On the Convergence of Martingales and the "Littlewood-Paley" Operators

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Let $f=\{f_n,\ \mathfrak{F}_n;\ n\geqq 1\}$ be a martingale on a probability space $(\Omega,\ \mathfrak{F},\ P)$, where $\mathfrak{F}_1\subset\mathfrak{F}_2\subset\cdots\cdots$ are sub-σ-fields of $\mathfrak{F}$. Denote $d_n=f_n-f_{n-1}$ $(n\geqq 2)$, $d_1=f_1$, and $s_n=\frac{1}{n}(f_1+\cdots\cdots+f_n)$. We define the so-called "Littlewood-Paley" operators $\lambda(f)$ and $\pi(f)$ by

$$\lambda(f)=\sum_{n=1}^{\infty}\frac{(f_n-s_n)^2}{n} \tag{1}$$

and

$$\pi(f)=\sum_{k=1}^{\infty}(f_{n_k}-s_{n_k})^2 \tag{2}$$

where $n_1<n_2<\cdots$ are positive integers such that $1<q_1\leqq\frac{n_{k+1}}{n_k}\leqq q_2$ $(k=1,2,\cdots)$ with constants q_1 and q_2. Tsuchikura ([2] Th. 2.4) showed that if $E[\sup_n|d_n|]<\infty$ and one of the expectations $E[\lambda(f)^{\frac{1}{2}}]$ and $E[\pi(f)^{\frac{1}{2}}]$ is finite, then $\lim_{n\to\infty} f_n$ exists almost surely. In this note we show that the above theorem of Tsuchikura is valid without the assumption $E[\sup_n|d_n|]<\infty$.

Burkholder and Gundy [1] introduced more wide class of operators on martingales. Let a_{jk} be a $\mathfrak{F}_{k-1}$-measurable function, and for all $k\geqq 1$,

$$C_1\leqq\sum_{j=1}^{\infty}a_{jk}{}^2\leqq C_2$$

where C_1 and C_2 are positive constants. Then an operator M:

$$M(f)=\left\{\sum_{j=1}^{\infty}(\limsup_{n\to\infty}\left|\sum_{k=1}^{n}a_{jk}\ d_k\right|)^2\right\}^{\frac{1}{2}} \tag{3}$$

is called to be of matrix type. Let $f^n=\{f^n{}_1,\ f^n{}_2,\cdots\}$ be a martingale f stopped at n, that is $f^n{}_k=f_k(k\leqq n)$, $=f_n(k>n)$. It was shown ([1] Th. 6.1) that if $\sup_n E[M(f^n)]<\infty$, then $\lim_{n\to\infty} f_n$ exists almost surely, where M is of matrix type.

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Theorem. *Let* $f=\{f_n,\ \mathfrak{F}_n;\ n\geqq 1\}$ *be a martingale.*
If $E[\lambda(f)^{\frac{1}{2}}]<\infty$ *or* $E[\pi(f)^{\frac{1}{2}}]<\infty$, *then* $\lim_{n\to\infty} f_n$ *exists almost surely.*

Proof. Note that

$$\lambda(f)=\sum_{n=1}^{\infty}\frac{(f_n-s_n)^2}{n}=\sum_{n=1}^{\infty}\frac{1}{n}\Big(\sum_{k=1}^{n}\frac{k-1}{n}d_k\Big)^2$$
$$=\sum_{n=2}^{\infty}\Big(\sum_{k=2}^{n}\frac{k-1}{n^{\frac{3}{2}}}d_k\Big)^2.$$

We put

$$a_{jk}=\begin{cases}1 & \text{if } j=k=1,\\ j^{-\frac{3}{2}}(k-1) & \text{if } 2\leqq k\leqq j,\\ 0 & \text{otherwise}\end{cases}$$

in (3), and denote the resulting operator of matrix type by $\Lambda(f)$:

$$\Lambda(f)=\Big[d_1^2+\sum_{j=2}^{\infty}\Big\{\mathop{lim\ sup}_{n\to\infty}\Big|\sum_{k=2}^{n\wedge j}j^{-\frac{3}{2}}(k-1)\,d_k\Big|\Big\}^2\Big]^{\frac{1}{2}}$$
$$=\Big[d_1^2+\sum_{j=2}^{\infty}\Big|\sum_{k=2}^{j}j^{-\frac{3}{2}}(k-1)d_k\Big|^2\Big]^{\frac{1}{2}}$$

Now,

$$\Lambda(f^n)=\Big\{(d^n{}_1)^2+\sum_{j=2}^{\infty}\Big|\sum_{k=2}^{j}j^{-\frac{3}{2}}(k-1)\,d^n{}_k\Big|^2\Big\}^{\frac{1}{2}}$$
$$=\Big\{d_1^2+\sum_{j=2}^{n}\Big|\sum_{k=2}^{j}j^{-\frac{3}{2}}(k-1)\,d_k\Big|^2+\sum_{j=n+1}^{\infty}\Big|\sum_{k=2}^{n}j^{-\frac{3}{2}}(k-1)\,d_k\Big|^2\Big\}^{\frac{1}{2}}$$
$$\leqq\Big\{d_1^2+\sum_{j=2}^{\infty}\Big(\sum_{k=2}^{j}j^{-\frac{3}{2}}(k-1)\,d_k\Big)^2+\frac{1}{n^2}\Big(\sum_{k=2}^{n}(k-1)\,d_k\Big)^2\Big\}^{\frac{1}{2}}$$
$$=\{d_1^2+\lambda(f)+(f_n-s_n)^2\}^{\frac{1}{2}}$$
$$\leqq|d_1|+\lambda(f)^{\frac{1}{2}}+|f_n-s_n|$$

