On some Theorems concerning the Summability $|\bar{N}, p_n|$

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1. Definitions and notations

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \cdots + p_n,\ \ P_{-1} = p_{-1} \equiv 0.$$

The sequence-to-sequence transformation:

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu \quad (P_n \neq 0) \tag{1}$$

defines the sequence $\{t_n\}$ of Nölund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n)[1] to the sum s if $\lim_{n\to\infty} t_n$ exits and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$ if the sequence $\{t_n\}$ is of bounded variation, that is

$$\sum_n |t_n - t_{n-1}| < \infty,$$

or symbolically $\{t_n\} \in BV$.

In the special case in which $p_n = 1$, the (N, p_n) mean reduces to the familiar $(C, 1)$ mean.

Also, the sequence-to-sequence transformation:

$$y_n = \frac{1}{2}(s_{n-1} + s_n),\ \ n \geqq 0,\ \ s_{-1} \equiv 0 \tag{2}$$

defines the sequence $\{y_n\}$ of the (Y)-means of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be summable (Y)[2] to the sum s if $\lim_{n\to\infty} y_n$ exists and is equal to s, and is said to be absolutely summable (Y), or summable $|Y|$ if the sequence $\{y_n\}$ is of bounded variation.

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Now, we definie the sequence–to–sequence transformation:

$$\bar{t}_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \qquad (P_n \neq 0). \tag{3}$$

The series $\sum a_n$ is said to be summable $(\bar{N}, p_n)$[1] to the sum s if $\lim_{n\to\infty} \bar{t}_n$ exists and is equal to s, and is also said to be absolutely summable $(\bar{N}, p_n)$, or summable $|\bar{N}, p_n|$ if the sequence $\{\bar{t}_n\}$ is of bounded variation, that is

$$\sum_n |\bar{t}_n - \bar{t}_{n-1}| < \infty,$$

or symbolically $\{\bar{t}_n\} \in BV$.

The product of the $(\bar{N}, p_n)$ matrix with a $(C, 1)$ matrix defines the matrix $(\bar{N}, p_n)\cdot(C, 1)$. Thus the $(\bar{N}, p_n)\cdot(C, 1)$ matrix is given by

$$d_{nk} = \frac{1}{P_n} \sum_{\nu=k}^{n} \frac{p_\nu}{\nu+1}.$$

The series $\sum a_n$ is said to be absolutely summable $(\bar{N}, p_n)\cdot(C, 1)$, or summable $|(\bar{N}, p_n)\cdot(C, 1)|$ if the sequence $\{u_n\}$ is of bounded variation, where

$$u_n = \sum_{k=0}^{n} d_{nk} s_k = \frac{1}{P_n} \sum_{\nu=0}^{n} \frac{p_\nu}{\nu+1} \sum_{k=0}^{\nu} s_k.$$

Then, the absolutely summable $(\bar{N}, p_n)\cdot(Y)$, or summable $|(\bar{N}, p_n)\cdot(Y)|$ is defined in the same way.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesque over $(0, 2\pi)$ and let its Fourier series be

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t).$$

We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}; \quad \varphi^*(t) = \varphi(t) - s;$$

$$A_n^*(t) = \frac{1}{n}(s_n - s), \ n = 1, 2, 3, \cdots,$$

where $s_n = \sum_{k=0}^{n} A_k(t)$ and s is an appropriate number independent of n.

Finally, as usual, K denotes a positive constant not necessarily the same at each occurrence.

2. Introduction

Concerning the $|N, p_n|$-summability of the series associated with a Fourier series, that is $\sum A_n^*(x)$, H. P. Dikshit proved the following:

Theorem D.[3] *If* $|\varphi^*(t)|/t$ *is integrable in* $(0, \pi)$ *and* $\{p_n\}$ *is a positive sequence such that* $\{(n+1)p_n/P_n\} \in BV$, $\{P_n^{-1}\sum_{k=0}^{n} P_k/(k+1)\} \in BV$, *then* $\sum_{n=1}^{\infty} A_n^*(x)$ *is summable* $|N, p_n|$.

In the part I of this note, we shall show that the same holds true in the summability $|\overline{N}, p_n|$ of such a Fourier series.

U. Kakkar[4] has obtained a number of results concerning the absolute summability of the iteration product.

In the part II of this note, we shall give that some analogous results are true in the summability $|\overline{N}, p_n|$.

3. Part I

We state our result as follows:

Theorem 1. *If* $|\varphi^*(t)|/t$ *is integrable in* $(0, \pi)$ *and* $\{p_n\}$ *is a positive sequence such that*

$$\{(n+1)p_n/P_n\} \in BV, \qquad \{P_n^{-1}\sum_{k=0}^{n} P_k/(k+1)\} \in BV,$$

then $\sum_{n=1}^{\infty} A_n^*(x)$ *is summable* $|\overline{N}, p_n|$.

