On Contraction of Walsh Fourier Transforms

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§ 1. We begin with some notations and definitions:

Let $\{\phi_n(x)\}$ $n=1, 2, \cdots,$ be the system of Rademacher functions, i. e.

$$\phi_0(x)=\begin{cases} 1(x\in[0,\ 1/2)) \\ -1(x\in[1/2,\ 1)), \end{cases} \qquad \phi_0(x+1)=\phi_0(x),\ \ \phi_n(x)=\phi_0(2^n\cdot x),$$

and let

$$\psi_n(x)=1 \text{ for } n=0, \text{ and}$$
$$\psi_n(x)=\phi_{n(1)}(x)\phi_{n(2)}(x)\cdots\ \phi_{n(r)}(x) \text{ for } n=2^{n(1)}+2^{n(2)}+\cdots+2^{n(r)}$$

where $$n(1)>n(2)>\cdots>n(r)>0.$$

The functions thus defined are called the Walsh functions, which form a complete orthonormal system over the unit interval. For basic properties of Walsh functions, the reader is referred to N. J. Fine [2].

Following A. Beurling, $g(x)$ is called a contraction of $f(x)$ if

$$|g(x)-g(x')|\leqq|f(x)-f(x')| \text{ for } x,\ x'\in(0,\ 1).$$

A sequence $\{a_n\}$ is called a contraction of a sequence $\{c_n\}$ if

$$|a_m-a_n|\leqq|c_m-c_n| \quad \text{for every m and n.}$$

A denotes a positive absolute constant not always the same.

We have previously proved the following two theorems (see [5]):

Theorem 1. Let

$$f(x)\sim\sum_{n=0}^{\infty}c_n\psi_n(x)$$

and

$$g(x)\sim\sum_{n=0}^{\infty}a_n\psi_n(x)\in L(0,\ 1).$$

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Suppose that

$$\int_0^1 |g(x \dotplus h) - g(x)|^2 dx \leqq \int_0^1 |f(x \dotplus h) - f(x)|^2 dx$$

for any h, and suppose that there exists a positive sequence $\{\gamma_n\}$ such that

(i) $$|c_n| \leqq \gamma_n \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n{}^p < \infty$$

(ii) $$\sum_{n=1}^{\infty} n^{-3p/2} \Big(\sum_{\nu=1}^{n} \nu^2 \gamma_\nu{}^2 \Big)^{p/2} + \sum_{n=1}^{\infty} n^{-p/2} \Big(\sum_{\nu=n+1}^{\infty} \gamma^2{}_\nu \Big)^{p/2} < \infty,$$

then

$$\sum_{n=0}^{\infty} |a_n|^p < \infty$$

where $2/3 < p \leqq 2$.

Theorem 2. Let

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x).$$

For a given $\{a_n\}$, if $a_n \to 0$ and

$$\sum_{n=0}^{\infty} |a_{m(n,j)} - a_n|^2 \leqq \sum_{n=0}^{\infty} |c_{m(n,j)} - c_n|^2 \quad \text{for every integer } j$$

where $m(n, j) = n \dotplus 2^j$, and if there exists a function $\gamma(x)$ such that

(i) $$|f(x)| \leqq \gamma(x) \text{ and } \gamma(x) \in L^p(0, 1)$$

(ii) $$\int_0^1 x^{-3p/2} \Big(\int_0^x t^2 \gamma^2(t) dt \Big)^{p/2} dx + \int_0^1 x^{-p/2} \Big(\int_x^1 \gamma^2(t) dt \Big)^{p/2} dx < \infty$$

where $1 \leqq p \leqq 2$, *then there exists a function* $g(x)$ *belonging to* $L^p(0, 1)$ such that

$$g(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

Theorem 2 is the dual for Theorem 1. The case $p = 1$ of Theorem 1 was first proved by prof. C. Watari [8]. For the corresponding results for Fourier series, see also [1] and [4]. Prof. G. Sunouchi [7] proved the following proposition 1:

Proposition 1. For $1 \leqq p < 2$, the convergence of the series

$\sum_{n=1}^{\infty} n^{-3p/2}\Big(\sum_{\nu=1}^{n} \nu^2 a_\nu^2\Big)^{p/2}$ is equivalent to the convergence of the series

$\sum_{n=1}^{\infty} n^{-p/2}\Big(\sum_{\nu=n}^{\infty} a^2\nu\Big)^{p/2}.$

Hence the hypotheses of our theorems may be modified.

