On the Absolute Summability Factors of Fourier Series

Kazuo IWAI*

(Received 26 October 1971)

1. Definitions and Notations. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$.

Let the Fourier series of $f(t)$ be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t).$$

We shall use throughout this note the following notations

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$\Phi(t) = \int_0^t |\varphi(u)|\,du,$$

$$s_n(t) = \sum_{\nu=0}^{n} A_\nu(t).$$

Let $s_n^\alpha(t)$ denote the nth Cesàro means of order α $(\alpha > -1)$ of the series $\sum A_\nu(t)$. The series $\sum A_\nu(t)$ is said to be absolute summable (C, α) with index k, or simply summable $|C, \alpha|_k$ $(k \geqq 1,\ \alpha > -1)$, at $t=x$, if,

$$\sum_n n^{k-1} |s_n^\alpha(x) - s_{n-1}^\alpha(x)|^k < \infty.$$

Summability $|C, \alpha|_1$ is the same as ordinary absolute Cesàro summability of order α $(\alpha > 1)$.

Let $\lambda = \{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leqq 1$ and $\lambda_1 = 1$.

The sequence-to-sequence transformation

$$V_n(\lambda) = V_n(\lambda; t) = \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+1}^{n} s_\nu(t)$$

* Assistant Professor, Institute of Mathematics, Faculty of Engineering, Shinshu University, Nagano-shi.

defines the sequence $\{V_n(\lambda)\}$ of generalized de la Vallée Poussin means of the sequence $\{s_n(t)\}$ generated by the sequence $\{\lambda_n\}$.

The series $\sum_n A_n(t)$ is said to be summable $|V, \lambda|$, at $t = x$, if the series

$$\sum_{n=1}^{\infty} | V_{n+1}(\lambda;\, t) - V_n(\lambda;\, t) |$$

is convergent at $t = x$.

A sequence $\{\lambda_n\}$ is said to be convex when

$$\Delta^2\lambda_n \geqq 0, \quad n = 1, \ 2, \ 3, \cdots\cdots$$

with the notation

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}, \quad \Delta^2\lambda_n = \Delta(\Delta\lambda_n).$$

K denotes a positive constant not necessarily the same at each occurrence.

2. Introduction. Regarding the absolute Cesàro summability factors of the Lebesgue Fourier series, various theorems are known. For example Fu Cheng Hsiang established the following theorem.

Theorem A.[1] *If*

$$\int_0^t |\varphi(u)|\,du = O(t),$$

as $t \to 0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{A_n(t)}{n^{\alpha}}$$

is summable $|C, 1|$, *at* $t = x$, *for every* $\alpha > 0$.

In this note we shall show that the theorem due to Fu Cheng Hsiang mentioned above, is also extended to the theorems concerning $|C, 1|_k$ $(k \geqq 1)$ and $|V, \lambda|$ summability factors, and further is a particular case of more general theorem due to N. Singh.[2]

Also, P. L. Sharma and B. L. Gupta[3] have given a theorem concerning $|C, 1|$ summability factors of Fourier series.

In the follwing section we shall show that it is to be easily obtained from two well-know theorems.[4),5)]

3. We shall prove first the following theorem.

Theorem 1. *If*

$$\int_0^t |\varphi(u)|\,du = O(t), \tag{1}$$

as $t \to +0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{A_n(t)}{n^{\alpha}}$$

is summable $|C, 1|_k (k \geqq 1)$ *for every* $\alpha > 0$.

In order to prove this theorem we require the following lemmas.

Lemma 1.[6] *Let* $\alpha > -1$ *and let* $\tau_n^{\alpha}(t)$ *be the n-th Cesàro mean of order* α *of the sequence* $\{nA_n(t)\}$, *then*

$$\tau_n^{\alpha}(t) = n\{s_n^{\alpha}(t) - s_{n-1}^{\alpha}(t)\}. \tag{2}$$

Lemma 2.[1] *Write*

$$U_n(t) = \sum_{k=0}^{n} (n+2-k)\cos(n+2-k)t,$$

then

$$U_n(t) = O\begin{cases} nt^{-1} & (nt \geqq 1), \\ n^2 & (\textit{for all } t). \end{cases} \tag{3}$$