We require the following lemma for the proof of our theorem 1.

Lemma. *If* $\{p_n\}$ *is a positive sequence such that* $\{(n+1)p_n/P_n\} \in BV$, $\{P_n^{-1}\sum_{k=0}^{n} P_k/(k+1)\} \in BV$, *then uniformly in* $0 < t \leqq \pi$,

$$\sum_{n=1}^{\infty} \left| \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n-1} \frac{P_{k-1}}{k} \sin (k+1/2)t \right| < K.$$

For the proof, reader should refer to the authur's note [5] in the "References" shown below.

Proof of theorem 1. For $\sum_{n=1}^{\infty} A_n^*(x)$ we have

$$\bar{t}_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k^* = \frac{1}{P_n}\sum_{k=0}^{n} (P_n - P_{k-1}) A_k^*(x),$$

where

$$s_k = \sum_{n=1}^{k} A_n^*(x).$$

Therefore, we have

$$\begin{aligned}
\overline{t}_n - \overline{t}_{n-1} &= \frac{1}{P_n} \sum_{k=0}^{n} (P_n - P_{k-1}) A_k^*(x) - \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} (P_{n-1} - P_{k-1}) A_k^*(x) \\
&= \sum_{k=0}^{n} A_k^*(x) - \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} A_k^*(x) - \sum_{k=0}^{n-1} A_k^*(x) + \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} P_{k-1} A_k^*(x) \\
&= A_n^*(x) + \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \sum_{k=0}^{n-1} P_{k-1} A_k^*(x) - \frac{1}{P_n} P_{n-1} A_n^*(x) \\
&= \frac{p_n}{P_n} A_n^*(x) + \frac{p_n}{P_{n-1} P_n} \sum_{k=0}^{n-1} P_{k-1} A_k^*(x).
\end{aligned}$$

But, It is easy to see that [6]

$$A_n^*(x) = \frac{1}{k\pi} \int_0^{\pi} \varphi^*(t) \frac{\sin(k+1/2)t}{\sin t/2} dt.$$

Hence, we have

$$\overline{t}_n - \overline{t}_{n-1} = \frac{p_n}{P_n} \frac{1}{n\pi} \int_0^{\pi} \varphi^*(t) \frac{\sin(n+1/2)t}{\sin t/2} dt + \frac{2}{\pi} \int_0^{\pi} \varphi^*(t) \sum_{k=1}^{n-1} \frac{p_n}{P_{n-1} P_n} P_{k-1} \frac{1}{k} D_k(t)\, dt$$

$$\equiv I + J, \text{ say,}$$

where

$$D_k(t) = \frac{\sin(k+1/2)t}{\sin t/2} = \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos kt.$$

Obviously, in order to prove the theorem 1, we have to establish that

$$\sum_n |\overline{t}_n - \overline{t}_{n-1}| \leqq \sum_n |I| + \sum_n |J| < \infty.$$

Under the hypotheses of the theorem, we have

$$\sum_n |I| \leqq \frac{1}{\pi} \sum_n \frac{p_n}{nP_n} \int_0^{\pi} \frac{|\varphi^*(t)|}{t} \frac{t}{\sin t/2} dt$$

$$\leqq K\sum_n \frac{p_n}{nP_n}$$

$$\leqq K\sum_n \frac{1}{n(n+1)}\left|\frac{(n+1)p_n}{P_n}\right|$$

$$\leqq K\sum_n \frac{1}{n(n+1)}$$

$$<\infty.$$

Let us write

$$g(n,t)=\sum_{n=1}^{\infty}\frac{p_n}{nP_n}\sum_{k=1}^{n-1}\frac{P_{k-1}}{k}\sin(k+1/2)t.$$

Then, we have, by lemma,

$$|g(n,t)|<\infty, \qquad \text{uniformly in } 0<t\leqq\pi.$$

Thus, we obtain

$$\sum_{n=1}^{\infty}|J|\leqq\int_0^{\pi}\frac{|\varphi^*(t)|}{t}\frac{t/2}{\sin t/2}|g(n,t)|dt<\infty.$$

Consequently, collecting the above estimations we obtain

$$\sum_n|\bar{t}_n-\bar{t}_{n-1}|<\infty,$$

that is,

$$\sum_{n=1}^{\infty}A^*_n(x) \text{ is summable } |\bar{N}, p_n|.$$

This completes the proof of our theorem 1.

4. **Part II**

O. Szász[2] obtained a number of results concerning (Y)-summability. Recently K. Ishiguro[7] has also obtained some results for (Y)-summability.

In what follows, we shall prove a result concerning the summability $|\bar{N}, p_n|$.