Now, N. J. Fine [3] introduced the generalized Walsh function

$$\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$$

where $[x]$ = the greatest integer in x. Then for each z, we have $\psi_x(y \dot{+} z) = \psi_x(y)\psi_x(z) a.e.\, y.$ Further, by symmetry, we have $\psi_y(x) = \psi_x(y)$.

If $f(x)$ is integrable on $(0, \infty)$, then its Walsh Fourier transform is defined by

$$\hat{f}(y) = Tf(y) = \int_0^\infty f(x)\psi_y(x)dx.$$

It is well-known that the following facts are true;

(1°) $$(\tau_z f)(x) = f(x \dot{+} z) \to T(\tau_z f) = \psi_z \cdot \hat{f}$$

(2°) $$\widehat{f\cdot\psi_z} = \tau_z \hat{f}$$

(3°) Plancheral theorem.

We refer the reader to N. J. Fine [3], R. G. Selfridge [6] and C. Watari [9] for detailed properties of Walsh Fourier transforms.

The purpose of this paper is to prove the Walsh analogue of theorem on contraction of Fourier transforms (see [4]). The author thanks prof. C. Watari for his valuable suggestions and encouragements in the preparation of this paper.

Our result is as follows:

Theorem 3. Suppose that $F(x) \in L^p(0, \infty)$, where $1 \leqq p \leqq 2$, and that its Walsh Fourier transform is $f(x)$. Further, suppose that $g(x)$ is a contration of $f(x)$, that is,

$$\int_0^\infty |g(x \dot{+} h) - g(x)|^2 dx \leqq \int_0^\infty |f(x \dot{+} h) - f(x)|^2 dx \qquad \text{for any } h,$$

satisfying either, for the case $p = 1$,

$$\lim_{x\to\infty} g(x) = 0$$

or, for the case $1 < p < 2$,

$$g(x) \in L^q(A, \ \infty)$$

where A is a positive finite real number and $q = p/(p-1)$.
If there exists a function $\gamma(x)$ such that

(i) $$|F(x)| \leqq \gamma(x) \text{ and } \gamma(x) \in L^p(0, \ \infty)$$

(ii) $$t^2\gamma^2(t) \in L(0, \ \delta) \text{ and } \gamma^2(t) \in L(\delta, \ \infty) \text{ for any } \delta > 0$$

(iii) $$\int_0^\infty x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx < \infty \quad \text{or} \quad \int_0^\infty x^{-p/2}\left(\int_x^\infty \gamma^2(t)dt\right)^{p/2}dx < \infty,$$

then $g(x)$ is Walsh Fourier transform of a function $G(x)$ which belongs to $L^p(0, \infty)$, where $1 \leqq p \leqq 2$.

As this theorem is proved along the line of [4], its proof is somewhat more tedious.

§ **2.** We begin with the equivalency of the convergence of the two integrals of the assumption (iii) in Theorem 3.

Theorem 4. For $1 \leqq p < 2$, the convergence of the integral

$$\int_0^\infty x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx$$

is equivalent to the convergence of the integral $\int_0^\infty x^{-p/2}\left(\int_x^\infty \gamma^2(t)dt\right)^{p/2}dx.$

Proof. First, we shall prove the inequality

$$\int_0^\infty x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx \leqq A\int_0^\infty x^{-p/2}\left(\int_x^\infty \gamma^2(t)dt\right)^{p/2}dx.$$

By dissecting the range of integration and using Jensen's inequality, we have

$$\begin{aligned}\int_0^\infty x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx &= \sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}} x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx \\ &\leqq A\sum_{n=-\infty}^{\infty} 2^{-n(3p/2-1)}\left(\sum_{k=-\infty}^{n} 2^{2k}\int_{2k}^{2k+1}\gamma^2(t)dt\right)^{p/2} \\ &\leqq A\sum_{n=-\infty}^{\infty} 2^{-n(3p/2-1)}\sum_{k=-\infty}^{n} 2^{kp}\left(\int_{2k}^{2k+1}\gamma^2(t)dt\right)^{p/2} \\ &\leqq A\sum_{k=-\infty}^{\infty} 2^{kp}\left(\int_{2k}^{2k+1}\gamma^2(t)dt\right)^{p/2}\sum_{n=k}^{\infty} 2^{-n(3p/2-1)}\end{aligned}$$

$$\leqq A\sum_{k=-\infty}^{\infty}2^{kp}\cdot 2^{-k(3p/2-1)}\left(\int_{2k}^{\infty}\gamma^2(t)dt\right)^{p/2}=A\sum_{k=-\infty}^{\infty}2^{-k(p/2-1)}\left(\int_{2k}^{\infty}\gamma^2(t)dt\right)^{p/2}$$

$$\leqq A\sum_{k=-\infty}^{\infty}\int_{2k-1}^{2k}x^{-p/2}\left(\int_{x}^{\infty}\gamma^2(t)dt\right)^{p/2}dx=A\int_0^{\infty}x^{-p/2}\int_x^{\infty}\gamma^2(t)dt\Big)^{p/2}dx.$$

