On the Absolute Cesàro Summability Factors of Fourier Series

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1. Definitions and Notations

Let $\sum a_n$ be a given infinite series and let s_n denote its n–th partial sum. Let s_n^α denote the n–th Cesàro means of order α ($\alpha > -1$) of the sequences $\{s_n\}$.

The series $\sum a_n$ is said to be absolutely summable (C, α), or summable $|C, \alpha|$, if $\{s_n^\alpha\} \in BV$, that is to say,

$$\sum | s_n^\alpha - s_{n-1}^\alpha | < \infty.$$

Also, the series $\sum a_n$ is said to be absolutely summable (C, α) with index k, or simply summable $|C, \alpha|_k$ ($k \geqq 1$, $\alpha \geqq -1$), if

$$\sum n^{k-1} | s_n^\alpha - s_{n-1}^\alpha |^k < \infty.$$

Summability $|C, \alpha|_1$ is the same as summability $|C, \alpha|$.

A sequence $\{\lambda_n\}$ is said to be convex if $\Delta^2\lambda_n \geqq 0$, $n = 1, 2, \cdots\cdots$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesque over $(-\pi, \pi)$.

Let the Fourier series of $f(t)$ be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t),$$

where we can assume, without loss of generality, that $a_0 = 0$.

We shall use throughout this note the following notations.

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\},$$

$$s_n(x) = \sum_{\nu=0}^{n} A_\nu(x).$$

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Throughout this note K will denote positive constant which will not necessarily be the same at different occurrences.

2. Introduction

Recently Pati has proved the following theorems.

Theorem A [1: Theorem 1]. *if* $\{\lambda_n\}$ *be a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$, *then a necessary and sufficient condition for* $\sum \lambda_n A_n(t)$ *to be summable* $|C, 1|$, *when*

$$\int_0^t |\varphi(u)|\, du = o(t), \tag{1}$$

is that

$$\sum n^{-1}\lambda_n |s_n(t) - f(t)| < \infty. \tag{2}$$

Theorem B [1: Theorem 2]. *If* $\{\lambda_n\}$ *be a convex sequence such that* $\sum n^{-1}\lambda_n (\log n)^{1/2} < \infty$, *then at every point* $t = x$ *at which* (1) *holds,* $\sum \lambda_n A_n(t)$ *is summable* $|C, 1|$.

Subsequently Singh[2] obtained the following result which generalizes Theorem A to the theorem concerning summability $|C, 1|_k$.

Theorem C [2: § 1. 4]. *If* $\{\lambda_n\}$ *be a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$, *then a necessary and sufficient condition for* $\sum \lambda_n A_n(t)$, *at* $t = x$, *to be summable* $|C, 1|_k$, $k \geqq 1$, *when*

$$\int_0^t |\varphi(u)|^k\, du = o(t), \qquad as\ t \to 0, \tag{3}$$

is that

$$\sum n^{-1} \lambda_n^k\ |s_n(x) - f(x)|^k < \infty.$$

For $k = 1$, it may be observed that theorem A of Pati mentioned above is a particular case of Theorem C.

Now, in this note we shall show that theorem B of Pati mentioned above is also generalized to the theorem concerning summability $|C, 1|_k$.

3. Theorem and Proof

In what follows, we shall prove the following theorem.

Theorem. *If* $\{\lambda_n\}$ *be a convex sequence such that,*

$$\sum n^{-1}\lambda_n(\log n)^{1-k/2} < \infty \qquad for\ 1 \leqq k < 2, \tag{4}$$

and

$$\sum n^{-1}\lambda_n < \infty \qquad for \quad k \geqq 2 \tag{5}$$

then at every point $t = x$ *at which* (3) *holds,* $\sum \lambda_n A_n(t)$ *is summable* $|C, 1|_k$ *for every* $k \geqq 1$.

For the proof of our theorem we shall require a number of lemmas.