Lemma 3.[1]

$$\left|\frac{1}{n+1}\left\{\sum_{\nu=1}^{n} U_\nu(t)\,\Delta\frac{1}{(\nu+2)^{\alpha}}\right\}\right| \leqq \begin{cases} \dfrac{K}{tn^{\alpha}} + \dfrac{K}{nt^{2-\alpha}} & (nt \geqq 1), \\ kn^{1-\alpha} & (\textit{for all } t). \end{cases} \tag{4}$$

Proof of Theorem 1. We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \varphi(t)\cos nt\,dt.$$

Let $\tau_n(x)$ be the nth Cesàro mean of first order of the sequence $\{nA_n(x)/n^{\alpha}\}$. To prove our theorem it is evident by Lemma 1 that we have only to

$$\sum_{n=1}^{\infty} \frac{|\tau_n(x)|^k}{n} < \infty. \tag{5}$$

Now we have

$$\frac{\pi}{2}\tau_n(x)=\int_0^{\pi}\varphi(t)\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{(\nu+2)\cos(\nu+2)t}{(\nu+2)^{\alpha}}dt.$$

By Abel's transformation, we get

$$\frac{\pi}{2}\tau_n(x)=\int_0^{\pi}\varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^{n}U_\nu(t)\Delta\frac{1}{(\nu+2)^{\alpha}}\right\}dt$$

$$+\int_0^{\pi}\varphi(t)\frac{1}{n+1}\frac{U_n(t)}{(n+3)^{\alpha}}dt$$

$$=I_{1n}+I_{2n},\ \text{say.}$$

Thus, on writing

$$I_{1n}=\int_0^{1/n}+\int_{1/n}^{\pi}=I_{3n}+I_{4n},\ \text{say,}$$

we see that, by Lemma 3 and condition (1) of our theorem

$$I_{3n}=O\left(n^{1-\alpha}\int_0^{1/n}|\varphi(t)|dt\right)$$

$$=O(n^{-\alpha}),$$

and

$$I_{4n}=O\left(\frac{1}{n^{\alpha}}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}dt\right)+O\left(\frac{1}{n}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^{2-\alpha}}dt\right).$$

Also, we have by (1)

$$\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}dt=\left\{\frac{\Phi(t)}{t}\right\}_{1/n}^{\pi}+\int_{1/n}^{\pi}\frac{\Phi(t)}{t^2}dt$$

$$=O(1)+O\left(\int_{1/n}^{\pi}\frac{1}{t}dt\right)$$

$$=O(\log n),$$

and hence

$$\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t^{2-\alpha}}dt\leqq n^{1-\alpha}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}dt$$

$$=O(n^{1-\alpha}\log n).$$

Consequently, we have

$$I_{4n} = O(\log n/n^{\alpha}).$$

On the other hand, we write

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n}, \text{ say.}$$

Then, by Lemma 2 and condition (1)

$$I_{5n} = O\left(n^{1-\alpha}\int_0^{1/n} |\varphi(t)|dt\right)$$
$$= O(n^{-\alpha}),$$

and

$$J_{6n} = O\left(n^{-\alpha}\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}dt\right)$$
$$= O(\log n/n^{\alpha}).$$

From the above analysis we obtain that for every $\alpha > 0$,

$$\sum_{n=1}^{\infty} \frac{|I_{in}|^k}{n} < \infty \ (i = 3,\ 4,\ 5,\ 6),\ k \geqq 1.$$

Therefore, by Minkowski's inequality we have

$$\sum_{n=1}^{\infty} \frac{|\tau_n(x)|^k}{n} < \infty.$$

This proves theorem 1.

4. We shall prove here the following theorem.

Theorem 2. *Let $\mu(x)$ $(x \geqq 0)$ be a function monotone decreasing and satisfying the conditon*

$$\sum_{n=2}^{\infty} \frac{\mu(n)\log n}{\lambda_n} < \infty. \tag{6}$$

Let $\{\log n/\lambda_n\}$ be an ultimately non-increasing sequence.
If

$$\int_0^t |\varphi(u)|du = O(t), \tag{7}$$

as $t \to 0$, *then the series*

$$\sum_n \mu(n) A_n(x)$$

is summable $|V, \lambda|$ *at the point* x.

