*$ is immediate by Corollary 2, and $d_{a_n} \uparrow d_a$ is a mere con sequence of the continuity of a measure. Hence $\lim_{n\to\infty} a_n^*(s) \leqq a^*(s)$ $(s \in N)$ is immediate. To obtain the opposite side inequality, assume $a^*(s) > \lambda$. Then we have $d_a(\lambda) = \lim_{n\to\infty} d_{a_n}(\lambda) > s-1$ by (3.7), which implies the existence of an integer m so that $d_{a_n}(\lambda) > s-1$ holds for any $n > m$. Hence $a_n^*(s) > \lambda$ for any $n > m$ and $\lim_{n\to\infty} a_n^*(s) \geqq \lambda$ hold. That is $\lim_{n\to\infty} a_n^* \geqq a^*$. Thus we have completed the proof.

Now we shall extend the preorders of Hardy–Littlewood–Pólya in R^n to the sequennces belonging to $(c_0)_+$.

DEFINITION 3. If a, $b \in (c_0)_+$, then we write

$$a \ll b \text{ if and only if } \sum_{i=1}^{k} a_i^* \leqq \sum_{i=1}^{k} b_i^* \tag{3.9}$$

for any $k \in N$, and

$$a \prec b \text{ if and only if } a \ll b \text{ and } \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i, \tag{3.10}$$

here we write $\sum_{i=1}^{\infty} b_i = \infty$, whenever $\sum_{i=1}^{\infty} b_i$ is divergent. We say a is *weakly* (*strongly*) *majorized* by b if $a \ll b$ $(a \prec b)$.

It should be noted that (3.9) is a generalization of the preorder of Markus [7, p. 103]. It is clear that $a \sim b$ is equivalent to $a \ll b$ and $b \ll a$, and that $a \ll b$ $(a \prec b)$ is equivalent to $a^* \ll b^*$ $(a^* \prec b^*)$.

PROPOSITION 2. *If* $0 \leqq a_n \uparrow a \in (c_0)_+$ *and* $0 \leqq b_n \uparrow b \in (c_0)_+$ *with* $a_n \ll b_n$ $(a_n \prec b_n)$ *for any* $n \in N$, *then* $a \ll b$ $(a \prec b)$ *necessarily holds.*

Proof. $a_n \ll b_n$ $(a_n \prec b_n)$ is equivalent to $a_n^* \ll b_n^*$ $(a_n^* \prec b_n^*)$. Hence $a^* \ll b^*$ $(a^* \prec b^*)$ is readily seen.

LEMMA 1. *If* $a \prec b$ $(a \ll b)$, *then there exist two sequences* $\{a_n\} \subset S_+$ *and* $\{b_n\} \subset S_+$ *such that* $a_n \prec b_n$ $(a_n \ll b_n)$ *and* $a_n \uparrow a$, $b_n \uparrow b$ *hold.*

Proof. If a, $b \in S_+$, then our theorem is clear. In the other case, firstly we shall prove that there are two sequences $\{\boldsymbol{a}_n^*\} \subset S_+$ and $\{\boldsymbol{b}_n^*\} \subset S_+$ such that $\boldsymbol{a}_n^* \uparrow a^*, \boldsymbol{b}_n^* \uparrow b^*$, and $\boldsymbol{a}_n^* \prec \boldsymbol{b}_n^*$ $(n \in N)$ hold. If $b \neq 0$ and $b \in S_+$, then there exsists a unique $k \in N$ so that $b_k^* > 0$, and $b_{k+1}^* = 0$ hold. For this k, choose any $j \in N$ so that $a_1^* + \cdots + a_j^* > b_1^* + \cdots + b_{k-1}^*$ holds, and put $j_0 = j$, $\boldsymbol{a}_n^* = (a_1^*, \ldots, a_{j_0}^*, \ldots, a_{j_0+n}^*, 0, 0, \ldots)$ and $\boldsymbol{b}_n^* = (b_1^*, \ldots, b_{k-1}^*, \sum_{i=1}^{j_0+n} a_i^* - \sum_{i=1}^{k-1} b_i^*, 0, 0, \ldots)$. On the other hand, if $b \notin S_+$, then there exists an unique $k_n \in N$ so that $a_1^* + \cdots + a_{k_n}^* \leqq b_1^* + \cdots + b_n^*$ and $a_1^* + \cdots + a_{k_n+1}^* > b_1^* + \cdots + b_n^*$ hold, for any $n \in N$, and set $\boldsymbol{b}_n^* = (b_1^*, \ldots, b_n^*, 0, 0, \ldots)$ and $\boldsymbol{a}_n^* = (a_1^*, \ldots, a_{k_n}^*, \sum_{i=1}^{n} b_i^* - \sum_{i=1}^{k_n} a_i^*, 0, 0, \ldots)$. Then $\{\boldsymbol{a}_n^*\}$ and $\{\boldsymbol{b}_n^*\}$ satisfy our requirements. Secondly, according to Definition 1, if $a \notin S_+$, then there exists a one-to-one mapping ϕ: $N \to A_+$ which satisfies (3. 2), and we define $\bar{a}_n(i) = \bar{a}_n(\phi(j)) = \boldsymbol{a}_n^*(j)$ for any $i \in A_+$, and $\bar{a}_n(i) = 0$ for any $i \notin A_+$, where n is any positive integer. On the other hand, if a belong to S_+, then there exists a permutation Π over N such that $a(\Pi(j)) = a^*(j)$ holds. For this case, set $\bar{a}_n(i) = \bar{a}_n(\Pi(j)) = \boldsymbol{a}_n^*(j)$. If we define $\bar{b}_n$ similarly as above, $\{\bar{a}_n\}$ and $\{\bar{b}_n\}$ satisfy the whole requirements in our theorem. Finally, if $a \ll b$, then a proof of our theorem is obtained similarly as above.

