On the Nörlund Summability of Laguerre Series

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1. Definitions and Notations

Let $f(t)$ be a Lebesgue-measurable function such that the integral

$$\int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x)\, dx, \quad \alpha > -1 \tag{1}$$

exists, where $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order α.

The Laguerre series corresponding to this function $f(x)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{2}$$

in which

$$a_n = \frac{1}{\Gamma(\alpha+1) A_n^\alpha} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha)}(y)\, dy \tag{3}$$

and

$$A_n^\alpha = \binom{n+\alpha}{n} \sim n^\alpha.$$

Let $\sum a_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. A sequence $\{s_n\}$ is said to be summable by harmonic means,[1] if

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_{n-k}}{k+1}$$

exists.

Let $\{p_n\}$ be a sequence of real constants such that $p_0 > 0$, $p_n \geqq 0$ and let us write

$$P_n = p_0 + p_1 + \cdots + p_n, \quad P_{-1} = p_{-1} = 0.$$

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The sequence-to-sequence transformations :

$$\tau_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_{n-k} (P_n \neq 0) \tag{4}$$

defines the sequence of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable (N, p_n)[1] to the sum s if $\lim_{n\to\infty} \tau_n$ exists and is equal to s, and further is said to be regular[1] if it sums every convergent series to its ordinary sum.

In the special case in which $p_n = \frac{1}{n+1}$, the Nörlund mean reduces to the harmonic mean stated above.

2. Introduction

Recently B. S Pandey has proved the following theorem.

Theorem A.[2] *For* $-\frac{1}{2} > \alpha \geqq -\frac{5}{6}$, *the series* $\sum_n a_n L_n^{(\alpha)}(x)$ *is summable to sum s by harmonic means at the point* $x = 0$, *provided,*

$$\int_0^t |\varphi(y)| dy = o(t^{\alpha+1}), \quad as \quad t \to +0,$$

$$\int_w^n e^{y/2} y^{-\alpha/2-3/4} |\varphi(y)| dy = o(n^{-\alpha/2-1/4})$$

and

$$\int_n^\alpha e^{y/2} y^{-1/3} |\varphi(y)| dy = o(1),$$

where

$$\varphi(y) = \frac{1}{\Gamma(\alpha+1)} e^{-y} \{f(y) - s\} y^\alpha.$$

In this note we shall prove a theorem concerning Nörlund summability which includes, as a particular case, theorem A stated avove.

3. The main theorem

We establish the following theorem which includes, as a particular case, the theorem due to B. S. Pandey.[2]

Theorem. *we write*

$$\varphi(y) = \frac{1}{\Gamma(\alpha+1)} e^{-y} \{f(y) - s\} y^\alpha,$$

and let $e^{y/2}y^{-1/3}\varphi(y)$ *be Lebesgue integrable over* $(1, \infty)$.

If

$$\int_0^t |\varphi(y)|\,dy = o(t^{\alpha+1}) \tag{5}$$

as $t \to +0$, *and*

$$\int_\omega^n e^{y/2}y^{-\alpha/2-3/4}|\varphi(y)|\,dy = o(n^{-\alpha/2-1/4}) \tag{6}$$

as $n \to \infty$, *then for* $-\dfrac{1}{2} > \alpha \geqq -\dfrac{5}{6}$ *Laguerre series*

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)$$

is summable to the sum s *by regular Nörlund means* (N, p_n) *at the point* $x=0$.

4. Preliminary lemmas

Lemma 1.[3] *Let* α *be arbitrary and real,* c *and* ω *be fixed positive constants. Then for* $n \to \infty$

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4}\, O(n^{\alpha/2-1/4}) & \text{if } \dfrac{c}{n} \leqq x \leqq \omega, \\ O(n^{\alpha}) & \text{if } 0 \leqq x \leqq \dfrac{c}{n}. \end{cases}$$

Lemma 2.[3] *Let* α *be arbitrary and real,* $\omega > 0$, $0 < \eta < 4$. *We have for* $n \to \infty$

$$\max\, e^{-x/2}x^{\alpha/2+1/4}|L_n^{(\alpha)}(x)| \sim \begin{cases} n^{\alpha/2-1/4} & \text{if } \omega \leqq x \leqq (4-\eta)n, \\ n^{\alpha/2-1/12} & \text{if } x \geqq \omega. \end{cases}$$

5. Proof of the Theorem.

Let s_n denote the nth partial sum of the series $\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)$ at the point $x=0$, and τ_n denote the Nörlund means of the sequence $\{s_n\}$.