It is easy to see that, by taking $\mu(n) = n^{-\alpha}$ $(\alpha > 0)$ and $\lambda_n = n$, the result of Fu Cheng Hsiang[1] follows from our Theorem 2. Further, L. Leindler[7] has proved the theorems (i. e., Leindler [8], Theorems 3 and 4) concerning $|V, \lambda|$ summability factors, but the result[1] of before-mentioned Fu Cheng Hsiang's theorem is not obtained from them.

Proof of Theorem 2. Let $V_n(\lambda)$ denote the nth de la Vallée Poussin mean of the series $\sum \mu(n) A_n(x)$ and let $\tau_n(x) = | V_{n+1}(\lambda;\, x) - V_n(\lambda; x) |$.

In order to prove the theorem, it is sufficient to demonstrate that

$$\sum_n |\tau_n(x)| < \infty.$$

As easy computation gives that

$$V_{n+1}(\lambda) - V_n(\lambda) = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n \} \mu(k) A_k(x).$$

Using that

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \cos nt \, dt,$$

we have

$$\tau_n(x) = \frac{2}{\pi} \left| \int_0^{\pi} \frac{\varphi(t)}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n \} \mu(k) \cos kt \, dt \right|$$

$$\leqq \frac{2}{\pi} \left\{ \left| \int_0^{1/n} \right| + \left| \int_{1/n}^{\pi} \right| \right\} = \frac{2}{\pi} \left\{ \tau_n^1(x) + \tau_n^2(x) \right\}, \text{ say.}$$

By the hypotheses we get that

$$\tau_n^1(x) = O(1) \frac{1}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n \} \mu(k) \int_0^{1/n} |\varphi(t)| \, dt$$

$$= O(1) \frac{1}{n\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n) \mu(k).$$

Let $\sum_n{}'$ be the summation over all n satisfying $\lambda_{n+1} = \lambda_n$; and $\sum_n{}''$ the summation over all n where $\lambda_{n+1} > \lambda_n$.

Then, by the hypotheses we have

$$\sum_n{}' \tau_n^1(x) = O(1)\sum_n{}' \frac{1}{n\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} \lambda_n \mu(k)$$

$$= O(1)\sum_{k=2}^{\infty} \mu(k) \sum_{n=k-1}^{k+\lambda_k-1} \frac{1}{n\lambda_n}$$

$$= O(1)\sum_{k=2}^{\infty} \frac{\mu(k)}{k-1}\frac{\lambda_k+1}{\lambda_k-1}$$

$$= O(1)\sum_{k=2}^{\infty} \frac{\mu(k)(\lambda_{k-1}+2)}{\lambda^2{}_{k-1}}$$

$$= O(1)\sum_{k=2}^{\infty} \frac{\mu(k-1)}{\lambda_{k-1}}$$

$$= O(1).$$

On the other hand, in the case $\lambda_{n+1} > \lambda_n$,

$$(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n = \lambda_n + k - n - 1,$$

and $n - k + \lambda_k \geqq \lambda_n$ $(n > k)$, so we get that

$$\sum_n{}'' \tau_n^1(x) = O(1)\sum_n{}'' \frac{1}{n\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} \{(\lambda_{n+1}-\lambda_n)(k-n-1)+\lambda_n\}\mu(k)$$

$$= O(1)\sum_n{}'' \frac{1}{n\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n+1} \lambda_k \mu(k)$$

$$= O(1)\sum_{k=2}^{\infty} \lambda_k \mu(k) \sum_{n\geqq n}{}'' \frac{1}{n\lambda_n^2}.$$

