On the absolute convergence of Walsh Fourier series

Yasuo OKUYAMA*

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Synopsis

The object of this note is to prove that the extension of Szasz's theorem (1) on the absolute convergence of Fourier series also holds for the Walsh Fourier series, and we consider some applications for the theorem.

I. Introduction

We shall begin with some notations and definitions:

The Rademacher functions are defined by

$$\phi_0(x) = 1\,(0 \leqq x < \frac{1}{2}),\ \ \phi_0(x) = -1\,(\frac{1}{2} \leqq x < 1)$$

$$\phi_0(x) = \phi_0(x+1),\ \ \phi_k(x) = \phi_0(2^k \cdot x)\ \ (k = 1, 2, \cdots).$$

The Walsh functions are then given by

$$\psi_0(x) \equiv 1,\ \ \psi_k(x) = \phi_{k(1)}(x)\phi_{k(2)}(x) \cdots \phi_{k(v)}(x)$$

for $k = 2^{k(1)} + 2^{k(2)} + \cdots + 2^{k(v)} \geqq 1$, where the integers $k(i)$ are uniquely determined by $k(i+1) < k(i)$.

For basis properties of Walsh functions, the reader is referred to N. J Fine (2).

Let us write, for the integral modulus of continuity and the best approximation respectively,

$$w^{(p)}(\delta) = w^{(p)}(\delta, f) = \sup_{0 \leqq y \leqq \delta} \left\{ \int_0^1 |f(x \dot{+} y) - f(x)|^p\, dx \right\}^{1/p}$$

$$E_n^{(p)} = E_n^{(p)}(f) = \inf \left\{ \int_0^1 |f(x) - p_n(x)|^p dx \right\}^{1/p}$$

where the infimum is taken over all Walsh polynomials P_n of degree not exceeding n.

Let

$$w(\delta) = w(\delta, f) = \sup_{0 \leqq h \leqq \delta} |f(x \dot{+} h) - f(x)|$$

* Lecturer of Mathematics, Faculty of Engineering, Shinshu University, Nagano.

If for some $\alpha > 0$ we have $w(\delta) \leqq C\delta^{\alpha}$, with C independent of δ, we shall say that $f(x)$ belongs to the class Λ_{α} ; in symbols

$$f(x) \in \Lambda_{\alpha}.$$

If $w^{(p)}(\delta) = 0(\delta^{\alpha})$, we write $f(x) \in \Lambda_{\alpha}^{p}$.

Finally, A denotes a positive absolute constant that is not always the same in each occurrence.

II. Several theorems

Our main theorem now reads as follows;

Theorem 1. Let $1 \leqq p \leqq 2$. Suppose that $f(x) \in L^{p}(0, 1)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

If the series $\sum_{n=1}^{\infty} n^{\gamma-1/q} w^{(p)}\left(\frac{1}{n}, f\right)$ converges, then the series $\sum_{n=1}^{\infty} n^{\gamma} |a_n|$ converges for $\gamma < {}^1/_q$ where ${}^1/_p + {}^1/_q = 1$. However, the conclusion ceases to be true if $2 < p < \infty$.

To prove this theorem, we require following two statements;

Theorem (F. Riesz). Let $1 < p \leqq 2$. If $f(x) \in L^{p}(0, 1)$, then the Walsh Fourier coefficients

$$a_k = \int_0^1 f(x)\psi_k(x)dx, \ k = 0, 1, 2, \cdots$$

satisfy the inequality

$$\left\{\sum_{k=0}^{\infty} |a_k|^q\right\}^{1/q} \leqq \left\{\int_0^1 |f(x)|^p dx\right\}^{1/p}$$

where ${}^1/_p + {}^1/_q = 1$.

This theorem is well known (see(6)).

Lemma. The following inequality holds :

$$\int_0^{1/n} (\psi_k(y) - 1)^2 dy \geqq \frac{1}{n} \text{ for } k \geqq 2n.$$

Proof

If the dyadic expansion of the integer n is

$$n = 2^{n(1)} + 2^{n(2)} + \cdots + 2^{n(v)} \geqq 1, \ \ n(1) > n(2) > \cdots > n(v) \geqq 0,$$

then we have

$$2n \geqq 2^{n(1)+1} > 2^{n(1)}.$$

By the definitions of Walsh functions, we always have

$$\int_0^{1/2^{n(1)+1}} \psi_k(y)dy = 0.$$

Thus

$$\int_0^{1/n} \psi_k(y)dy = \int_0^{1/2^{n(1)+1}} \psi_k(y)dy + \int_{1/2^{n(1)+1}}^{1/n} \psi_k(y)dy$$
$$= \int_{1/2^{n(1)+1}}^{1/n} \psi_k(y)dy$$

and

$$\left|\int_0^{1/n} \psi_k(y)\, dy\right| \leqq \int_{1/2^{n(1)+1}}^{1/n} dy = \frac{1}{n} - \frac{1}{2^{n(1)+1}} \leqq \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

