On a Theorem Concerning Minimization Problems on a Network

Yukio TAKEUCHI*

(Received May 31, 1978)

In this paper, we give another proof for Iri's theorem concerning minimization problems on a network. This proof asserts that a sum of integrals of characteristic curves which depend on flows, is minimized by some flow on a network, if the characteristic curves of branches satisfy the following conditions:

(1) every characteristic curve is monotonically increasing outside of some finite interval;

(2) if a characteristic curve has an upper bound, then the curve is a horizontal line in the right outside of some finite interval;

(3) if a characteristic curve has a lower bound, then the curve is a horizontal line in the left outside of some finite interval;

(4) every characteristic curve is bounded on any finite interval.

The proof is short in process and is natural in method. Furthermore, the proof does not require a deeper understanding of graph theory.

1 Introduction

A network-flow problem is, mathematically, a special case of mathematical programming problems, in which the constraint relations imposed on variables are intimately connected with a graph. M. Iri proved a useful theorem concerning minimization problems on a network in his book[1] and his proof requires a deeper understanding of graph theory. In this paper, we shall prove this theorem and the proof is slightly simpler than that of Iri's book. Furthermore, the proof does not require a deeper understanding of graph theory.

The author would like to express his hearty thanks to Professor Y. NAKAMURA for his many valuable advices and encouragements in the course of preparing this paper, and he is also grateful to professor W. TAKAHASHI for his enlightening comments and discussions.

2 Preliminaries

We give here the notational conventions and definitions to be used through this paper.

*Assistant, Department of Information Engineering

The set of real numbers will be denoted by R, and R^n will be denoted n-dimensional Euclidian space.

Let $A=(a_{ij})$ be an $m \times n$ matrix which satisfies the following three conditions:

(1) *for all* i, j, $|a_{ij}|=1$ *or* $|a_{ij}|=0$;

(2) *for each* j, *there exists exactly one* i *such that* $a_{ij}=1$;

(3) *for each* j, *there exists exactly one* i *such that* $a_{ij}=-1$.

This matrix A can be thought of as a continuous linear transformation of R^m into R^n. The null space of this transformation will be denoted by X.

Let $f_{j(j=1,2,\cdots,n)}$ be functions of R into R, which are continuous expect on finitepoints, and which satisfy the following three conditions:

(1) $\lim_{t\to\infty} f_j(t)=\infty$ *or there exist* $M_j>0$ *and* $N_j\in R$ *such that* $f_j(t)=N_j$ *for all* $t\geq M_j$;

(2) $\lim_{t\to-\infty} f_j(t)=-\infty$ *or there exist* $K_j>0$ *and* $L_j\in R$ *such that* $f_j(t)=L_j$ *for all* $t\leq -K_j$;

(3) *for every* $c>0$, *there exists* $M_c>0$ *such that* $|f_j(t)|\leq M_c$ *for all* $-c\leq t\leq c$.

We put $N=\sum_{\lim_{t\to\infty} f_j(t)\neq\infty}(M_j+|N_j|)+\sum_{\lim_{t\to-\infty} f_j(t)\neq-\infty}(K_j+|L_j|)$. Then for each j satisfying $\lim_{t\to\infty} f_j(t)=\infty$, there exists $M_j'>0$ such that $f_j(t)\geq N+1$ for all $t\geq M_j'$, and for each j satisfying $\lim_{t\to-\infty} f_j(t)=-\infty$, there exists $K_j'>0$ such that $f_j(t)\leq -(N+1)$ for all $t\leq -K_j'$.

We set $L=\sum_{\lim_{t\to\infty} f_j(t)=\infty} M_j' + \sum_{\lim f_j(t)=-\infty} K_j'+(N+1)$.

Let $g_{j(j=1,2,\cdots,n)}$ be functions of R into R, defined by $g_j(t)=\int_0^t f_j(s)\,ds$ and let F be a function of R^n into R, defined by the formula $F(x)=\sum_{j=1}^n g_j((x)_j)$ where $(x)_j$ is the j-th coordinate of x.

Let $I_{j(j=1,2,\cdots,n)}$ be closed intervals which take one of the forms $[a, b]$, $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$. We set $I=X\cap(I_1\times I_2\times\cdots\times I_n)$, and assume $I\neq\emptyset$.

3 A theorem concerning minimization problems on a network

THEOREM. *If* $\inf_{x\in I} F(x)=a>-\infty$, *then there exists a point* $x_0\in I$ *such that* $F(x_0)=a$.

Proof. By our assumption of $\inf_{x\in I} F(x)=a>-\infty$, there exists a sequence $\{x_k\}$ in I such that $F(x_k)\downarrow a$. If $\{x_k\}$ has only finitely many points, the conclusion of our theorem is obvious, so we may assume that $\{x_k\}$ has infinitely many distinct points. Let $B_1=\{j;$ *the sequence* $\{(x_k)_j\}$ *in* I_j *has a limit point*$\}$. It is clear that there exists $M>L$ which satisfies the following two conditions:

(1) *for each* $j \in B_1$, *there exists a limit point of* $\{(x_k)_j\}$ *in the closed interval* $[-(M-1), M-1]$;

(2) *for all* j, $[-(M-1), M-1] \cap I_j \neq \emptyset$.