Because in $\sum''$ there are only the indices n having the property $\lambda_{n+1} > \lambda_n$, it holds

$$\sum_{n\geqq k}{}'' \frac{1}{n\lambda_n^2} \leqq \sum_{\nu=\lambda_k}^{\infty} \frac{1}{k}\frac{1}{\nu^2}$$

$$= O\left(\frac{1}{k}\frac{1}{\lambda_k}\right),$$

so it is easy to see that

$$\sum_n'' \tau_n^1(x) = O(1)\sum_{k=2}^{\infty}\frac{\mu(k)}{k}$$

$$= O(1)\sum_{k=2}^{\infty}\frac{\mu(k)}{\lambda_k}$$

$$= O(1).$$

From the above analysis we obtain that

$$\sum_n \tau_n^1(x) < \infty.$$

In the following steps we shall prove that

$$\sum_n |\tau_n^2(x)| < \infty.$$

Let

$$\alpha_k^{(n)} = \{(\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n\}\frac{\mu(k)}{k}.$$

By Abel's transformation, we get

$$\tau_n^2(x) = \left|\int_{1/n}^{\pi}\frac{\varphi(t)}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}\{(\lambda_{n+1} - \lambda_n)(k - n - 1) + \lambda_n\}\ k\ \cos kt\frac{\mu(k)}{k}dt\right|$$

$$= \left|\int_{1/n}^{\pi}\frac{\varphi(t)}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n+1}(k\cos kt)\,\alpha_k^{(n)}\,dt\right|$$

$$\leqq \int_{1/n}^{\pi}\frac{|\varphi(t)|}{\lambda_n\lambda_{n+1}}\sum_{k=n-\lambda_n+2}^{n}S_k(t)\,\varDelta\alpha_k^{(n)}\,dt$$

$$+ \int_{1/n}^{\pi}\frac{|\varphi(t)|}{\lambda_n\lambda_{n+1}}s_{n-\lambda n+1}(t)\,\alpha_{n-\lambda n+2}^{n}\,dt$$

$$+ \int_{1/n}^{\pi}\frac{|\varphi(t)|}{\lambda_n\lambda_{n+1}}S_{n+1}(t)\,\alpha_{n+1}^{n}\,dt$$

$$= J_1 + J_2 + J_3, \text{ say,}$$

where

$$S_n(t) = \sum_{k=1}^{n}k\ \cos kt.$$

We now discuss $\sum J_1$.

We have by hypothesis (7)

$$\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} dt = \left\{\frac{\Phi(t)}{t}\right\}_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{|\Phi(t)|}{t^2} dt$$
$$= O(\log n).$$

Hence we have by (4) of § 3

$$\sum_n J_1 = \sum_n \int_{1/n}^{\pi} \frac{|\varphi(t)|}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n} S_k(t) \Delta\alpha_k^{(n)} dt$$
$$= O(1) \sum_n \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k \Delta\alpha_k^{(n)}$$
$$= O(1) \left\{\sum_n{}' + \sum_n{}''\right\} \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k\Delta\alpha_k^{(n)}$$
$$= O(1) (I_1' + I_1''), \text{ say.}$$

Then we have first by hypothesis (6)

$$I_1' = \sum_n{}' \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k \Delta\alpha_k^{(n)}$$
$$= \sum_n{}' \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} k \lambda_n \left\{\frac{\mu(k)}{k} - \frac{\mu(k+1)}{k+1}\right\}$$
$$\leqq \sum_n{}' \frac{\log n}{\lambda_n} \sum_{k=n-\lambda_n+2}^{n} \{\mu(k) - \mu(k+1)\}$$
$$\leqq \sum_n{}' \frac{\log n}{\lambda_n} \mu(n - \lambda_n + 2) + \sum_n{}' \frac{\log n}{\lambda_n} \mu(n+1)$$
$$\leqq \sum_n{}' \frac{\log(n - \lambda_n + 2)}{\lambda_{n-\lambda_n+2}} \mu(n - \lambda_n + 2) + \sum_n{}' \frac{\log n}{\lambda_n} \mu(n)$$
$$= O(1).$$