Concerning the converse inequality, we proceed with the same method. In fact, we have

$$\int_0^{\infty}x^{-p/2}\left(\int_x^{\infty}\gamma^2(t)dt\right)^{p/2}dx\leqq\sum_{n=-\infty}^{\infty}\int_{2^n}^{2^{n+1}}x^{-p/2}dx\left(\int_{2^n}^{\infty}\gamma^2(t)dt\right)^{p/2}$$

$$\leqq A\sum_{n=-\infty}^{\infty}2^{-n(p/2-1)}\left(\sum_{k=n}^{\infty}2^{-2k}\int_{2k}^{2k+1}t^2\gamma^2(t)dt\right)^{p/2}$$

$$\leqq A\sum_{n=-\infty}^{\infty}2^{-n(p/2-1)}\sum_{k=n}^{\infty}2^{-kp}\left(\int_{2k}^{2k+1}t^2\gamma^2(t)dt\right)^{p/2}$$

$$\leqq A\sum_{k=-\infty}^{\infty}2^{-kp}\left(\int_0^{2k+1}t^2\gamma^2(t)dt\right)^{p/2}\sum_{n=-\infty}^{k}2^{-n(p/2-1)}$$

$$\leqq A\sum_{k=-\infty}^{\infty}2^{-k(3p/2-1)}\left(\int_0^{2k+1}t^2\gamma^2(t)dt\right)^{p/2}$$

$$\leqq A\sum_{k=-\infty}^{\infty}\int_{2k+1}^{2k+2}x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx=A\int_0^{\infty}x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx.$$

Therefore this completes the proof of Theorem 4.

§ **3**. To prove Theorem 3, we collect here various preliminary results which we will need.

Proposition 2 [3]. If $f(x)$ is integrable on $(0,\ \infty)$, then

$$\lim_{y\to\infty}\hat{f}(y)=\lim_{y\to\infty}\int_0^{\infty}f(x)\psi_y(x)dx=0.$$

Proposition 3 [6]. If $0<a<2^n$, then

$$\int_0^{2^n}f(x)dx=\int_0^{2^n}f(a\dot{+}x)dx.$$

Proposition 4 [9]. If the function $\chi_{[0,2^n)}(x)$ is the characteristic functions of the interval $[0,\ 2^n)$, then we have

$$\hat{\chi}_{[0,2^n)}(y)=\int_0^{\infty}\chi_{[0,2^n)}(x)\psi_y(x)dx=2^n\chi_{[0,2^{-n})}(y)$$

where n is every integer.

Proposition 5 [6]. If $F(x) \in L^p(0, \infty)(1 < p \leqq 2)$, then there is a function $f(y) \in L^q(0, \infty)$ such that

$$f(y) = \underset{\omega\to\infty}{\overset{(q)}{\mathrm{l.i.m}}} \int_0^\omega F(x)\psi_y(x)dx \text{ and } \| f \|_q \leqq \| F \|_p$$

where $1/p + 1/q = 1$.

The following lemma is of particular importance in the proof of Theorem 3.

Lemma. Suppose that the function $F(x)$ belongs to $L^2(0, \infty)$ and $f(x) = \underset{\omega\to\infty}{\overset{(2)}{\mathrm{l.i.m}}} \int_0^\omega F(y)\psi_x(y)dy$. If the function $g(x)$ is a contraction of a function $f(x)$, then there exists a function $G(x)$ belonging to L^2 (ε, ∞) for any positive number ε such that

$$\underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}} \left\{ \int_0^{2^n} g(x)(\phi_j(x) - 1)\psi_y(x)dx \right\}$$

$$= G(y)(\phi_j(y) - 1).$$

Moreover, this function $G(x)$ satisfies the following inequality:

$$\int_0^t x^2 |G(x)|^2 dx \leqq A \left\{ \int_0^t |F(x)|^2 x^2 dx + t^2 \int_t^\infty |F(x)|^2 dx \right\}.$$