In order to prove the theorem, it is sufficient to demonstrated that

$$\tau_n - s = o(1), \quad \text{as } n \to \infty.$$

Since the integral in the left side of (6) increases with the increase of n, we have obviously $\alpha < -\dfrac{1}{2}$.

Now we have

$$s_n=\sum_{k=0}^{n}\frac{1}{\Gamma(\alpha+1)}\frac{1}{A^{\alpha}{}_k}L_k^{(\alpha)}(0)\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,L_k^{(\alpha)}(y)\,dy$$

$$=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,L_n^{(\alpha+1)}(y)\,dy.$$

Hence by the definition (4)

$$\tau_n-s=\frac{1}{P_n}\sum_{k=0}^{n}p_k(s_{n-k}-s)$$

$$=\frac{1}{P_n}\sum_{k=0}^{n-1}p_k(s_{n-k}-s)+\frac{p_n}{P_n}(s_0-s).$$

But, by (1)

$$s_0=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,L_0^{(\alpha+1)}(y)\,dy$$

$$=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}e^{-y}y^{\alpha}f(y)\,dy$$

$$=O(1),\quad \text{for } \alpha>-1,$$

and hence, by the regularity for Nörlund means, we have for $\alpha>-1$,

$$\frac{p_n}{P_n}(s_0-s)=o(1),\quad \text{as } n\to\infty.$$

Therefore we have

$$\tau_n-s=\frac{1}{P_n}\sum_{k=0}^{n-1}\frac{p_k}{\Gamma(\alpha+1)}\int_0^{\infty}e^{-y}y^{\alpha}\{f(y)-s\}\,L_{n-k}^{(\alpha+1)}(y)\,dy+o(1)$$

$$=\frac{1}{P_n}\sum_{k=0}^{n-1}p_k\int_0^{\infty}\varphi(y)\,L_{n-k}^{(\alpha+1)}(y)\,dy+o(1).$$

We now divide the integral into four parts such that

$$\int_0^{\infty}\varphi(y)\,L_{n-k}^{(\alpha+1)}(y)\,dy$$

$$=\left\{\int_0^{c/(n-k)}+\int_{c/(n-k)}^{\omega}+\int_{\omega}^{n-k}+\int_{n-k}^{\infty}\right\}\varphi(y)\,L_{n-k}^{(\alpha+1)}(y)\,dy$$

$$=A_1+B_1+C_1+D_1,\quad \text{say.}$$

Furthermore we set

$$\tau_n - s = \frac{1}{P_n}\sum_{k=0}^{n-1} p_k (A_1 + B_1 + C_1 + D_1) + o(1)$$

$$= A + B + C + D + o(1), \text{ say.}$$

In the estimation of A, we use Lemma 1 and our hypothesis (5), then

$$|A| \leqq \frac{1}{P_n}\sum_{k=0}^{n-1} p_k \int_0^{c/(n-k)} |\varphi(y)|\, L_{n-k}^{(\alpha+1)}(y)\,dy$$

$$= \frac{1}{P_n}\sum_{k=0}^{n-1} p_k\, O(n-k)^{\alpha+1}\, o(n-k)^{-(\alpha+1)}$$

$$= o(1), \text{ as } n \to \infty.$$

Similarily, by Lemma 1

$$|B| \leqq \frac{1}{P_n}\sum_{k=0}^{n-1} p_k \int_{c/(n-k)}^{\omega} |\varphi(y)|\, L_{n-k}^{(\alpha+1)}(y)\,dy$$

$$= \frac{1}{P_n}\sum_{k=0}^{n-1} p_k \int_{c/(n-k)}^{\omega} y^{-(\alpha+1)/2-1/4} |\varphi(y)|\, O(n-k)^{(\alpha+1)/2-1/4}\,dy$$

$$= \frac{1}{P_n}\sum_{k=0}^{n-1} p_k\, O(n-k)^{\alpha/2+1/4} \int_{c/(n-k)}^{\omega} |\varphi(y)|\, y^{-\alpha/2-3/4}\,dy. \tag{7}$$