Lemma 1 [3: Lemma 1]. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then* λ_n *is non-negative and decreasing,* $n\Delta\lambda_n = o(1)$ *and* $\lambda_n \log n = o(1)$, *as* $n \to \infty$.

Lemma 2. *Under the same conditions as in Lemma* 1, *for every* $k \geqq 1$,

$$\sum_{n=1}^{m} \Delta(\lambda_n^k) \log(n+1) < \infty, \quad as \quad n \to \infty. \tag{6}$$

The case $k = 1$ is referred to Pati and $k > 1$ is referred to Prasad, respectively, where references are given.

Lemma 3 [4: §4]. *If*

$$\int_0^t |\varphi(u)|\, du = o(t), \tag{7}$$

then

$$\sum_{\nu=1}^{n} |s_\nu(x) - f(x)|^k = \begin{cases} o\{n(\log n)^{k/2}\} & if \quad 1 \leqq k \leqq 2, & (8) \\ o\{n(\log n)^{k-1}\} & if \quad k \geqq 2. & (9) \end{cases}$$

Proof of the Theorem. In order to prove our theorem, we have to establish by Theorem C that, for $k \geqq 1$,

$$\sum n^{-1} \lambda_n^k |s_n(x) - f(x)|^k < \infty.$$

By Abel transformation, we have

$$\sum_{n-1}^{m} n^{-1}\lambda_n^k\, |s_n(x) - f(x)|^k$$

$$= \sum_{n=1}^{m-1} \Delta(n^{-1}\lambda_n^k) \sum_{\nu=1}^{n} |s_\nu(x) - f(x)|^k + m^{-1}\lambda_m^k \sum_{\nu=1}^{m} |s_\nu(x) - f(x)|^k$$

$$= \sum_{n=1}^{m-1} \{n(n+1)\}^{-1}\lambda_n^k \sum_{\nu=1}^{n} |s_\nu(x) - f(x)|^k$$

$$+ \sum_{n=1}^{m-1} (n+1)^{-1}\Delta\lambda_n^k \sum_{\nu=1}^{n} |s_\nu(x) - f(x)|^k + m^{-1}\lambda_m^k \sum_{\nu=1}^{m} |s_\nu(x) - f(x)|^k$$

$$\equiv I_1 + I_2 + I_3, \text{ say.}$$

By Hölder inequality we get at once (7) of Lemma 3 from our hypothesis (3).

Accordingly, it may be permitted to make use of Lemma 3 throughout in what follows.

Now, we consider two cases separately for index k.

Case (i): when $1 \leqq k < 2$.

From (8) of Lemma 3, we observe that there exist a positive number K such that

$$\sum_{\nu=1}^{n} | s_\nu(x) - f(x) |^k \leqq K n (\log n)^{k/2}, \quad (n = 1, 2, 3, \cdots\cdots).$$

Therefore, we have

$$\begin{aligned} I_1 &= \sum_{n=1}^{m-1} \{n (n+1)\}^{-1} \lambda_n^k \sum_{\nu=1}^{n} | s_\nu(x) - f(x) |^k \\ &\leqq \sum_{n=1}^{m-1} (n+1)^{-1} \lambda_n^k \, K (\log n)^{k/2} \\ &= K \sum_{n=1}^{m-1} n^{-1} \lambda_n \, \lambda_n^{k-1} (\log n)^{1-k/2} (\log n)^{k-1} \\ &= K \sum_{n=1}^{m-1} n^{-1} \lambda_n (\log n)^{1-k/2} (\lambda_n \log n)^{k-1} \\ &\leqq K \sum_{n=1}^{m-1} n^{-1} \lambda_n (\log n)^{1-k/2} \\ &< \infty, \text{ as } m \to \infty, \end{aligned}$$

by virtue of Lemma 1 and hypothesis (4) of our theorem.