Proof. Since $g(x)$ is a contraction of $f(x)$, we have, by Plancheral theorem,

$$\int_0^\infty |g(x \dotplus 2^j) - g(x)|^2 dx \leqq \int_0^\infty |f(x \dotplus 2^j) - f(x)|^2 dx$$

$$= \int_0^\infty |F(x)(\phi_j(x) - 1)|^2 dx < \infty.$$

Hence we have

$$(*) \qquad \int_0^\infty |g(x \dotplus 2^j) - g(x)|^2 dx < \infty \qquad \text{for every integer } j.$$

Put

$$G_n(y) = \int_0^{2^n} g(x)\psi_y(x)dx.$$

Then, using proposition 3 for $n > j$, we obtain

$$\int_0^{2^n} g(x \dotplus 2^j)\psi_y(x)dx = \int_0^{2^n} g(x)\psi_y(x \dotplus 2^j)dx$$

$$= \int_0^{2^n} g(x)\phi_y(x)\phi_y(2^j)dx = \phi_y(2^j)\int_0^{2^n} g(x)\phi_y(x)dx$$

$$= \phi_j(y)\int_0^{2^n} g(x)\phi_y(x)dx = \phi_j(y)G_n(y).$$

Thus we have

$$G_n(y)\Big\{\phi_j(y)-1\Big\} = \int_0^{2^n}\{g(x\dotplus 2^j)-g(x)\}\phi_y(x)dx.$$

Therefore, by (*)

$$\underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}}\Big\{G_n(y)(\phi_j(y)-1)\Big\} = \underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}}\Big\{\int_0^{2^n} g(x)\phi_y(x)(\phi_j(y)-1)dx\Big\}$$

which we can write as follows

$$\equiv G^{(j)}(y)(\phi_j(y)-1), \quad \text{say},$$

where $G^{(j)}(y)$ is defined for almost all y belonging to the set

$$E_j = \{y \mid k+(2\nu-1)/2^{j+1} \leqq y < k+2\nu/2^{j+1};\ \nu=1,\ 2,\ \cdots 2^j,\ k=0,\ 1,\ 2,\cdots\}.$$

Since

$$\phi_i(y)=\phi_j(y)=-1 \quad \text{for} \quad y\in E_i\cap E_j\,(i,j=1,\ 2,\cdots),$$

we have clearly

$$G^{(i)}(y)=G^{(j)}(y) \quad \text{for almost all} \quad y\in E_i\cap E_j\,(i\neq j).$$

As the union of the sets E_j is $(0,\ \infty)$, let us define the function $G(y)$ on the open interval $(0,\ \infty)$ according to the following rule :

$$G(y)=G^{(j)}(y) \quad \text{for} \quad y\in E_j(j=1,2,\cdots).$$

Thus $G(y)$ is well-defined almost everywhere in the open interval $(0,\ \infty)$. It is clear from the definition of the function $G(y)$ that

$$\underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}}\Big\{\int_0^{2^n} g(x)(\phi_j(x)-1)\phi_y(x)dx\Big\} = G(y)(\phi_j(y)-1),$$

$$\underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}}\Big\{\int_0^{2^n} G(y)(\phi_j(y)-1)\phi_x(y)dy\Big\} = g(x\dotplus 2^j)-g(x)$$

and

$$G(y) \in L^2(\varepsilon, \infty) \text{ for every positive number } \varepsilon.$$

To prove the inequality in the Lemma, the proof proceeds in two steps.

The first step. As the proof of this step is similar to that used by Y. Okuyama [5], we give here a proof for the sake of completeness. Since $g(x)$ is a contraction of $f(x)$ and

$$\underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}}\left\{\int_0^{2^n} F(x)(\phi_j(x)-1)\phi_y(x)dx\right\} = f(y \dotplus 2^j) - f(x),$$

we have

$$\int_0^\infty |G(x)(\phi_j(x)-1)|^2 dx = \int_0^\infty |g(x \dotplus 2^j) - g(x)|^2 dx$$

$$\leqq \int_0^\infty |f(x \dotplus 2^j) - f(x)|^2 dx = \int_0^\infty |F(x)(\phi_j(x)-1)|^2 dx.$$

Hence

$$\int_0^\infty |G(x)(\phi_j(x)-1)|^2 dx \leqq \int_0^\infty |F(x)(\phi_j(x)-1)|^2 dx.$$

