On the Absolute Nörlund Summability of Fourier Series

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(Received October 24, 1973)

1. Definitions and Notations

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive numbers, and let us write :

$$P_n = p_0 + p_1 + \cdots\cdots + p_n;\ \ P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \sum_{r=0}^{n} \frac{p_{n-r} s_r}{P_n} \tag{1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $(N,\ p_n)$ to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and said to be absolutely summable $(N,\ p_n)$, or summable $|N,\ p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, the series $\sum |t_n - t_{n-1}|$ is convergent.

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \qquad (\alpha > 0), \tag{2}$$

the Nörlund mean reduces to the familiar $(C,\ \alpha)$ mean.

Thus the summability $|N,\ p_n|$, where p_n is defined by (2), is the same as the summability $|C,\ \alpha|$.

Let $f(t)$ be a periodic function with period 2π, and integrable in the Lebesque sense over $(-\pi,\ \pi)$. We assume, without any loss of generality, that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0, \tag{3}$$

and

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$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{4}$$

We use the notation

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

Throughout this note, A will denote positive constants which will not necessarily be the same at different occurrences.

2. Introduction

T. Singh [6] has proved the following theorem.

Theorem A. *If the function $\varphi(t)$ is of bounded variation on the interval* $(0, \pi)$ *and if the sequence $\{p_n\}$ is non-increasing and convex, and satisfies the condition*

$$\sum_{k=1}^{n} \frac{P_k}{k} = O(P_n) \qquad \textit{for all } n \geqq 1, \tag{5}$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N, p_n|$.

The following theorem is due to Y. Okuyama [4].

Theorem B. *If the function $\varphi(t)$ is of bounded variation on the interval* $(0, \pi)$ *and if the sequence $\{p_n\}$ is non-increasing and convex, and satisfies the condition*

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{1}{P_n}\right) \qquad \textit{for all } n \geqq 1, \tag{6}$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N, p_n|$.

Recently M. Izumi and S. Izumi [3] have proved the following theorem.

Theorem C. *Let $a \geqq 0$ and $K > \pi$ be a constant. If the function $\varphi(t)$* $\cdot (\log K/t)^a$ *is of bounded variation over the interval* $(0, \pi)$ *and if the sequence $\{p_n\}$ is non-increasing and satisfies the condition*

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{(\log n)^a}{P_n}\right) \quad \textit{for all } n \geqq 2, \tag{7}$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N, p_n|$.

In the case $a = 0$ of this theorem, we have

Theorem D. *If the function $\varphi(t)$ is of bounbed variation over the interval* $(0, \pi)$ *and if the sequence $\{p_n\}$ is non-increasing and satisfies the condition*

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{1}{P_n}\right) \qquad \textit{for all } n \geqq 1, \tag{8}$$

then the Fourier series of $f(t)$, at $t=x$, is summable $|N, p_n|$.

Hence Theorem D is a generalization of Theorem B. Also, M. Izumi and S. Izumi have described in their paper [3] that Theorem D is a generalization of Theorem A because the condition of convexity of $\{p_n\}$ is dropped and the condition (5) implies the condition (8). However, we show in our note, by proving Theorem 1, that the condition (5) is equivalent to the condition (8). Thus the "point" in the result of Theorem D is the dropping of the condition of the convexity of $\{p_n\}$ from the hypotheses of Theorem A.

3. Theorems

Theorem 1. *Let $\{p_n\}$ be a sequence of positive numbers. Then the condition*

$$\sum_{k=1}^{n} \frac{P_k}{k} = O(P_n)$$

is equivalent to the condition

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{1}{P_n}\right).$$

Proof. First we assume that

$$\sum_{k=1}^{n} \frac{P_k}{k} = O(P_n).$$

Let M be any integer. By Abel's transformation, we have

$$\begin{aligned} P_n \sum_{k=n}^{M} \frac{1}{kP_k} &= P_n \sum_{k=n}^{M} \frac{P_k}{k} \frac{1}{P_k^2} \\ &= P_n \sum_{k=n}^{M-1} \left(\sum_{r=1}^{k} \frac{P_r}{r}\right)\left(\frac{1}{P_k^2} - \frac{1}{P_{k+1}^2}\right) + P_n \left(\sum_{k=1}^{n-1} \frac{P_k}{k}\right) \frac{1}{P_n^2} \\ &\qquad + P_n \left(\sum_{k=1}^{M} \frac{P_k}{k}\right) \frac{1}{P_M^2} \\ &\equiv I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

