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Approximation Property of Functions and the Absolute Nörlund Summability of Fourier Series

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1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$.

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \cdots\cdots + p_n;\ P_{-k} = p_{-k} = 0,\ \text{for } k \geqq 1.$$

The sequence $\{t_n\}$, given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} P_k a_{n-k},\quad (P_n \neq 0),$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence of constants $\{p_n\}$.

The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the series

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

is convergent.

In the special cases in which $p_n = A_n^{\delta-1} = \binom{n+\delta-1}{n}$, and $p_n = 1/(n+1)$, summability $|N, p_n|$ are the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively.

Let $f(x)$ be a periodic function with period 2π and integrable (L) over $(0, 2\pi)$. We assume that the Fourier series of $f(x)$ is given by

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

Throughout the paper, we use the following notations:

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$$\omega(\delta) = \sup_{0<|h|\leqq\delta} |f(x+h) - f(x)|,$$

$$\omega_p(\delta) = \sup_{0<|h|\leqq\delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx\right)^{1/p},$$

$$E_n^{(p)}(f) = \|f - T_n(f)\|_p$$

where $1 \leqq p \leqq \infty$ and $T_n(f)$ is a trigonometric polynomial of the best approximation of order n for $f(x)$ with respect to the corresponding norm.

Let p' be the conjugate index of p, i. e. $1/p + 1/p' = 1$.
We write $\Delta p_n = p_n - p_{n+1}$.

A will denote a positive constant which will not necessarily be the same at different occurrences.

2. M. and S. Izumi [3] and S. N. Lal [6] proved the following theorem.

Theorem A. *Let $\{p_n\}$ and $\{\Delta p_n\}$ are both non-negative and non-increasing sequences. If the conditions*

$$\sum_{n=1}^{\infty} p_n^p n^{p-2} < \infty \quad (1 < p \leqq 2)$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1/p'} P_n} < \infty$$

hold, then the Fourier series of $f(x)$ is summable $|N, p_n|$.

Supplementing the result of Theorem A, S. M. Shah [9] proved the following theorem, which is closely concerned with Zygmund's result [16; pp. 241–242].

Theorem B. *Let $\{p_n\}$ and $\{\Delta p_n\}$ are both non-negative and non-increasing sequences. Let $f(x)$ be a 2π-periodic function of bounded variation over $[0, 2\pi]$ and suppose that the condition*

$$c_1 n^{\gamma} \varphi(n) \leqq P_n \leqq c_2 n^{\gamma} \varphi(n), \quad 0 \leqq \gamma < 1/2$$

holds where c_1 and c_2 are fixed positive constants and $\varphi(x)$ is positive on $[0, \infty)$ and slowly oscillating in the sense of Karamata.

If the conditions

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)}{n} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\omega(1/n)^{1/2}}{n P_n} < \infty$$

hold, then the Fourier series of $f(x)$ is summable $|N, p_n|$.

On the other hand, S. N. Lal [4, 5] also proved the following theorem.

Theorem C. *Let $f(x)$ be a function belonging to L^p $(1 < p \leqq 2)$. Let $\{p_n\}$ and $\{\Delta p_n\}$ are both non-negative and non-increasing sequences.*

If the conditions

$$\sum_{n=1}^{\infty} p_n^p \, n^{p-2} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\omega_p(1/n)}{n^{1/p'} P_n}$$

hold, then the Fourier series of $f(x)$ is summable $|N, p_n|$ almost everywhere.

S. N. Lal assumes the convergence of the series $\sum_{n=1}^{\infty} P_n^p \, n^{-2}$ in place of the series $\sum_{n=1}^{\infty} p_n^p \, n^{p-2}$. But we can easily show that the convergence of the series $\sum_{n=1}^{\infty} P_n^p \, n^{-2}$ is equivalent to that of the series $\sum_{n=1}^{\infty} p_n^p \, n^{p-2}$ under the condition that $\{p_n\}$ is non-increasing.

Also, Theorem C is a generalization of Theorem D, which is due to N. Matsuyama [7].