Therefore we obtain

$$\int_{2^{-k-1}}^{2^{-k}} |G(x)(\phi_k(x)-1)|^2 dx \leqq \int_{2^{-k-1}}^{1} |F(x)(\phi_k(x)-1)|^2 dx$$

$$+ \int_1^\infty |F(x)(\phi_k(x)-1)|^2 dx.$$

By the definition of Rademacher function, we have

$$\phi_k(x) = 1 \quad (0 \leqq x < 2^{-k-1}), \quad \phi_k(x) = -1 \quad (2^{-k-1} \leqq x < 2^{-k}).$$

Thus we obtain

$$\int_{2^{-k-1}}^{2^{-k}} |G(x)|^2 dx \leqq \int_{2^{-k-1}}^{1} |F(x)|^2 dx + \int_1^\infty |F(x)|^2 dx$$

$$= \sum_{i=0}^{k} \int_{2^{-i-1}}^{2^{-i}} |F(x)|^2 dx + \int_1^\infty |F(x)|^2 dx.$$

If $x = 2^{-k}$, then we have

$$\int_0^{2^{-k}} t^2 |G(t)|^2 dt = \sum_{j=k}^{\infty} \int_{2^{-j-1}}^{2^{-j}} t^2 |G(t)|^2 dt \leqq \sum_{j=k}^{\infty} 2^{-2j} \int_{2^{-j-1}}^{2^{-j}} |G(t)|^2 dt$$

$$\leqq A\left\{\sum_{j=k}^{\infty} 2^{-2j} \sum_{i=0}^{k} \int_{2^{-i-1}}^{2^{-i}} |F(t)|^2 dt + \sum_{j=k}^{\infty} 2^{-2j} \int_1^\infty |F(t)|^2 dt\right\}$$

$$\leqq A\Big\{\sum_{j=k}^{\infty}\Big(\sum_{i=0}^{k-1}+\sum_{i=k}^{j}\Big)2^{-2j}\int_{2^{-i-1}}^{2^{-i}}|F(t)|^2dt+\sum_{j=k}^{\infty}2^{-2j}\int_1^{\infty}|F(t)|^2dt\Big\}$$

$$\leqq A\Big\{2^{-2k}\int_{2^{-k}}^{1}|F(t)|^2dt+\sum_{i=k}^{\infty}\sum_{j=i}^{\infty}2^{-2j}\int_{2^{-i-1}}^{2^{-i}}|F(t)|^2dt+\sum_{j=k}^{\infty}2^{-2j}\int_1^{\infty}|F(t)|^2dt\Big\}$$

$$\leqq A\Big\{2^{-2k}\int_{2^{-k}}^{1}|F(t)|^2dt+\sum_{i=k}^{\infty}\int_{2^{-i-1}}^{2^{-i}}t^2|F(t)|^2dt+2^{-2k}\int_1^{\infty}|F(t)|^2dt\Big\}$$

$$=A\Big\{\int_0^{2^{-k}}t^2|F(t)|^2dt+2^{-2k}\int_{2^{-k}}^{\infty}|F(t)|^2dt\Big\}.$$

Next, we suppose that

$$2^{-k-1}<x<2^{-k}.$$

Then we obtain

$$\int_0^x t^2|G(t)|^2dt\leqq\int_0^{2^{-k}}t^2|G(t)|^2dt$$

$$\leqq A\Big\{\int_0^{2^{-k}}t^2|F(t)|^2dt+2^{-2k}\int_{2^{-k}}^{\infty}|F(t)|^2dt\Big\}.$$

On the other hand, we have

$$\int_0^{2^{-k}}t^2|F(t)|^2dt=\int_0^x t^2|F(t)|^2dt+\int_x^{2^{-k}}t^2|F(t)|^2dt$$

$$\leqq\int_0^x t^2|F(t)|^2dt+2^{-2k}\int_x^{2^{-k}}|F(t)|^2dt.$$

Thus we have

$$\int_0^x t^2|G(t)|^2dt\leqq A\Big\{\int_0^x t^2|F(t)|^2dt+x^2\int_x^{\infty}|F(t)|^2dt\Big\}.$$

Hence the case $0<x\leqq 1$ is proved. We must prove the case $1\leqq x<\infty$.