On the other hand, Abel's transformation gives that

$$\sum_{k=n-\lambda_n+2}^{n} k \Delta\alpha_k^{(n)} \leqq \sum_{k=n-\lambda_n+2}^{n} \alpha_k^{(n)},$$

so we have by $n - \lambda_n \geqq k - \lambda_k$ $(n > k)$ and our hypotheses

$$I_1'' \leqq \sum_n'' \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \alpha_k^{(n)}$$

$$\leqq \sum_n'' \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} (\lambda_n + k - n - 1) \frac{\mu(k)}{k}$$

$$\leqq \sum_n'' \frac{\log n}{\lambda_n^2} \sum_{k=n-\lambda_n+2}^{n} \lambda_k \frac{\mu(k)}{k}$$

$$\leqq \sum_{k=2}^{\infty} \frac{\lambda_k \, \mu(k)}{k} \sum_{n \geqq k}'' \frac{\log n}{\lambda_n^2}$$

$$\leqq \sum_{k=2}^{\infty} \frac{\lambda_k \, \mu(k)}{k} \sum_{\nu=\lambda_k}^{\infty} \frac{\log(\nu + k)}{\nu^2}$$

$$= O(1) \sum_{k=2}^{\infty} \frac{\lambda_k \, \mu(k)}{k} \frac{\log(\lambda_k + k)}{\lambda_k}$$

$$= O(1) \sum_{k=2}^{\infty} \frac{\mu(k)}{\lambda_k} \log k$$

$$= O(1).$$

From the above analysis we obtain that

$$\sum_n J_1 = O(1).$$

Next we shall discuss $\sum J_2$.

$$\sum_n J_2 \leqq \sum_n \int_{1/n}^{\pi} \frac{|\varphi(t)|}{\lambda_n \lambda_{n+1}} S_{n-\lambda_n+1}(t) \, \alpha_{n-\lambda_n+2}^{(n)} dt$$

$$= O(1) \sum_n \frac{\log n}{\lambda_n^2} (n - \lambda_n + 1) \, \alpha_{n-\lambda_n+2}^{(n)}$$

$$= O(1) \left(\sum_n' + \sum_n'' \right) \frac{\log n}{\lambda_n^2} (n - \lambda_n + 1) \, \alpha_{n-\lambda_n+2}^{(n)}$$

$$= O(1) \, (I_2' + I_2''), \text{ say.}$$

Then

$$I_2' = \sum_n' \frac{\log n}{\lambda_n^2} (n - \lambda_n + 1) \, \lambda_n \frac{\mu(n - \lambda_n + 2)}{n - \lambda_n + 2}$$

$$\leqq {\sum_n}' \frac{\log n}{\lambda_n} \mu(n-\lambda_n+2)$$

$$\leqq {\sum_n}' \frac{\log(n-\lambda_n+2)}{\lambda_{n-\lambda_n+2}} \mu(n-\lambda_n+2)$$

$$\leqq \sum_{k=1}^{\infty} \frac{\log k}{\lambda_k} \mu(k)$$

$$= O(1).$$

Similiarly we have

$$I_2'' \leqq {\sum_n}'' \frac{\log n}{\lambda_n^2} (n-\lambda_n+1) \frac{\mu(n-\lambda_n+2)}{n-\lambda_n+2}$$

$$= O(1).$$

From the above analysis we obtain that

$$\sum_n J_2 = O(1).$$

Finally, we shall estimate $\sum J_3$.

Proceedings in the same way, using our hypotheses

$$\sum_n J_3 \leqq \sum_n \int_{1/n}^{\pi} \frac{|\varphi(t)|}{\lambda_n \lambda_{n+1}} S_{n+1}(t) \alpha_{n+1}^{n} dt$$

$$= O(1) \sum_n \frac{\log n}{\lambda_n^2} (n+1) \alpha_{n+1}^{(n)}$$

$$= O(1) \sum_n \frac{\log n}{\lambda_n} \mu(n+1)$$

$$= O(1).$$

Consequently we have also

$$\sum_n |\tau_n^2(x)| < \infty.$$

Collecting above estimations we have

$$\sum_n |\tau_n(x)| \leqq \sum_n |\tau_n^1(x)| + \sum_n |\tau_n^2(x)|$$

$$< \infty.$$

Hence we have

$$\sum_n \mu(n) A_n(t) \in |V, \lambda|, \text{ at } t = x.$$

This completes the proof of our Theorem 2.