Next, by integration by parts and hypothesis (5) we get

$$\int_{c/(n-k)}^{\omega} |\varphi(y)|\, y^{-\alpha/2-3/4}\,dy = \{\Phi(y)\, y^{-\alpha/2-3/4}\}_{c/(n-k)}^{\omega} + \left(\frac{\alpha}{2}+\frac{3}{4}\right)\int_{c/(n-k)}^{\omega} \Phi(y)\, y^{-\alpha/2-7/4}\,dy$$

$$= K + o\left(\frac{C}{n-k}\right)^{\alpha/2+1/4} + \int_{c/(n-k)}^{\omega} o(y^{\alpha/2-3/4})\,dy$$

$$= K + o(n-k)^{-\alpha/2-1/4}, \text{ as } n \to \infty,\ \alpha < -\frac{1}{2}, \tag{8}$$

where

$$\Phi(t) = \int_0^t |\varphi(y)|\,dy.$$

Hence we have for $\alpha < -\frac{1}{2}$

$$|B| \leqq \frac{1}{P_n} \sum_{k=0}^{n-1} p_k O(n-k)^{\alpha/2+1/4} \{K + o(n-k)^{-\alpha/2-1/4}\}$$

$$= o(1), \text{ as } n \to \infty,$$

by (7) and (8).

In the estimation of C, we use Lemma 2 and hypothesis (6), then

$$|C| \leqq \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \int_{\omega}^{n-k} |\varphi(y)| \, |L_{n-k}^{(\alpha+1)}(y)| \, dy$$

$$\leqq \frac{K}{P_n} \sum_{k=0}^{n-1} p_k \int_{\omega}^{n-k} |\varphi(y)| \, e^{y/2} y^{-(\alpha+1)/2-1/4} (n-k)^{(\alpha+1)/2-1/4} \, dy$$

$$= \frac{K}{P_n} \sum_{k=0}^{n-1} p_k \int_{\omega}^{n-k} |\varphi(y)| \, e^{y/2} y^{-\alpha/2-3/4} (n-k)^{\alpha/2+1/4} \, dy$$

$$\leqq \frac{K}{P_n} \sum_{k=0}^{n-1} p_k (n-k)^{\alpha/2+1/4} o(n-k)^{-\alpha/2-1/4}$$

$$= o(1), \text{ as } n \to \infty.$$

Lastly we shall estimate D.

By hypothesis on $\varphi(t)$ we get

$$\int_n^{\infty} e^{y/2} y^{-1/3} |\varphi(y)| \, dy = o(1), \text{ as } n \to \infty.$$

Hence we use Lemma 2, then for $-\frac{1}{2} > \alpha \geqq -\frac{5}{6}$,

$$|D| \leqq \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \int_{n-k}^{\infty} |\varphi(y)| \, |L_{n-k}^{(\alpha+1)}(y)| \, dy$$

$$= \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \int_{n-k}^{\infty} |\varphi(y)| \, e^{y/2} y^{-(\alpha+1)/2-1/4} (n-k)^{(\alpha+1)/2-1/12} \, dy$$

$$= \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \int_{n-k}^{\infty} |\varphi(y)| \, e^{y/2} y^{-\alpha/2-3/4} (n-k)^{\alpha/2+5/12} \, dy$$

$$= \frac{1}{P_n} \sum_{k=0}^{n-1} p_k (n-k)^{\alpha/2+5/12} \int_{n-k}^{\infty} |\varphi(y)| \, e^{y/2} y^{-1/3} \frac{1}{y^{\alpha/2+5/12}} \, dy$$

$$\leqq \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \int_{n-k}^{\infty} |\varphi(y)| \, e^{y/2} y^{-1/3} \, dy$$

$$= o(1), \text{ as } n \to \infty.$$

Collecting above estimations we have

$$\tau_n - s = o(1), \text{ as } n \to \infty.$$

This completes the proof of our theorem.

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References

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