The second step. Since $g(x)$ is a contraction of $f(x)$, we have clearly

$$\int_0^{\infty}|G(t)(\psi_z(t)-1)|^2dt=\int_0^{\infty}|g(t\dot{+}z)-g(t)|^2dt$$

$$\leqq\int_0^{\infty}|f(t\dot{+}z)-f(t)|^2dt=\int_0^{\infty}|F(t)(\psi_z(t)-1)|^2dt.$$

Thus

$$\int_0^{\infty}|G(t)|^2(\psi_z(t)-1)^2dt\leqq\int_0^{\infty}|F(t)|^2(\psi_z(t)-1)^2dt.$$

We integrate both the sides of this inequality with respect to z within the limits $[2^{-k-1}, 2^{-k})$ and we obtain

$$\int_{2^{-k-1}}^{2^{-k}} dz \int_0^\infty |G(t)|^2(\psi_z(t)-1)^2 dt \leqq \int_{2^{-k-1}}^{2^{-k}} dz \int_0^\infty |F(t)|^2(\psi_z(t)-1)^2 dt. \tag{1}$$

Using proposition 4, we see that

$$\int_{2^{-k-1}}^{2^{-k}} (\psi_z(t)-1)^2 dz = 2\int_{2^{-k-1}}^{2^{-k}} (1-\psi_z(t))dz = 2\cdot(2^{-k-1} - \int_{2^{-k-1}}^{2^{-k}} \psi_t(z)dz)$$

$$=2\cdot(2^{-k-1} - \int_0^\infty \chi_{[2^{-k-1},\,2^{-k})}(z)\psi_t(z)\,dz) = 2\cdot(2^{-k-1} - \hat{\chi}_{[2^{-k-1},\,2^{-k})}(t))$$

$$=\begin{cases} 0, & 0 \leqq t < 2^k \\ 2^{-k+1}, & 2^k \leqq t < 2^{k+1} \\ 2^{-k}, & 2^{k+1} \leqq t < \infty. \end{cases}$$

Hence we have by (1)

$$\int_{2^k}^{2^{k+1}} |G(t)|^2 dt \leqq A\int_{2^k}^\infty |F(t)|^2 dt = A\Big\{\int_{2^k}^{2^n} |F(t)|^2 dt + \int_{2^n}^\infty |F(t)|^2 dt\Big\} \tag{2}$$

$$= A\Big\{\sum_{j=k}^{n-1} \int_{2^j}^{2^{j+1}} |F(t)|^2 dt + \int_{2^n}^\infty |F(t)|^2 dt\Big\}.$$

If $x = 2^n$, then we have

$$\int_0^{2^n} t^2|G(t)|^2 dt = \int_0^1 t^2|G(t)|^2 dt + \sum_{k=0}^{n-1}\int_{2^k}^{2^{k+1}} t^2|G(t)|^2 dt = I_1 + I_2, \qquad \text{say.}$$

According to the proof of the first step, we obtain

$$I_1 = \int_0^1 t^2|G(t)|^2 dt \leqq A\Big\{\int_0^1 t^2|F(t)|^2 dt + \int_1^\infty |F(t)|^2 dt\Big\} \tag{3}$$

$$\leqq A\Big\{\int_0^{2^n} t^2|F(t)|^2 dt + 2^{2n}\int_{2^n}^\infty |F(t)|^2 dt\Big\}.$$

On the other hand, we have by (2)

$$I_2 = \sum_{k=0}^{n-1}\int_{2^k}^{2^{k+1}} t^2|G(t)|^2 dt \leqq A\Big\{\sum_{k=0}^{n-1} 2^{2k}\int_{2^k}^{2^{k+1}} |G(t)|^2 dt\Big\}$$

$$\leqq A\Big\{\sum_{k=0}^{n-1} 2^{2k}\sum_{j=k}^{n-1}\int_{2^j}^{2^{j+1}} |F(t)|^2 dt + \sum_{k=0}^{n-1} 2^{2k}\int_{2^n}^\infty |F(t)|^2 dt\Big\}$$

$$\leqq A\left\{\sum_{j=0}^{n-1}\int_{2j}^{2j+1}|F(t)|^2dt\sum_{k=0}^{j}2^{2k}+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\}$$

$$\leqq A\left\{\sum_{j=0}^{n-1}2^{2j}\int_{2j}^{2j+1}|F(t)|^2dt+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\} \tag{4}$$

$$\leqq A\left\{\sum_{j=0}^{n-1}\int_{2j}^{2j+1}t^2|F(t)|^2dt+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\}$$

$$=A\left\{\int_{1}^{2^n}t^2|F(t)|^2dt+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\}$$

$$\leqq A\left\{\int_{0}^{2^n}t^2|F(t)|^2dt+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\}.$$