Clearly, we have

$$I_2 = O(1) \text{ and } I_3 = O(1).$$

Next, we estimate

$$I_1 = P_n \sum_{k=n}^{M-1} \left(\sum_{r=1}^{k} \frac{P_r}{r}\right)\left(\frac{1}{P_k^2} - \frac{1}{P_{k+1}^2}\right)$$

$$\leqq P_n \sum_{k=n}^{M-1} O(P_k) \frac{p_{k+1}(P_{k+1}+P_k)}{P_k^2 P_{k+1}^2}$$

$$\leqq A P_n \sum_{k=n}^{M-1} \frac{p_{k+1}}{P_k P_{k+1}}$$

$$= A P_n \sum_{k=n}^{M-1} \left(\frac{1}{P_k} - \frac{1}{P_{k+1}} \right)$$

$$= O(1).$$

Since an integer M is arbitrary, we have

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{1}{P_n}\right).$$

Conversely we assume that

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{1}{P_n}\right).$$

Using the formula [1]

$$\sum_{k=m+1}^{n} \lambda_k u_k = \sum_{k=m+2}^{n} (\lambda_k - \lambda_{k+1}) \sum_{r=k}^{\infty} u_r + \lambda_{m+1} \sum_{r=m+1}^{\infty} u_r - \lambda_n \sum_{r=n+1}^{\infty} u_r,$$

we have

$$\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k} = \frac{1}{P_n} \sum_{k=1}^{n} P_k^2 \frac{1}{kP_k}$$

$$= \frac{1}{P_n} \left[\sum_{k=2}^{n} (P_k^2 - P_{k-1}^2) \sum_{r=k}^{\infty} \frac{1}{rP_r} + P_1^2 \sum_{r=1}^{\infty} \frac{1}{rP_r} + P_n^2 \sum_{r=n+1}^{\infty} \frac{1}{rP_r} \right]$$

$$= \frac{1}{P_n} \sum_{k=2}^{n} (P_k^2 - P_{k-1}^2) \sum_{r=k}^{\infty} \frac{1}{rP_r} + \frac{P_1^2}{P_n} \sum_{r=1}^{\infty} \frac{1}{rP_r} + P_n \sum_{r=n+1}^{\infty} \frac{1}{rP_r}$$

$$\equiv J_1 + J_2 + J_3, \text{ say.}$$

It is clear that

$$J_2 = O(1) \text{ and } J_3 = O(1).$$

Also, we have

$$J_1 = \frac{1}{P_n} \sum_{k=2}^{n} (P_k^2 - P^2_{k-1}) \sum_{r=k}^{\infty} \frac{1}{rP_r}$$

$$=\frac{1}{P_n}\sum_{k=2}^{n} p_k(P_k+P_{k-1})\sum_{r=k}^{\infty}\frac{1}{rP_r}$$

$$\leqq A\frac{1}{P_n}\sum_{k=0}^{n} p_k$$

$$=O(1).$$

Collecting the above estimations, we have

$$\sum_{k=1}^{n}\frac{P_k}{k}=O(P_n).$$

Thus, by Theorem 1, we see that Theorem A and Theorem B are equivalent to each other. Also, we have the following theorem, which is equivalent to Theorem D.

Theorem 2. *If the function* $\varphi(t)$ *is of bounded variation over the interval* $(0, \pi)$ *and if the sequence* $\{p_n\}$ *is non-increasing and satisfies the condition*

$$\sum_{k=1}^{n}\frac{P_k}{k}=O(P_n) \qquad \textit{for all } n\geqq 1,$$

then the Fourier series of $f(t)$, *at* $t=x$, *is summable* $|N, p_n|$.

4. Remarks

Using the results by O. P. Varshney [7] and T. Pati [5], H. P. Dikshit [2] has explained that Theorem 1 holds under the condition of boundedness of the sequence $\{(n+1)p_n/P_n\}$, where $\{p_n\}$ is a positive sequence.

O. P. Varshney [7] has proved the following theorem.

Theorem E. If $\varphi(t)\in BV(0, \pi)$ *and* $\{p_n\}$ *is a positive sequence such that* $P_n\to\infty$, *as* $n\to\infty$,

(i) $$\left\{\frac{(n+1)p_n}{P_n}\right\}\in BV$$

and

(ii) $$\sum_{k=n}^{\infty}\frac{1}{(k+2)P_k}=O\left(\frac{1}{P_n}\right),$$

then the Fourier series of $f(t)$, *at* $t=x$, *is summable* $|N, p_n|$.

H. P. Dikshit [2] has obtained the following theorem.

Theorem F. *If* $\varphi(t)\in BV(0, \pi)$, *and* $\{p_n\}$ *is a positive sequence such that the condition*

(i) $$\left\{\frac{(n+1)p_n}{P_n}\right\} \in BV$$

and

(ii) $$\sum_{k=1}^{n} \frac{P_k}{k+1} = O(P_n),$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N, p_n|$.

Thus, by Theorem 1, we see that Theorem E and Theorem F are equivalent to each other.

References

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