Theorem D. *If $f(x)$ belongs to L^p $(1 < p \leqq 2)$ and the series*

$$\sum_{n=1}^{\infty} \frac{\omega_p(1/n)}{n^{1/p'+\delta}}$$

converges, then the Fourier series of $f(x)$ is summable $|C, \delta|$ almost everywhere, where $-1 < \delta < 1/p$.

N. Matsuyama established this theorem as the analogue of Hyslop's theorem [2], which is easily deduced from Theorem A.

In this note, the author deduces several results from Theorem C by the same method as that used by C. Watari and Y. Okuyama [14].

3. The following theorem is well-known.

Theorem of Denjoy-Lusion. *If the $\sum_{n=0}^{\infty} A_n(x)$ converges absolutely for x belonging to a set P of positive measure, then $\sum_{n=0}^{\infty} (|a_n| + |b_n|)$ converges.*

Using a result by A. Zygmund [15], G. Sunouchi [10] proved the following Theorem.

Theorem E. *There exists a function in L^2 which is summable $|C, 1|$ in (a, b) in $(0, 2\pi)$, but not summable $|C, 1|$ almost everywhere in the complementary interval.*

To extend this theorem, we require Tsuchikura's theorem [12, 13], which is as follows.

Theorem F. *If* $f(x)$ *belongs to* L^p $(1<p\leqq 2)$ *and if, for some* $\varepsilon>0$,

$$\int_0^t |f(x+t)+f(x-t)-2f(x)|^p dx = O\left(|t|\left(\log\frac{1}{|t|}\right)^{-p-\varepsilon}\right)$$

as $t\to 0$, *at a point* x, *then the Fourier series of* $f(x)$ *is summable* $|C, \delta|$ $(\delta>1/p)$ *at the point.*

Following Sunouchi's argument and applying Tsuchikura's Theorem, we can obtain the following theorem.

Theorem 1. *For* $\delta>1/2$, *there exists a function in* L^2 *which is summable* $|C, \delta|$ *in* (a, b) *in* $(0, 2\pi)$, *but not summable* $|C, \delta|$ *almost everywhere in the complementary interval.*

Thus this theorem shows that the $|C, \delta|$-analogue of the Denjoy-Dusin theorem does not hold for $\delta>1/2$. This theorem is open for $0<\delta\leqq 1/2$.

4. One of the fundamental theorems of the constructive theory of function is a reciprocal relation between $E_n^{(p)}(f)$ and $\omega_p(1/n)$, that is to say,

$$E_n^{(p)}(f)\leqq A\omega_p(1/n)$$

and conversely

$$\omega_p(1/n)\leqq A\ n^{-1}\sum_{k=0}^{n} E_k^{(p)}(f).$$

Suppose that $\sum_{n=k}^{\infty} 1/n^{1+1/p'}\ P_n = O(1/k^{1/p'}P_k)$. Then we have

$$\sum_{n=1}^{\infty}\frac{\omega_p(1/n)}{n^{1/p'}P_n}\leqq A\sum_{n=1}^{\infty}\frac{1}{n^{1+1/p'}P_n}\sum_{k=0}^{n}E_k^{(p)}(f)$$

$$\leqq A\sum_{k=1}^{\infty}E_k^{(p)}(f)\sum_{n=k}^{\infty}\frac{1}{n^{1+1/p'}P_n}$$

$$\leqq A\sum_{k=1}^{\infty}\frac{E_k^{(p)}(f)}{k^{1/p'}P_k}.$$

Hence, if $P_n=A_n^{\delta}$, we assume that $\delta>-1/p'$.
Thus Theorem C can be restated as follows.

Theorem 2. *Let* $\{p_n\}$ *and* $\{\Delta p_n\}$ *are both non-negative and non-increasing sequences.*

If the conditions

$$\sum_{n=1}^{\infty} p_n^p\, n^{p-2} < \infty \ (1 < p \leqq 2)$$

and

$$\sum_{n=1}^{\infty} \frac{E_n^{(p)}(f)}{n^{1/p'} P_n} < \infty$$

hold, then the Fourier series of $f(x)$ is summable $|N,\ p_n|$ almost everywhere.

To reduce several theorems from this theorem, we need a lemma, which is due to C. Watari and Y. Okuyama [14].

