Operational Method for Various Continuous Beams

By

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1. PREFACE

In the analysis of structural mechanics, the operational method was proposed by one of the writers with the papers "Operational Method for Continuous Beams," and "Operational Method for Pin-Jointed Trusses," both of which have been published on the Proceedings of the ASCE Structural Division, on December, 1964, and June, 1966, respectively.[1),2)] Since then, this method has been developed for the analysis of various structural systems.

Presented herein is the operational method applied to various continuous beams, such as ordinary beams, beams on elastic foundation, and beams with axial loads.

2. BASIC CONCEPTS

The basic concepts of the operational method are summarized as follows:

1. The structural system is considered as the assemblage of topological units, each of which is composed of several constituent members.

2. The parameters characterizing the mechanical behavior of the constituent unit are arranged in a column vector, and defined as the "eigenmatrix" of the unit. In general, for rigidly connected structures, the eigenmatrix is composed of integration constants of geverning differential equations, while, for pin connected structures, the assemblage of member forces and nodal displacements are taken as the eigenmatrix, which can also be the state vector of truss systems.

3. The connection conditions, compatibility and equilibrium conditions, between two consecutive units are treated by perfectly classified matrix form. After this treatment, a certain shift formula or recursion one for eigenmatrices

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between two consecutive units is obtained. It is composed of a shift operator and a feed operator; the former shifting the eigenmatrix of a unit to that of the adjacent unit, and the latter introducing the influence of external loads on the shift formula.

4. In virtue of the recursion formula, the eigenmatrix of a unit becomes current to the entire system, and hence, it is called the "current-matrix."

5. The current-matrix is determined by both extreme boundary conditions of the system, and therefore, the system can be solved. In this treatment, the operational matrices, perfectly corresponding to the boundary conditions, appear, so that they are called the "boundary matrices."

6. From the viewpoint of the matrix algebra, the operational method permits a simplified analysis by making use of the above operators.

The matrix analysis for structural mechanics should be based on the pure and complete classification of data, which leads to readiness and simplification in philosophy and computation.

3. KEY EQUATIONS

Herein are shown the key equations to ordinary beams, beams on elastic foundation, and beams with axial loads. The non-dimensional abscissas are for convenience adopted for use to denote the current and load abscissas of a memder, that is to say, taking $L=$ the beam length (m), $x=$ the current abscissa (m), and $\xi=$ the load abscissa (m), we write $\rho=x/L=$ the non-dimensional current abscissa, and $\kappa=\xi/L=$ the non-dimensional load abscissa, providing their positive abscissas are directed rightwards.

The state vector of a beam at abscissa ρ is given by the following equation:

$$\mathbf{W}(\rho)=\mathbf{DP}(\rho)[\mathbf{N}+\mathbf{K}(\rho)]. \tag{1}$$

Here, $\mathbf{W}(\rho)=$ the state vector, $\mathbf{D}=$ the coefficient matrix, $\mathbf{P}(\rho)=$ the abscissa matrix, $\mathbf{N}=$ the eigenmatrix, and $\mathbf{K}=$ the load matrix. They are defined as follows:

(1) **State Vector $\mathbf{W}(\rho)$.**

The physical quantities at abscissa ρ in the beam are arranged in a column vector as follows and called the "state vector." That is to say,

$$\mathbf{W}(\rho)=\{w \quad \theta \quad M \quad S\}_\rho. \tag{2}$$

Here, $w=$ the deflection, $\theta=$ the slope angle, $M=$ the bending moment, and $S=$ the shearing force at abscissa ρ, respectively.

(2) Coefficient Matrix $\mathbf{D}$.

The coefficients to be attached to respective physical quantities are arranged in a diagonal matrix, and called the "coefficient matrix." For respective beams, they are given as follows:

(a) *Ordinary beams*,

$$\mathbf{D} = \mathrm{diag}\left[\frac{L^3}{6EI} \quad \frac{L^2}{6EI} \quad -\frac{L}{3} \quad -1\right], \tag{3}$$

in which, E = modulus of elasticity, and I = the moment of inertia.