It is also clear that this sequence has a subsequence $\{y_k\}$ which satisfies the following conditions:

(1) $F(y_1)-a<1$;

(2) *for all* $j \in B_1$ *and* k, $(y_k)_j \in [-M, M]$;

(3) *for all* $j \notin B_1$, $M<(y_k)_j \uparrow \infty$ *or* $-M>(y_k)_j \downarrow -\infty$.

Let $B_2=\{j \notin B_1; f_j((y_1)_j)=f_j((y_2)_j)=\cdots\}$ *and* $B_3=\{j; j \notin (B_1 \cap B_2)\}$, i. e., $B_3=\{j \notin B_1; f_j((y_k)_j)\uparrow \infty$ *or* $f_j((y_k)_j)\downarrow -\infty\}$.

We now define D and I_0 by $D=[-n^nM, n^nM]$, $I_0=I \cap D$ and prove that there exists a sequence $\{z_k\}$ in I_0 which satisfies the condition $F(z_k)\leq F(y_k)$ for all k. Let k be a natural number which is fixed throughout this paragraph. We now construct such a z_k. If $|(y_k)_{j_1}|>n^nM$ for some j_1, then by the definition of A there exist i_1 and i_2 such that $a_{i_2j_1}=-a_{i_1j_1}\neq 0$. If there exists no j_2 such that $j_1 \neq j_2, a_{i_2j_2}\neq 0$ and $|(y_k)_{j_2}|>n^{n-1}M$, then

$$\left|\sum_{j=1}^{n} a_{i_2j}(y_k)_j\right| \geq \left| a_{i_2j_1}(y_k)_{j_1}\right| - \left|\sum_{j\neq j_1} a_{i_2j}(y_k)_j\right| > n^nM-(n-1)n^{n-1}M>0,$$

which contradicts $y_k \in X$. We continue this operation in this manner, and we have $E=\{j_v, j_{v+1}, \cdots, j_u\}$ and $E'=\{i_v, i_{v+1}, \cdots, i_u\}_{(1\leq v\leq u\leq n)}$ which satisfy the following two conditions:

(1) *for all* $j \in E$, $|(y_k)_j|>M$;

(2) $a_{i_vj_v}=-a_{i_{v+1}j_v}\neq 0$, $a_{i_{v+1}j_{v+1}}=-a_{i_{v+2}j_{v+1}}\neq 0$, $\cdots$, $a_{i_uj_u}=-a_{i_vj_u}\neq 0$.

Let $\alpha=\min_{j\in E}|(y_k)_j|-M+1$ and let y_k' be a point in R^n defined by

$$(y_k')_j=\begin{cases}(y_k)_j-(y_k)_j\alpha/|(y_k)_j| & if\ j\in E,\\ (y_k)_j & if\ j\notin E.\end{cases}$$

The fact $y_k' \in I$ follows from the definition of M and

$$\sum_{j=1}^{n} a_{ij}(y_k)_j-\sum_{j=1}^{n} a_{ij}(y_k')_j=\sum_{j\in E} a_{ij}((y_k)_j\alpha/|(y_k)_j|)=0.$$

We now show $E \subset B_2$. It is clear that $E\cap B_1=\emptyset$. We therefore assume that there exists a $j_0 \in E\cap B_3$, and we deduce a contradiction from this assumption. By the definition of N, we have

$$F(y_k)-F(y_k')$$

$$=\sum_{j\in E\cap B_2}(g_j((y_k)_j)-g_j((y_k')_j))+\sum_{j\in E\cap B_3}(g_j((y_k)_j)-g_j((y_k')_j))$$

$$\geq -N\alpha + (g_{i_0}((y_k)_{j_0}) - g_{j_0}((y_k')_{j_0}))$$
$$\geq -N\alpha + (N+1)\alpha = \alpha > 1,$$

which implies that $F(y_k') < F(y_k) - 1 < a$. By this contradiction, we have $E \subset B_2$. We now prove that $F(y_k') \leq F(y_k)$. We do this by assuming that $F(y_k') - F(y_k) = b > 0$ and by deriving a contradictory equation $\inf_{x \in I} F(x) = -\infty$ from this assumption

We define $y^t{}_{k\,(t=1,\,2,\,\cdots)}$ by

$$(y^t{}_k)_j = \begin{cases} (y_k)_j + (y_k)_j \alpha t / |(y_k)_j| & if\ j \in E; \\ (y_k)_j & if\ j \notin E. \end{cases}$$

$y^t{}_k \in I$ is obvious. By $E \subset B_2$, it is easy to see that

$$F(y^t{}_k) = F(y_k) - bt \downarrow -\infty \quad as\ t \to \infty.$$

If we continue this process at most n times, we get a point $z_k \in I_0$ such that $F(z_k) \leq F(y_k)$. This fact shows that $\inf_{x \in I_0} F(x) = a$.

We are now in a position to complete the proof of our theorem. Since I_0 is compact by Tychonoff's theorem[2] and since F is continuous, there exists a point $x_0 \in I_0$ such that $F(x_0) = a$.

Referrences

1) Iri, M., Network Flow, Transportation and Scheduling: Theory and Algorithms. Academic Press, (1968).
2) Dunford, N, and J. T. Schwartz, Linear Operator I. Interscience, New York, (1958).
3) Rockafellar, R. T., Convex Analysis. Princeton University Press, (1969).