Hence we have

$$\int_0^{1/n} (\psi_k(y) - 1)^2 dy = 2\int_0^{1/n} dy - 2\int_0^{1/n} \psi_k(y)dy \geqq 2\cdot\frac{1}{n} - 2\cdot\frac{1}{2n} = \frac{1}{n}.$$

Proof of Theorem 1.
The case $p = 1$ is clear, and so we suppose that $1 < p \leqq 2$.
Let

$$f(x) \sim \sum_{k=0}^{\infty} a_k \psi_k(x).$$

Then

$$f(x \dotplus y) \sim \sum_{k=0}^{\infty} a_k \psi_k(y)\psi_k(x).$$

Hence

$$f(x \dotplus y) - f(x) \sim \sum_{k=0}^{\infty} a_k(\psi_k(y) - 1)\psi_k(x).$$

By F. Riesz's theorem, we have

$$\left\{\sum_{k=0}^{\infty} |a_k|^q |\psi_k(y) - 1|^q\right\}^{1/q} \leqq \left\{\int_0^1 |f(x \dotplus y) - f(x)|^p\, dx\right\}^{1/p}.$$

From this

$$\sum_{k\ 0}^{\infty} |a_k|^q |\psi_k(y) - 1|^q \leqq \left\{\int_0^1 |f(x \dotplus y) - f(x)|^p dx\right\}^{q/p}.$$

Thus

$$\sum_{k=2n}^{\infty} |a_k|^q |\phi_k(y) - 1|^q \leqq \left\{\int_0^1 |f(x \dotplus y) - f(x)|^p dx\right\}^{q/p}.$$

We integrate both the sides of this inequality with respect to y within the limits $(0, {}^1/_n)$ and we have

$$\sum_{k=2n}^{\infty} |a_k|^q \int_0^{1/n} |\phi_k(y) - 1|^q dy \leqq \int_0^{1/n} \left\{\int_0^1 |f(x \dotplus y) - f(x)|^p dx\right\}^{q/p} dy. \tag{1}$$

On the other hand, if $1 < p \leqq 2$, then $2 \leqq q$. Thus since $|\phi_k(x) - 1| = 0$ or 2, we have

$$|\phi_k(y) - 1|^q \geqq |\phi_k(y) - 1|^2.$$

By lemma, we have

$$\int_0^{1/n} |\phi_k(y) - 1|^q dy \geqq \int_0^{1/n} |\phi_k(y) - 1|^2 dy \geqq \frac{1}{n} \quad \text{for } k \geqq 2n. \tag{2}$$

Therefore, we find out from (1) and (2) that

$$\frac{1}{n^{1/q}}\left(\sum_{k=2n}^{\infty} |a_k|^q\right)^{1/q} \leqq \frac{1}{n^{1/q}} w^{(p)}\left(\frac{1}{n}, f\right). \tag{3}$$

We now have

$$\frac{k}{4} < \left[\frac{k}{2}\right] \leqq \frac{k}{2}.$$

where $\left[\frac{k}{2}\right]$ as usual denotes the largest integer $\leqq {}^k/_2$.

Hence by changing the order of summation and using Hölder's inequality, we have

$$\begin{aligned}
\sum_{k=2}^{\infty} k^r |a_k| &= \sum_{k=2}^{\infty} \sum_{n=1}^{\left[\frac{k}{2}\right]} \frac{k^r |a_k|}{\left[\frac{k}{2}\right]} \leqq A \sum_{k=2}^{\infty} \sum_{n=1}^{\left[\frac{k}{2}\right]} \frac{k^r |a_k|}{k} \\
&= A \sum_{n=1}^{\infty} \sum_{k=2n}^{\infty} k^{r-1} |a_k| \leqq A \sum_{n=1}^{\infty} \left(\sum_{k=2n}^{\infty} k^{(r-1)p}\right)^{1/p} \left(\sum_{k=2n}^{\infty} |a_k|^q\right)^{1/q} \\
&\leqq A \sum_{n=1}^{\infty} n^{r-1/q} \left(\sum_{k=2n}^{\infty} |a_k|^q\right)^{1/q} \leqq A \sum_{n=1}^{\infty} n^{r-1/q} w^{(p)}\left(\frac{1}{n}, f\right) \qquad \text{by (3)}
\end{aligned}$$

which is convergent by the assumption of Theorem.