(b) *Beams on elastic foundation*,

$$\mathbf{D} = \mathrm{diag}\left[\frac{L^3}{2\beta^3 EI} \quad \frac{L^2}{2\beta^2 EI} \quad -\frac{L}{\beta} \quad -1\right], \tag{4}$$

in which

$$\beta = \sqrt[4]{\frac{kL^4}{4EI}}, \qquad k = \text{modulus of foundation.} \tag{5}$$

(c) *Beams with axial loads*,

$$\mathbf{D} = \mathrm{diag}\left[\frac{L^3}{\alpha^3 EI} \quad \frac{L^2}{\alpha^2 EI} \quad -\frac{L}{\alpha} \quad -1\right], \tag{6}$$

in which

$$\alpha = \sqrt{\frac{QL^2}{EI}}, \qquad Q = \text{axial compressive force.} \tag{7}$$

(3) Abscissa Matrix $\mathbf{P}(\rho)$.

Corresponding to respective physical quantities, the abscissa functions are arranged in 4-by-4 square matrix, and called the "abscissa matrix." For respective beams, they are given as follows:

(a) *Ordinary beams*,

$$\mathbf{P}(\rho) = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ 0 & 1 & 2\rho & 3\rho^2 \\ 0 & 0 & 1 & 3\rho \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

(b) *Beams on elastic foundation*,

$$\mathbf{P}(\rho) = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1 - \phi_2 & \phi_1 + \phi_2 & -\phi_3 - \phi_4 & \phi_3 - \phi_4 \\ -\phi_2 & \phi_1 & \phi_4 & -\phi_3 \\ -\phi_1 - \phi_2 & \phi_1 - \phi_2 & \phi_3 - \phi_4 & \phi_3 + \phi_4 \end{bmatrix}, \tag{9}$$

in which

$$\phi_1 = e^{\beta\rho}\cos\beta\rho, \quad \phi_2 = e^{\beta\rho}\sin\beta\rho, \quad \phi_3 = e^{-\beta\rho}\cos\beta\rho, \quad \phi_4 = e^{-\beta\rho}\sin\beta\rho. \tag{10}$$

(c) *Beams with axial loads,*

$$\boldsymbol{P}(\rho) = \begin{bmatrix} 1 & \alpha\rho & \cos\alpha\rho & \sin\alpha\rho \\ 0 & 1 & -\sin\alpha\rho & \cos\alpha\rho \\ 0 & 0 & -\cos\alpha\rho & -\sin\alpha\rho \\ 0 & 0 & \sin\alpha\rho & -\cos\alpha\rho \end{bmatrix}. \tag{11}$$

(4) Eigenmatrix $\boldsymbol{N}$.

Four integration constants of a governing differential equation are arranged in a column vector as follows, and defined as the "eigenmatrix" of the beam. That is to say,

$$\boldsymbol{N} = \{A \quad B \quad C \quad D\}, \tag{12}$$

in which A, B, C, D = the integration constants of governing differential equation. Then, all the mechanical behavior of the beam are characterized by the eigenmatrix, which perfectly corresponds to the state vector defined in Eq. 2. In general, this correspondence holds for usual structural systems; however, in the higher structural analysis such as the recursive finite element method (unpublished), it has been found that the perfect correspondence fails.

(5) Load-Matrix $\boldsymbol{K}$.

The load-matrix is obtained from the treatment of the connection condition of state vectors at the loaded point. Then the influence of external load can be expressed by mere addition of corresponding load matrix to the eigenmatrix $\boldsymbol{N}$. Referring to Fig. 1, the state vectors at respective domains are given as follows:

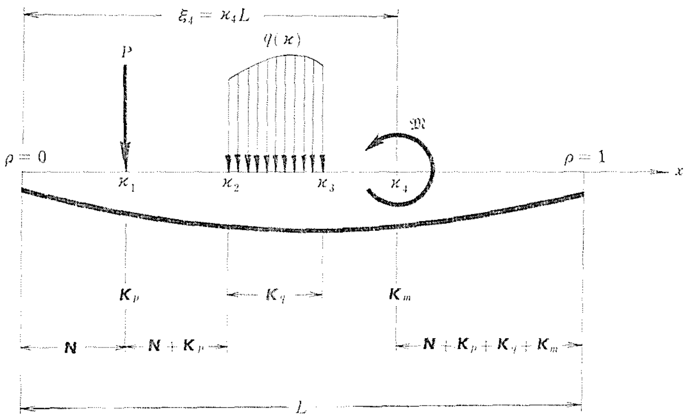


Fig. 1. Load Matrix.