Lemma. *Let $f(x)=\sum_{n=1}^{\infty} A_n(x) \in L^p$ and let $E_n^{(p)}(f)=O(n^{-\alpha})$ for some $\alpha>0$. Then, for any $\beta<\alpha$, there exists a function $f^{[\beta]}(x)$ belonging to L^p such that*

$$f^{[\beta]}(x) \sim \sum_{n=1}^{\infty} n^{\beta} A_n(x)$$

and

$$E_n^{(p)}(f^{[\beta]}) = O(n^{-\alpha+\beta}).$$

Theorem 3. *If $E_n^{(2)}(f)=O(n^{-\alpha}), \alpha>0$ and $\beta<\alpha$, then the series $\sum_{n=1}^{\infty} n^{\beta} A_n(x)$ is summable $|C,\ \delta|$ almost everywhere, where $\delta>1/2+\beta-\alpha$ and $\delta>-1/2$.*

Proof. By lemma, we have

$$E_n^{(2)}(f^{[\beta]}) = O(n^{-\alpha+\beta})$$

and

$$f^{[\beta]}(x) \sim \sum_{n=1}^{\infty} n^{\beta} A_n(x).$$

Thus we obtain

$$\sum_{n=1}^{\infty} \frac{E_n^{(2)}(f^{[\beta]})}{n^{1/2} P_n} = O\left(\sum_{n=1}^{\infty} \frac{1}{n^{1/2+\delta+\alpha-\beta}}\right) < \infty \quad \text{for } \delta > 1/2+\beta-\alpha.$$

By Theorem 2, Theorem 3 is completed.

The case $\beta=0$ of this theorem is an analogue of the result due to J. M. Hyslop [2].

Theorem 4. *Let $\beta<\alpha$. If $E_n^{(1)}(f)=O(1/n)$ and $E_n^{(\infty)}(f)=O(n^{-\alpha})$, then the series $\sum_{n=1}^{\infty} n^{\beta} A_n(x)$ is summable $|C,\ \delta|$ almost everywhere, where $\delta>\beta-\alpha/2$ and $\delta>-1/2$.*

Proof. By Lemma, we have

$$E_n^{(1)}(f^{[\beta]}) = O(n^{-1+\beta}),\quad E_n^{(\infty)}(f^{[\beta]}) = O(n^{-\alpha+\beta})$$

and

$$f^{[\beta]}(x) \sim \sum_{n=1}^{\infty} n^{\beta} A_n(x).$$

Since $\left(E_{2n}^{(2)}(f^{[\beta]})\right)^2 \leqq A\, E_n^{(1)}(f^{[\beta]})\, E_n^{(\infty)}(f^{[\beta]})$(see [14]), we have

$$\sum_{n=1}^{\infty} \frac{E_n^{(2)}(f^{[\beta]})}{n^{1/2} P_n} = O\left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta-\beta+\alpha/2}}\right) < \infty \quad \textit{for } \delta > \beta - \alpha/2.$$

Hence Theorem 4 is proved by Theorem 2.

Since $f(x) \in BV(0,\ 2\pi)$ implies $E_n^{(1)}(f) = O(1/n)$, the case $\beta=0$ of Theorem 4 is an analogue of results due to H. C. Chow [1].

Theorem 5. *Let* $1 < p \leqq 2$. *If* $E_n^{(p)}(f) = O(n^{-\alpha})$ $(\alpha > 0)$ *and* $\beta < \alpha$, *then the series* $\sum_{n=1}^{\infty} n^{\beta} A_n(x)$ *is summable* $|C,\ \delta|$ *almost everywhere, where* $\delta > 1/p + \beta - \alpha$ *and* $\delta > -1/p'$.

Proof. By Lemma, we have

$$E_n^{(p)}(f^{[\beta]}) = O(n^{-\alpha+\beta})$$

and

$$f^{[\beta]}(x) \sim \sum_{n=1}^{\infty} n^{\beta} A_n(x).$$

Thus we obtain

$$\sum_{n=1}^{\infty} \frac{E_n^{(p)}(f^{[\beta]})}{n^{1/p'} P_n} = O\left(\sum_{n=1}^{\infty} \frac{1}{n^{1-1/p+\delta-\beta+\alpha}}\right) < \infty \quad \textit{for } \delta > 1/p + \beta - \alpha.$$

Hence we establish Theorem 5 by Theorem 2.