Combining (3) and (4), we have

$$\int_0^{2^n}t^2|G(t)|^2dt\leqq A\left\{\int_{0}^{2^n}t^2|F(t)|^2dt+2^{2n}\int_{2^n}^{\infty}|F(t)|^2dt\right\}.$$

Finally, we suppose that

$$2^n<x<2^{n+1}.$$

Then we obtain

$$\int_0^{x}t^2|G(t)|^2dt\leqq\int_0^{2^{n+1}}t^2|G(t)|^2dt$$

$$\leqq A\left\{\int_{0}^{2^{n+1}}t^2|F(t)|^2dt+2^{2(n+1)}\int_{2^{n+1}}^{\infty}|F(t)|^2dt\right\}.$$

On the other hand, we have

$$\int_0^{2^{n+1}}t^2|F(t)|^2dt=\int_0^{x}t^2|F(t)|^2dt+\int_x^{2^{n+1}}t^2|F(t)|^2dt$$

$$\leqq\int_0^{x}t^2|F(t)|^2dt+2^{2(n+1)}\int_x^{2^{n+1}}|F(t)|^2dt.$$

Thus we obtain

$$\int_0^{x}t^2|G(t)|^2dt\leqq A\left\{\int_{0}^{x}t^2|F(t)|^2dt+2^{2(n+1)}\int_{x}^{\infty}|F(t)|^2dt\right\}$$

$$\leqq A\left\{\int_{0}^{x}t^2|F(t)|^2dt+x^2\int_{x}^{\infty}|F(t)|^2dt\right\},$$

Q. E. D.

§ 4. **Proof of Theorem 3.** Now we are in position to prove our Theorem. We may suppose that $1 \leqq p < 2$. By the assumption of Theorem 3, we have

$$F(x) \in L^2\ (\varepsilon,\ \infty) \quad \text{for every positive number } \varepsilon.$$

Hence

$$F(x)(\phi_j(x)-1) \in L^2\ (0,\ \infty) \quad \text{for every integer } j.$$

Since $g(x)$ is a contraction of $f(x)$, we have, by Plancheral theorem,

$$\int_0^\infty |g(x \dot{+} 2^j) - g(x)|^2 dx \leqq \int_0^\infty |f(x \dot{+} 2^j) - f(x)|^2 dx$$

$$= \int_0^\infty |F(x)(\phi_j(x)-1)|^2 dx < \infty.$$

Thus we have

$$\int_0^\infty |g(x \dot{+} 2^j) - g(x)|^2 dx < \infty \qquad \text{for every integer } j.$$

Therefore, by Lemma, we have a function $G(x)$ which is well-defined almost everywhere on the open interval $(0, \infty)$. Now we shall show that this function $G(x)$ belongs to $L^p(0, \infty)$.

We put

$$\varphi_p(x) = \int_0^x |t|^p |G(t)|^p dt.$$

By Hölder's inequality and the lemma, we have

$$\varphi_p(x) \leqq x^{1-p/2}\Big(\int_0^x |t|^2 |G(t)|^2 dt\Big)^{p/2}$$

$$\leqq Ax^{1-p/2}\Big\{\int_0^x |F(t)|^2 t^2 dt + x^2 \int_x^\infty |F(t)|^2 dt\Big\}^{p/2}$$

$$\leqq Ax^{1-p/2}\Big\{\int_0^x \gamma^2(t) t^2 dt + x^2 \int_x^\infty \gamma^2(t) dt\Big\}^{p/2}, \text{ by (i)} \tag{5}$$

$$\leqq Ax^{1-p/2}\Big(\int_0^x \gamma^2(t) t^2 dt\Big)^{p/2} + Ax^{1+p/2}\Big(\int_x^\infty \gamma^2(t) dt\Big)^{p/2}$$

by Jensen's inequality.