Theorem 2. *If $\{p_n\}$ is a non-negative sequence of bounded variation and if $\sum_{\nu=1}^{\infty}|\alpha_\nu|/P\nu<\infty$, then the sequence $\{\alpha_\nu\}$ is summable $|\bar{N}, p_n|$.*

Proof of theorem 2. We have, by the definition,

$$\bar{t}_n-\bar{t}_{n-1}=\frac{1}{P_n}\sum_{\nu=0}^{n}p_\nu\alpha_\nu-\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1}p_\nu\alpha_\nu$$

$$=\frac{1}{P_n}p_n\alpha_n-\left(\frac{1}{P_n}-\frac{1}{p_n}\right)\sum_{\nu=0}^{n-1}p_\nu\alpha_\nu$$
$$\equiv I_1-I_2,\ \text{say.}$$

But, by the hypotheses we have

$$\sum_{n=1}^{\infty}|I_1|\leqq K\sum_{n=1}^{\infty}\frac{|\alpha_n|}{P_n}<\infty.$$

While, for every $m\geqq 1$, we have

$$\sum_{n=1}^{m}|I_2|\leqq\sum_{n=1}^{m}\left|\frac{1}{P_{n-1}}-\frac{1}{P_n}\right|\left|\sum_{\nu=0}^{n-1}p_\nu\alpha_\nu\right|$$
$$\leqq K\sum_{n=1}^{m}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}|\alpha_\nu|$$
$$\leqq K\sum_{\nu=0}^{m-1}|\alpha_\nu|\sum_{n=\nu+1}^{m}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)$$
$$=K\sum_{\nu=0}^{m-1}|\alpha_\nu|\left(\frac{1}{P_\nu}-\frac{1}{P_m}\right)$$
$$\leqq K\sum_{\nu=0}^{m-1}\frac{|\alpha_\nu|}{P_\nu}$$
$$<\infty.$$

Hence,

$$\sum_n|I_2|<\infty.$$

Consequently, collecting our estimations we have

$$\sum_n|\bar{t}_n-\bar{t}_{n-1}|=\sum_n|I_1-I_2|<\infty.$$

That is, the sequence $\{\alpha_\nu\}$ is summable $|\bar{N}, p_n|$.

This completes the proof of our theorem 2.

Next, in analogy to U. Kakkar's results,[4] we shall give some results concerning the iteration product, for the convenience of the reader and for the sake of completeness.

Theorem 3. *If* $\{p_n\}$ *is a non-negative sequence of bounded variation and sequence* $\{a_n\}$ *is summable* $|\bar{N}, p_n|$ *then*

$$|(\bar{N}, p_n)\cdot(Y)|\ \rightleftarrows\ |\bar{N}, p_n|.$$

The result follows from the identity

$$y_n = s_n - \frac{1}{2}a_n.$$

Theorem 4. *If* $\{p_n\}$ *is a non-negative sequence of bounded variation and* $\sum |a_n|/P_n$ *is convergent, then*

$$|(\overline{N}, p_n)\cdot(Y)| \rightleftarrows |\overline{N}, p_n|.$$

The result follows from the Theorems 2 and 3.

Theorem 5. *Let* $\{p_n\}$ *be a non-negative sequence of bounded variation and*

$$\sum |s_\nu|/\{(\nu+1)P_\nu\} < \infty.$$

Then,

$$|(\overline{N}, p_n)\cdot T(y)| \rightleftarrows |(\overline{N}, p_n)\cdot(C, 1)|,$$

where T *is the transformation from the sequence* $\{y_n\}$ *to* $\{v_n\}$ *and*

$$v_n = \frac{1}{n+1}\left(\frac{1}{2}y_0 + y_1 + \cdots + y_n\right).$$

Proof of theorem 5. By the definition, we have, obviously, $y_n = s_n - \frac{1}{2}a_n$, so that,

$$v_n + \frac{s_n}{2(n+1)} = \frac{y_0 + 2y_1 + \cdots + 2y_n}{2(n+1)} + \frac{s_n}{2(n+1)}$$

$$= \frac{s_0 + (s_0 + s_1) + (s_1 + s_2) + \cdots + (s_{n-1} + s_n) + s_n}{2(n+1)}$$

$$= \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

Hence, we have

$$\sigma_n \equiv \frac{s_0 + s_1 + \cdots + s_n}{n+1} = v_n + \frac{s_n}{2(n+1)}.$$

Obviously T is absolutely regular (*i.e.*, transforms every absolutely convergent sequence into a seqence absolutely convergent to the same limit). Hence to prove the result we have to show that the sequence $\{s_n/(n+1)\}$ is $|\overline{N}, p_n|$ summable. Now the result directly follows from the application of our theorem 2 with $\alpha_\nu = s_\nu/(\nu+1)$.

References

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