5. We shall show in this section that Theorem A due to Fu Cheng Hsiang is a particular case of the following Singh's theorem.

Theorem B.[2] *If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n n^{-1} (\log n)^{1/2}$ is convergent, then the series*

$$\sum_n \frac{\lambda_n}{(\log n+1)^{\beta}} A_n(t)$$

at $t=x$, is summable $|C, 1|$, provided that

$$\int_0^t |\varphi(u)| du = O\left\{t\left(\log\frac{1}{t}\right)^{\beta}\right\}, \text{ as } t \to +0, \ \beta \geqq 0.$$

In Singh's theorem stated above, if we set $\lambda_n = n^{-\alpha}$, the sequence $\{\lambda_n\}$ is clearly convex, furthermore

$$\sum_n \frac{\lambda_n}{n} (\log n)^{1/2} = \sum_n \frac{(\log n)^{1/2}}{n^{1+\alpha}} < \infty.$$

Consequently, Theorem A due to Fu Cheng Hsiang is a particular case in which $\beta = 0$ of Theorem B stated above.

6. P. L. Sharma and B. L. Gupta demonstrated the following theorem.

Theorem C.[3] *If $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent then the series $\sum \lambda_n A_n(t)$ at $t = x$, is summable $|C, 1|$, provided that*

$$\int_0^t |\varphi(u)| du = O\left\{t/ (\log\frac{1}{t})^{\alpha}\right\}, \text{ as } t \to +0, \ \alpha > \frac{1}{2}.$$

We shall show in this section that this theorem is to be easily obtained from the following theorems.

Theorem D.[4] *If, for some $\alpha > \frac{1}{2}$*

$$\int_0^t |\varphi(u)|\,du = O\,\{t/(\log\frac{1}{t})^{\alpha}\},$$

as $t \to +0$, *then*

$$\sum_{\nu=0}^{n} |s_\nu(x) - f(x)|^2 = O(n).$$

If we replace the small order by capital order in the Wang's theorem,[4) we get this theorem.

Theorem E.[5) *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, and*

$$\sum_{\nu=1}^{n} |s_\nu(x) - f(x)| = O\{n(\log n)^k\},\quad k \geqq 0$$

as $n \to \infty$, *then the series*

$$\sum_n \{\log(n+1)\}^{-k}\lambda_n A_n(x)$$

is summable $|C,\ 1|$.

Proof of Theorem C. By the help of Cauchy's inequality and Theorem D, we have

$$\sum_{\nu=0}^{n} |s_\nu(x) - f(x)| \leqq \Big(\sum_{\nu=0}^{n} |s_\nu(x) - f(x)|^2\Big)^{1/2}\Big(\sum_{\nu=0}^{n} 1\Big)^{1/2}$$

$$= O(n^{1/2})\,n^{1/2}$$

$$= O(n).$$

Accordingly, Theorem C is clearly a particular case in which $k=0$ of Theorem E stated above.

References

1) Fu C. HSIANG, *"On |C, 1| summability factors of Fourier series,"* Pacific J. of Math., vol. 33, 1970, pp. 139–147.

2) N. SINGH, *"On the absolute Cesàro summability of factored Fourier series,"* Riv. Mat. Univ. Parma, vol. 8, 1967, pp. 181–188.

3) P. L. SHARMA and B. L. GUPTA *"Absolute Cesàro summability of the factored Fourier series,"* Kōdai Math. Sem. Rep., vol. 22, 1970, pp. 61–64.

4) Fu C. WANG, *"Note on H_2 summability of Fourier series,"* J. of London Math. Soc., Vol. 19, 1944, pp. 208–208.

5) B. N. PRASAD and S. N. BHATT, *"The summability factors of a Fourier series,"* Duke Math. J., vol. 24, 1957, pp. 103–117.

6) E. KOGBETLIANTZ, *"Sur les séries absolument sommables par la méthode des moyennes arithmétique,"* Bull. des Sc. Math., (2), 1925, pp. 234–256.

7) L. LEINDLER, *"On the absolute summability factors of Fourier series,"* Acta Sci. Math. de Szeged., Tom. 28 (1967), pp. 323–336.