Also, the case $\beta = 0$ of this theorem is an analogue of results due to H. C. Chow [1].

In the case $\beta=0$, the reader is referred to H. C. Chow [1] and L. Mcfadden [8] for the results on the absolute Cesàro summability of Fourier series of a function which belongs to the class $Lip(\alpha, p)$.

We see from Lusin-Denjoy's theorem that the following Corollaries 1, 2 and 3 are the results which are deduced from Theorems 3, 4 and 5, respectively.

Corollary 1. $E_n^{(2)}(f) = O(n^{-\alpha})$, $\alpha > 0$ *and* $\beta < \alpha$ *imply* $\sum_{n=1}^{\infty} n^{\beta-1/2}(|a_n| + |b_n|) < \infty$.

Corollary 2. $E_n^{(1)}(f) = O(1/n)$ *and* $E_n^{(\infty)}(f) = O(n^{-\alpha})$ *together imply, for* $(0<)$ $\beta<\alpha$, $\sum_{n=1}^{\infty} n^{\beta/2}(|a_n| + |b_n|) < \infty$.

Corollary 3. $E_n^{(p)}(f) = O(n^{-\alpha})$ $(\alpha > 0)$, $1 < p \leqq 2$, *and* $\beta < \alpha - 1/p$ *imply* $\sum_{n=1}^{\infty} n^{\beta}(|a_n| + |b_n|) < \infty$.

These corollaries are due to C. Watari and Y. Okuyama [14]. Also see A. Zygmund [16].

Next, we can obtain the analogue of Theorem B. Our results read as follows:

Theorem 6. *Let* $\{p_n\}$ *and* $\{\Delta p_n\}$ *are both non-negative and non-increasing sequences. Let* $f(x)$ *be a* 2π*-periodic function of bounded variation over* $[0, 2\pi]$ *and suppose that the conditions*

$$\sum_{n=1}^{\infty} p_n^2 < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{E_n^{(\infty)}(f)^{1/2}}{nP_n} < \infty$$

hold, then the Fourier series of $f(x)$ *is summable* $|N, p_n|$ *almost everywhere.*

Since a funcion of bounded variation has the property $E_n^{(1)}(f) = O(1/n)$, Theorem 6 is contained in the following theorem.

Theorem 7. *Let* $\{p_n\}$ *and* $\{\Delta p_n\}$ *are both non-negative and non-increasing sequence. If the conditions*

$$E_n^{(1)}(f) = O(1/n),$$

$$\sum_{n=1}^{\infty} p_n^2 < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{E_n^{(\infty)}(f)^{1/2}}{nP_n} < \infty$$

hold, then the Fourier series of $f(x)$ *is summable* $|N, p_n|$ *almost everywhere.*

Proof. As stated above, we have

$$\left(E_{2n}^{(2)}(f) \right)^2 \leqq E_n^{(1)}(f) \ E_n^{(\infty)}(f).$$

Thus we have

$$\sum_{n=1}^{\infty}\frac{E_n^{(2)}(f)}{n^{1/2}P_n}\leqq A\sum_{n=1}^{\infty}\frac{E_n^{(\infty)}(f)^{1/2}}{nP_n}<\infty.$$

Hence Theorem 7 is proved by Theorem 2.

5. We assume that a sequence $\Phi=\{\varphi_n\}$ forms a complete orthonormal system over a set of finite measure and Φ_n is the linear space spanned by the first n elements of Φ. Morever, we suppose that the system Φ under consideration has the following properties:

(1°) (Nikolsky property) For $1\leqq p<q\leqq\infty$, and $P\in\Phi_n$, we have $||P||_q\leqq An^{\alpha}||P||_p$, $\alpha=1/p-1/q$.