LEMMA 2. *If* ϕ: $R_+ \to R$ *is convex with* $\phi(0) = 0$, *then* $\sum_{i=1}^{\infty} \phi(a_i)$ *is defined for any* $a \in (c_0)_+$, *and the next holds*:

$$0 \leqq a_n \uparrow a \in (c_0)_+ \text{ implies } \lim_{n\to\infty} \sum_{i=1}^{\infty} \phi(a_n(i)) = \sum_{i=1}^{\infty} \phi(a(i)) \tag{3.11}$$

Proof. If ϕ: $R_+ \to R$ is convex with $\phi(0)=0$, then the next four cases occur:

(C_1) $\phi(t)$ is non-decreasing on R_+, and hence continuous at $t=0$,

(C_2) $\phi(t)$ is non-increasing on R_+,

(C_3) there exist t_1, $t_2>0$ so that $\phi(t_1)$ $\phi(t_2)<0$,

and

(C_4) $\phi(t)$ is non-decreasing on $(0, \infty)$ and non continuous at $t=0$.

We recall that

$$a \in (c_0)_+ \text{ is equivalent to } d_a(\lambda) = \text{Card } \{i: a(i)>\lambda\} < \infty \tag{3.12}$$

for any $\lambda>0$; hence $\sum_{i=1}^{\infty} \phi(a(i))$ is defined for all convex functions ϕ with $\phi(0)=0$, which ma be $+\infty$ or $-\infty$. Besides, $\phi(\cdot)$ is necessarily continuous at any $t>0$, therefore it is easy to see that

$$0 \leqq a_n \uparrow a \in (c_0)_+ \text{ implies } \lim_{n\to\infty} \phi(a_n(i)) = \phi(a(i)) \tag{3.13}$$

for any $i \in N$. In the case C_1 or C_2, (3.11) follows from Levi's Monotone Convergence Theorem with (3.13), and in the case C_3, there exists an $\alpha>0$ such that $\phi(t)$ is non-increasing on $[0, \alpha]$, and non-decreasing on $[\alpha, \infty)$, Set $A_1 = \{i: a(i) \leqq \alpha\}$ and $A_2 = \{i: a(i)>\alpha\}$. Then A_2 is a finite set of indices; hence follows

$$\lim_{n\to\infty} \sum_{i\in A_2} \phi(a_n(i)) = \sum_{i\in A_2} \phi(a(i)). \tag{3.14}$$

On the other hand, if $i \in A_1$, then

$$\phi(a(i)) \leqq \phi(a_{n+1}(i)) \leqq \phi(a_n(i)) \leqq 0 \qquad (n \in N)$$

holds, and we have

$$\lim_{n\to\infty} \sum_{i\in A_1} \phi(a_n(i)) = \sum_{i\in A_1} \phi(a(i)), \tag{3.15}$$

again by Levi's theorem. Consequenty, (3.11) follows from (3.14) and (3.15). Finally, in the case C_4, there exists an $\alpha_0>0$ so that $\phi(\alpha_0)=0$, set $B_1 = \{i: 0 < a(i) \leqq \alpha_0\}$ and $B_2 = \{i: a(i) > \alpha_0, \}$, where B_2 is also a finite set of indices. If we note that $\phi(a_n(i)) \leqq \phi(a_{n+1}(i)) \leqq \phi(a(i)) \leqq 0$ holds for any $i \in B_1$, it is easy to see that $\sum_{i\in B_1} \phi(a_n(i)) = -\infty$ follows from $\sum_{i\in B_1} \phi(a(i)) = -\infty$. Moreover,

$$\phi(t) \leqq -\frac{\phi(0_+)}{\alpha_0} t + \phi(0_+) \leqq 0 \tag{3.16}$$

holds for any $t \in (0, \alpha_0]$, so we can claim that B_1 is again a finite set of indices, provided $\Sigma \phi(a(i)) \neq -\infty$. The rest of the proof is easy.

THEOREM 4. *Suppose* $a, b \in (c_0)_+$, *then*,

(1) $$a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} \phi(a) \leqq \sum_{i=1}^{\infty} \phi(b) \tag{3.17}$$

for all non-decreasing convex functions ϕ: $R_+ \to R$ *with* $\phi(0)=0$. *In particular*,

(2) $$a \ll b \text{ is equivalent to } \sum_{i=1}^{\infty} (a_i - u)^+ \leqq \sum_{i=1}^{\infty} (b_i - u)^+ \tag{3.18}$$

for all positive real numbers u.