By the assumption (iii) and Theorem 4, for each $\alpha \in (0, \infty)$ we obtain

$$A \geqq \int_\alpha^{2\alpha} x^{-3p/2}\Big(\int_0^x t^2 \gamma^2(t) dt\Big)^{p/2} dx$$

$$\geqq \int_{\alpha}^{2\alpha} x^{-3p/2}dx\left(\int_0^{\alpha} t^2\gamma^2(t)dt\right)^{p/2}$$

$$= A_p\alpha^{1-3p/2}\left(\int_0^{\alpha} t^2\gamma^2(t)dt\right)^{p/2}.$$

Hence

(6) $$\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2} = 0(x^{3p/2-1}).$$

Also

$$A \geqq \int_{\alpha/2}^{\alpha} x^{-p/2}\left(\int_x^{\infty} \gamma^2(t)dt\right)^{p/2}dx$$

$$\geqq \int_{\alpha/2}^{\alpha} x^{-p/2}dx\left(\int_{\alpha}^{\infty} \gamma^2(t)dt\right)^{p/2}$$

$$= A_p\alpha^{1-p/2}\left(\int_{\alpha}^{\infty} \gamma^2(t)dt\right)^{p/2},$$

and hence we have

(7) $$\left(\int_x^{\infty} \gamma^2(t)dt\right)^{p/2} = 0(x^{p/2-1}).$$

Combining (6) and (7), we have

(8) $$\varphi_p(x) = 0(x^p).$$

Now we can show that this function $G(x)$ belongs to L^p $(0, \infty)$; that is,

$$\int_0^{\infty} |G(x)|^p dx = \left[x^{-p}\varphi_p(x)\right]_0^{\infty} + \int_0^{\infty} x^{-p-1}\varphi_p(x)dx$$

$$= K_1 + K_2, \quad \text{say}$$

where K_1 is $0(1)$ by (8) and K_2 is, by (5), less than

$$A\int_0^{\infty} x^{-p-1}.\, x^{1-p/2}\left\{\int_0^x \gamma^2(t)t^2dt + x^2\int_x^{\infty} \gamma^2(t)dt\right\}^{p/2}dx$$

$$\leqq A\left\{\int_0^{\infty} x^{-3p/2}\left(\int_0^x t^2\gamma^2(t)dt\right)^{p/2}dx + \int_0^{\infty} x^{-p/2}\left(\int_x^{\infty} \gamma^2(t)dt\right)^{p/2}dx\right\}$$

which is finite because of the assumption (iii) and Theorem 4. Therefore we see that $G(x)$ belongs to $L^p(0, \infty)$.

Using proposition 5 and the definition of Walsh-Fourier transform, we define the Walsh-Fourier transform $g^*(y)$ of $G(x)$ in the following way respectively:

$$g^*(y) = \underset{n\to\infty}{\overset{(q)}{\mathrm{l.i.m}}} \int_0^{2^n} G(x)\psi_y(x)dx, \qquad \text{for} \quad 1 < p < 2$$

and

$$g^*(y) = \int_0^{\infty} G(x)\psi_y(x)dx, \qquad \text{for} \quad p = 1.$$

Since $G(x)(\phi_j(x) - 1) \in L^2(0, \infty)$, we have

$$g^*(y \dotplus 2^j) - g^*(y) = \underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}} \int_0^{2^n} G(x)(\phi_j(x) - 1)\psi_y(x)dx.$$

On the other hand, we get, by Plancheral theorem

$$g(y \dotplus 2^j) - g(y) = \underset{n\to\infty}{\overset{(2)}{\mathrm{l.i.m}}} \int_0^{2^n} G(x)(\phi_j(x) - 1)\psi_y(x)dx.$$

Hence we obtain

$$g^*(y \dotplus 2^j) - g^*(y) = g(y \dotplus 2^j) - g(y) \ a.e.$$

For the case $p = 1$, by proposition 2, we have

$$\lim_{j\to\infty} g^*(y \dotplus 2^j) = 0 \quad \text{and} \quad \lim_{j\to\infty} g(y \dotplus 2^j) = 0.$$

Therefore

$$g^*(y) = g(y) \ a.e.$$

For the case $1 < p < 2$, we have to recall the facts that

$$g(y) \in L^q \ (A. \ \infty) \quad \text{and} \quad g^*(y) \in L^q \ (0, \ \infty).$$

Take any finite interval (a, b). If $y \in (a, b)$, then $y \dotplus 2^j \in (2^j - b, \ 2^j + b)$. Thus we have

$$\int_a^b |g(y) - g^*(y)|^q dy = \int_a^b |g(y \dotplus 2^j) - g^*(y \dotplus 2^j)|^q dy$$

$$\leqq \int_{2^j - b}^{2^j + b} |g(y) - g^*(y)|^q dy$$

which tends to zero as $j \to \infty$. Therefore we have

$$g(y) - g^*(y) = 0 \ a. e.$$

which completes the proof of Theorem 3.

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