Next we suppose that $2 < p < \infty$. It is known that there exists a function $f(x)$ satisfying the Lipschitz condition of order $\alpha = \frac{1}{2}$ such that the Walsh Fourier series is not absolutely convergent (see (4) and (5)).
Using this function $f(x)$ and putting $\gamma = 0$ in Theorem 1, we obtain

$$\sum_{n=1}^{\infty} \frac{w^{(p)}(\frac{1}{n}, f)}{n^{1/q}} = 0\Big(\sum_{n=1}^{\infty} \frac{1}{n^{1/q+1/2}}\Big) < \infty,$$

and so Theorem does not hold.
Applying Theorem 1, we can prove the following two theorems, which are the Walsh-analogues of results due to A. Zygmund (6);

Theorem 2. Suppose that $f(x) \in \Lambda_\alpha$, $0 < \alpha \leqq 1$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

Then $\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}} |a_n|$ converges for $\beta < \alpha$.

Proof

If we put $\gamma = \beta - \frac{1}{2}$ and $p = 2$ in Theorem 1, then we get $\gamma < {}^1/_2$. Thus we can apply Theorem 1 to Theorem 2. Hence we have

$$\sum_{n=1}^{\infty} n^{\gamma-\frac{1}{2}} w^{(2)}\Big(\frac{1}{n}, f\Big) = \sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}-\frac{1}{2}}. \ n^{-\alpha} = \sum_{n=1}^{\infty} n^{\beta-\alpha-1} < \infty,$$

which implies the convergence of the series $\sum_{n=1}^{\infty} n^{\beta-\frac{1}{2}} |a_n|$.

Theorem 3. Let $1 \leqq p \leqq 2$. Suppose that $f(x) \in \Lambda_p^\alpha$ for $0 < \alpha \leqq 1$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

Then $\sum_{n=1}^{\infty} n^\gamma |a_n|$ converges for $\gamma < \alpha - {}^1/_p$.

Proof

Since $0 < \alpha \leqq 1$, then $\gamma < \alpha - {}^1/_p \leqq {}^1/_q$. Hence we can use Theorem 1. If we put $\gamma = \alpha - \frac{1}{p} - \varepsilon$ $(\varepsilon > 0)$, then we have

$$\sum_{n=1}^{\infty} n^{\gamma-1/q} w^{(p)}\Big(\frac{1}{n}, f\Big) = \sum_{n=1}^{\infty} n^{\alpha-1/p-\varepsilon-1/q}. \ n^{-\alpha} = \sum_{n=1}^{\infty} n^{-1-\varepsilon} < \infty,$$

which implies the convergence of the series $\sum_{n=1}^{\infty} n^{\gamma}|a_n|$.

In the case $\gamma=0$ of Theorem 1, we particularly have the following theorem;

Theorem 4. Let $1 \leqq p \leqq 2$. Suppose that $f(x) \in L^p(0, 1)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

If the series

$$\sum_{n=1}^{\infty} n^{-1/q} w^{(p)}\left(\frac{1}{n}, f\right)$$

converges for any number p, then the series $\sum_{n=0}^{\infty} |a_n|$ converges.

Recently Prof. C. Watari proved

Theorem 5. For $1 \leqq p \leqq q < \infty$, there is a constant $A = A(p, q)$ such that

$$\sum_{n=1}^{\infty} n^{-1/q'} E_n^{(q)}(f) \leqq A \sum_{n=1}^{\infty} n^{-1/p'} E_n^{(p)}(f)$$

where ${}^1/_p + {}^1/_{p'} = 1$ and ${}^1/_q + {}^1/_{q'} = 1$.

Also see M. H. Taibleson (3).

Since the finiteness of $\sum_{n=1}^{\infty} n^{-1/p'} w^{(p)}\left(\frac{1}{n}, f\right)$ is equivalent to the finiteness of $\sum_{n=1}^{\infty} n^{-1/p'} E_n^{(p)}(f)$ (see (5)), we see from Theorem 5 that Theorem 4 is really a corollary to the Walsh-analogue of Szasz's theorem (see (1)) which is the case $p=2$ in Theorem 4. By Theorem 2, we have the following corollay;

Corollary. If $f(x) \in \Lambda_\alpha, \alpha > {}^1/_2$, then the series $\sum_{n=0}^{\infty} |a_n|$ converges.

This is well known (see (2)).

Finally we note that Theorem 1 can be extended

Theorem 6. Let $1 \leqq p \leqq 2$. Suppose that $f(x) \in L^p(0, 1)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x).$$

If the series $\sum_{n=1}^{\infty} n^{\gamma-\beta/q}\left(w^{(p)}\left(\frac{1}{n}, f\right)\right)^{\beta}$ converges, then the series $\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta}$ converges for $\gamma < {}^{\beta}/_q$ where ${}^1/_p + {}^1/_q = 1$ and $\beta < q$.

The proof can be proved by the same argument of Theorem 1.

References

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