(3) $$a \prec b \text{ is equivalent to } \sum_{i=1}^{\infty} \phi(a_i) \leqq \sum_{i=1}^{\infty} \phi(b_i) \tag{3.19}$$

for all convex functions ϕ: $R_+ \to R$ *with* $\phi(0)=0$.

Proof. According to Lemma 1, if $a, b \in (c_0)_+$ satisfy $a \ll b$, then there exist two sequences $\{a_n\}$ and $\{b_n\} \subset S_+$ which satisfy

$$a_n \uparrow a, \ b_n \uparrow b, \text{ and } a_n \ll b_n.$$

Then $\sum_{1=1}^{\infty} \phi(a_n(i)) \leqq \sum_{1=1}^{\infty} \phi(b_n(i))$ follows from (2.3), where ϕ is any non-decreasing convex function on R_+, and the necessary conditions in (3.17), and in (3.18) follow from Lemma 2. Now we recall that

$$(x-u-v)^+ = ((x-u)^+ - v)^+$$

holds for any $u, v > 0$. If $\sum_{i=1}^{\infty} (a_i - u)^+ \leqq \sum_{i=1}^{\infty} (b_i - u)^+$ is valid for any $u > 0$, then

$$\sum_{i=1}^{\infty} ((a_i - u)^+ - v))^+ \leqq \sum_{i=1}^{\infty} ((b_i - u)^+ - v))^+ \tag{3.20}$$

is so, for any $u, v > 0$. Since $(a-u)^+ = ((a_1-u)^+, (a_2-u)^+, \ldots)$ and $(b-u)^+ = ((b_1-u)^+, (b_2-u)^+, \ldots)$ belong to S_+, $(a-u)^+ \ll (b-u)^+$ follows from (2.4) and

(3.20), and Proposition 2 implies $a \ll b$. Thus (3.17) and (3.18) are obtained. The sufficient condition in (3.19) is easily obtained if we put $\phi(t) = -t$, and the converse implication is also obtained similarly as above.

COROLLARY 3. *If* ϕ: $R_+ \to R_+$ *is non-decreasing and convex, with* $\phi(0) = 0$, *then*

$$a \ll b \text{ implies } (\phi(a_1), \phi(a_2), \cdots) \ll (\phi(b_1), \phi(b_2), \cdots).$$

Proof. If we put $\psi(t) = (t - u)^+ \circ \phi(t)$, then ψ is again a non-decreasing convex function with $\psi(0) = 0$, and (3.17) and (3. 18) imply

$$(\phi(a_1), \phi(a_2), \cdots) \ll (\phi(b_1), \phi(b_2), \cdots)^*.$$

EXAMPLE 1.

(1) If $a \ll b$, then $||a||_p \leqq ||b||_p$**

necessarily holds for any $p \geqq 1$, where $||\cdot||_p$ denotes the (l^p) norm, whether the right side is finite or infinite.

(2) If $a \prec b$, then $||b||_q \leqq ||a||_q$

necessarily holds for any $0 < q \leqq 1$, where $||\cdot||_q$ denotes the formal (l^q) norm, whether the right side is finite or infinite.

EXAMPLE 2. Suppose $a \prec b$, then $h(b) \leqq h(a)$ necessarily holds, where $h(a) = -\sum_{i=1}^{\infty} a_i \log a_i$ denotes an entropy of $a \in (c_0)_+$, provided $0 \cdot \log 0 = 0$.

References

[1] K. M. Chong and N. M. Rice, Equimeasurable rearrangements of functions, Queen's Papers in Pure and Applied Mathematics, No. 28, 1971.

[2] K. M. Chong, Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications, Can. J. Math. 26 (1974), 1321–1340.

[3] ———, An induction theorem for rearrangements, Can. J. Math. 28 (1976), 154–160.

[4] ———, A genral induction theorem for rearrangements of n-tuples, J. Math. Anal. Appli. 53 (1976), 426–437.

[5] G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities", Cambridge University Press, London and New York, 1967.

[6] W. A. J. Luxemburg, Rearrangement invariant Banach function spaces, Queen's

* This argument is borrowed from Chnog [2, P. 1330].
** If a $\in l^1$, then our example is easily obtained from [10, Examples (1), p. 19].

Papers in Pure and Applied Mathematics 10 (1967), 83–144.

[7] A. S Markus, Eigenvalues and singular values of the sum and product of linear operators, Uspehi Mat. Nauk 19, No. 4, 93–123. (Russian Math. Surveys 19 (1964), 91-120.

[8] A. W. Marshall and I. Olkin, "Inequalities: Theory of majorization and its applications", Mathematics in Science and Engineering, Vol. 143, Academic Press, 1979.

[9] G. Pólya, Remark on Weyl's note: Inequalities between two kinds of eigenvalues of a linear transformation, Proc. Nat. Acad. Sci., 36 (1950), 49–51.

[10] Y. Takeuchi, A new rearrangement and the theorem of Hardy–Littlewood–Pólya, J. Fac. Eng., Shinshu Univ., No. 41 (1976), 1–21.