Similarly, we have

$$\begin{aligned} I_2 &\leqq \sum_{n=1}^{m-1} (n+1)^{-1} \Delta \lambda_n^k \sum_{\nu=1}^{n} | s_\nu(x) - f(x) |^k \\ &\leqq K \sum_{n=1}^{m-1} (n+1)^{-1} \Delta \lambda_n^k \, n (\log n)^{k/2} \\ &\leqq K \sum_{n=1}^{m-1} \Delta \lambda_n^k (\log n)^{k/2} \\ &\leqq K \sum_{n=1}^{m-1} \Delta \lambda_n^k (\log n) \\ &< \infty, \text{ by Lemmas 3 and 2.} \end{aligned}$$

Next, by Lemmas 3 and 1 we have

$$I_3 \leqq m^{-1} \lambda_m^k \sum_{\nu=1}^{m} | s_\nu(x) - f(x) |^k$$

$$\leqq K m^{-1} \lambda_m^k \, m \, (\log m)^{k/2}$$

$$\leqq K \lambda_m^k \, (\log m)^{k/2}$$

$$= o(1), \text{ as } m \to \infty.$$

Collecting above estimations, we have, for $1 \leqq k < 2$,

$$\sum n^{-1} \lambda_n^k \; | s_n(x) - f(x) |^k < \infty.$$

This fact proves our theorem in the case $1 \leqq k < 2$.

Case (ii): When $k \geqq 2$.

From (9) of Lemma 3, we observe that there exist a positive number K such that

$$\sum_{\nu=1}^{n} | s_\nu(x) - f(x) |^k \leqq K n \, (\log n)^{k-1}.$$

Hence, we have

$$I_1 \leqq K \sum_{n=1}^{m-1} n^{-1} \lambda_n^k \, (\log n)^{k-1}$$

$$\leqq K \sum_{n=1}^{m-1} n^{-1} \lambda_n (\lambda_n \log n)^{k-1}$$

$$\leqq K \sum_{n=1}^{m-1} n^{-1} \lambda_n$$

$$< \infty, \text{ as } m \to \infty,$$

by virtue of Lemma 1 and hypothesis (5) of our theorem.

Next, we have

$$I_2 \leqq K \sum_{n=1}^{m-1} \Delta \lambda_n^k \, (\log n)^{k-1}$$

$$\leqq K \Big[\sum_{n=1}^{m} \lambda_n^k \, \Delta \{(\log n)^{k-1}\} + \lambda_m^k (\log m)^{k-1} \Big].$$

But, by virtue of Lemma 1,

$$\lambda_m^k (\log m)^{k-1} \leqq \frac{(\lambda_m \log m)^k}{\log m} = o(1), \quad \text{as} \quad m \to \infty.$$

Otherwise, we have

$$\sum_{n=1}^{m} \lambda_n^k \Delta \{(\log n)^{k-1}\} \leqq K \sum_{n=1}^{m} \lambda_n^k n^{-1} (\log n)^{k-2}$$

$$= K \sum_{n=1}^{m} \frac{(\lambda_n \log n)^k}{n(\log n)^2}$$

$$\leqq K \sum_{n=1}^{m} \frac{1}{n(\log n)^2}$$

$$< \infty,$$

by virtue of Lemma 1 again.

Finally, from (9) of Lemma 3 we have

$$I_3 \leqq K \lambda_m^k (\log m)^{k-1}$$

$$\leqq K \frac{(\lambda_m \log m)^k}{\log m}$$

$$= o(1), \quad \text{as} \quad m \to \infty.$$

Collecting above estimations, we have, for $k \geqq 2$,

$$\sum n^{-1} \lambda_n^k |s_n(x) - f(x)|^k < \infty.$$

This proves the theorem for the case $k \geqq 2$.

Thus, we obtained that at every point $t = x$ at which (3) holds, $\sum \lambda_n A_n(t)$ is summable $|C, 1|_k$, for $k \geqq 1$.

This completes the proof of our theorem.

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References

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