(2°) (de la Vallee Poussin property) There exists a sequence of linear operators $G_n : L^1\rightarrow\Phi_{2n}$ such that (i) bounded, (ii) G_n leaves the element of Φ_n invariant, (iii) For $1\leqq p\leqq\infty$, we have

$$||f-G_n f||_p\leqq A\ E_n^{(p)}(f)=A\ inf\Big\{||f-P||_p : P\in\Phi_n\Big\}.$$

It is well known that these properties are held by the system of trigonometric functions as well as that of Walsh functions.

Now, we have the following theorem, which is due to C. Watari and Y. Okuyama [14] for $P_n=1$.

Theorem 8. *Let Φ have the properties* (1°) *and* (2°) *and let* $1\leqq p<q\leqq\infty$. *If we suppose that a sequence $\{P_n\}$ is a positive sequence such that $\{n^{1/p'}P_n\}$ is non-decreasing and* $\sum_{k=0}^{n}2^{k/q}/P_{2^k}=O(2^{n/q}/P_{2^n})$, *then we have*

$$\sum_{n=1}^{\infty}\frac{E_n^{(q)}(f)}{n^{1/q'}P_n}\leqq A\sum_{n=1}^{\infty}\frac{E_n^{(p)}(f)}{n^{1/p'}P_n}.$$

Proof. *As* $1/p'<1/q'$, we see that $\{n^{1/q'}P_n\}$ is non-decreasing. Hence, by Cauchy's condensation principle, what we have to prove is

$$\sum_{n=0}^{\infty}\frac{2^{n/q}E_{2^n}^{(q)}(f)}{P_{2^n}}\leqq A\sum_{n=0}^{\infty}\frac{2^{n/p}E_{2^n}^{(p)}(f)}{P_{2^n}}.$$

This is reduced, by property (2°), to

$$\sum_{n=0}^{\infty}\frac{2^{n/q}}{P_{2^n}}||f-G_{2^n}f||_q\leqq A\sum_{n=0}^{\infty}\frac{2^{n/p}}{P_{2^n}}||f-G_{2^n}f||_p$$

or, by the subadditivity and the property (1°), we see that

$$\begin{aligned}||f-G_{2^n}f||_q &\leqq \sum_{k=n}^{\infty}||G_{2^{k+1}}f-G_{2^k}f||_q \\ &\leqq \sum_{k=n}^{\infty}2^{k\gamma}||G_{2^{k+1}}f-G_{2^k}f||_p \quad (\gamma=1/p-1/q) \\ &\leqq \sum_{k=n}^{\infty}2^{k\gamma}||f-G_{2^k}f||_p.\end{aligned}$$

Therefore we have $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n/q}}{P_{2^n}}||f-G_{2^n}f||_q$

$$\begin{aligned}&\leqq A\sum_{n=0}^{\infty}\frac{2^{n/q}}{P_{2^n}}\sum_{k=n}^{\infty}2^{k\gamma}||f-G_{2^k}f||_p \\ &\leqq A\sum_{k=0}^{\infty}2^{k\gamma}||f-G_{2^k}f||_p\sum_{n=0}^{k}\frac{2^{n/q}}{P_{2^n}} \\ &\leqq A\sum_{k=0}^{\infty}2^{k(1/p-1/q)}||f-G_{2^k}f||_p\frac{2^{k/q}}{P_{2^k}} \\ &\leqq A\sum_{k=0}^{\infty}\frac{2^{k/p}}{P_{2^k}}||f-G_{2^k}f||_p. \qquad \text{q. e. d.}\end{aligned}$$

Our results are stated in terms of the best approximation, but there is a rather complete parallelism between the modulus of continuity and the best approximation. Hence the above results can be stated in terms of the modulus of continuity. From theorem 8, we have

Corollary 4. *Let* $1 \leqq p < q \leqq \infty$. *Then*

$$\sum_{n=1}^{\infty}\frac{\omega_q(1/n)}{n^{1/q'+\delta}} \leqq A\sum_{n=1}^{\infty}\frac{\omega_p(1/n)}{n^{1/p'+\delta}}$$

where $1/q > \delta > -1/p'$.

This corollary shows that Matsuyama criterion is best possible at $p=2$ for $0<\delta<1/2$.

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