But by Lem. 2.3 [2], $\sup_n E[|f_n-s_n|]\leqq c\cdot E[\lambda(f)^{\frac{1}{2}}]$ where c is a positive constant. So,

$$\sup_n E[\Lambda(f^n)]\leqq E[|d_1|]+E[\lambda(f)^{\frac{1}{2}}]+\sup_n E[|f_n-s_n|]$$
$$\leqq E[|f_1|]+(c+1)\,E[\lambda(f)^{\frac{1}{2}}].$$

Therefore, if $E[\lambda(f)^{\frac{1}{2}}]<\infty$, then $\sup_n E[\Lambda(f^n)]<\infty$, and by the theorem of Burkholder and Gundy we can conclude that $\lim_{n\to\infty} f_n$ exists almost surely.

Now we turn to $\pi(f)$. Because

$$\pi(f)=\sum_{k=1}^{\infty}(f_{n_k}-s_{n_k})^2=\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{n_j}\frac{k-1}{n_j}d_k\Big)^2,$$

we put in (3)

$$a_{jk}=\begin{cases}1 & \text{if } j=k=1,\\ n_j^{-1}(k-1) & \text{if } 2\leqq k\leqq n_j,\\ 0 & \text{otherwise,}\end{cases}$$

and denote the matrix type operator from these a_{jk}'s by $\Pi(f)$:

$$\Pi(f)=\Big[d_1^2+\sum_{j=2}^{\infty}\Big\{\limsup_{n\to\infty}\Big|\sum_{k=2}^{n\wedge n_j}n_j^{-1}(k-1)d_k\Big|\Big\}^2\Big]^{\frac{1}{2}}$$
$$=\Big[d_1^2+\sum_{j=2}^{\infty}\Big|\sum_{k=2}^{n_j}n_j^{-1}(k-1)d_k\Big|^2\Big]^{\frac{1}{2}}.$$

Then,

$$\Pi(f^n)=\Big\{(d^n{}_1)^2+\sum_{j=2}^{\infty}\Big(\sum_{k=2}^{n_j}n_j^{-1}(k-1)d^n{}_k\Big)^2\Big\}^{\frac{1}{2}}$$
$$=\Big\{d_1^2+\sum_{j=2}^{n_j\leqq n}\Big(\sum_{k=2}^{n_j}n_j^{-1}(k-1)d_k\Big)^2+\sum_{j:n_j>n}\Big(\sum_{k=2}^{n}n_j^{-1}(k-1)d_k\Big)^2\Big\}^{\frac{1}{2}}$$
$$\leqq\Big\{d_1^2+\pi(f)+\sum_{j:n_j>n}\Big(\frac{n}{n_j}(f_n-s_n)\Big)^2\Big\}^{\frac{1}{2}}$$
$$\leqq|d_1|+\pi(f)^{\frac{1}{2}}+\Big(\frac{q_1^2}{q_1^2-1}\Big)^{\frac{1}{2}}|f_n-s_n|.$$

Note that $\{n(f_n-s_n),\ \mathfrak{F}_n;\ n\geqq 1\}$ is a martingale, so, by the submartingale inequality,

$$E[|f_n-s_n|]=\frac{1}{n}E[n|f_n-s_n|]\leqq\frac{1}{n}E[n_k|f_{n_k}-s_{n_k}|]$$
$$\leqq q_2E[|f_{n_k}-s_{n_k}|]\leqq q_2E[\pi(f)^{\frac{1}{2}}]$$

where $n_{k-1}<n\leqq n_k$. Therefore

$$E[\Pi(f^n)]\leqq E[|d_1|]+\Big(1+\frac{q_1q_2}{(q_1^2-1)^{\frac{1}{2}}}\Big)E[\pi(f)^{\frac{1}{2}}].$$

So, if $E[\pi(f)]^{\frac{1}{2}}<\infty$, then $\sup_n E[\Pi(f^n)]<\infty$ and $\lim_{n\to\infty} f_n$ exists almost surely.

q. e. d.

We give an example of a martingale f, such that $E[\lambda(f)^{\frac{1}{2}}]<\infty$ and

$E[\pi(f)^{\frac{1}{2}}]<\infty$, but $E[\sup_n |d_n|]=\infty$. Let $\Omega=(0,\ 1]$, $\mathfrak{F}$ be a family of all Borel sets on Ω, P be a Lebesgue measure on $\mathfrak{F}$, $\mathfrak{F}_n$ be a σ-field generated by the sets $(0,\ \frac{1}{2^n}]$, $(\frac{1}{2^n},\ \frac{1}{2^{n-1}}]$, $(\frac{1}{2^{n-1}},\ \frac{1}{2^{n-2}}]$, $\cdots$ $(\frac{1}{2},\ 1]$. We define $d_n's$ by $d_1\equiv 0$, $d_n=-\frac{2^n}{n-1}I_{(0,\frac{1}{2^n}]}+\frac{2^n}{n-1}I_{(\frac{1}{2^n},\ \frac{1}{2^{n-1}}]}(n\geqq 2)$. This gives a desired example.

References

[1] D. L. Burkholder and R. F. Gundy, Extrapolation and interpolation of quasi-linear operators of martingales, *Acta Mathematica*, 124(1970), 249–304.

[2] T. Tsuchikura, Sample properties of martingales and their arithmetic means, *Tôhoku Math. J.*, 20 (1968), 400–415.