$$\left.\begin{aligned}
0<\rho<\kappa_1 &: \boldsymbol{W}(\rho)=\boldsymbol{DP}(\rho)\boldsymbol{N},\\
\kappa_1<\rho<\kappa_2 &: \boldsymbol{W}(\rho)=\boldsymbol{DP}(\rho)[\boldsymbol{N}+\boldsymbol{K}_p],\\
\kappa_2<\rho<\kappa_3 &: \boldsymbol{W}(\rho)=\boldsymbol{DP}(\rho)[\boldsymbol{N}+\boldsymbol{K}_p+\boldsymbol{K}_q(\rho)],\\
\kappa_3<\rho<\kappa_4 &: \boldsymbol{W}(\rho)=\boldsymbol{DP}(\rho)[\boldsymbol{N}+\boldsymbol{K}_p+\boldsymbol{K}_q],\\
\kappa_4<\rho<1 &: \boldsymbol{W}(\rho)=\boldsymbol{DP}(\rho)[\boldsymbol{N}+\boldsymbol{K}_p+\boldsymbol{K}_q+\boldsymbol{K}_m].
\end{aligned}\right\} \tag{13}$$

Here, $\boldsymbol{K}_p$, $\boldsymbol{K}_q$, $\boldsymbol{K}_m$ = the load-matrices for the concentrated load, the distributed load, and the moment load. For respective beams, the load-matrices are given as follows:

(a) *Ordinary beams,*

$$\left.\begin{aligned}
\boldsymbol{K}_p &= P\{-\kappa_1^3 \quad 3\kappa_1^2 \quad -3\kappa_1 \quad 1\},\\
\boldsymbol{K}_q &= L\int_{\kappa_2}^{\kappa_3} q(\kappa)\{-\kappa^3 \quad 3\kappa^2 \quad -3\kappa \quad 1\}d\kappa,\\
\boldsymbol{K}_m &= \frac{3}{L}\mathfrak{M}\{\kappa_4^2 \quad -2\kappa_4 \quad 1 \quad 0\}.
\end{aligned}\right\} \tag{14}$$

(b) *Beams on elastic foundation,*

$$\left.\begin{aligned}
\boldsymbol{K}_p &= \frac{P}{4}\{-\phi_3-\phi_4 \quad \phi_3-\phi_4 \quad \phi_1-\phi_2 \quad \phi_1+\phi_2\}_{\kappa_1},\\
\boldsymbol{K}_q &= \frac{L}{4}\int_{\kappa_2}^{\kappa_3} q(\kappa)\{-\phi_3-\phi_4 \quad \phi_3-\phi_4 \quad \phi_1-\phi_2 \quad \phi_1+\phi_2\}d\kappa,\\
\boldsymbol{K}_m &= \frac{\beta\mathfrak{M}}{2L}\{-\phi_4 \quad \phi_3 \quad \phi_2 \quad -\phi_1\}_{\kappa_4}.
\end{aligned}\right\} \tag{15}$$

(c) *Beams with axial loads,*

$$\left.\begin{aligned}
\boldsymbol{K}_p &= P\{-\alpha\kappa_1 \quad 1 \quad \sin\alpha\kappa_1 \quad -\cos\alpha\kappa_1\},\\
\boldsymbol{K}_q &= L\int_{\kappa_2}^{\kappa_3} q(\kappa)\{-\alpha\kappa \quad 1 \quad \sin\alpha\kappa \quad -\cos\alpha\kappa\}d\kappa,\\
\boldsymbol{K}_m &= \frac{\alpha\mathfrak{M}}{L}\{1 \quad 0 \quad -\cos\alpha\kappa_4 \quad -\sin\alpha\kappa_4\}.
\end{aligned}\right\} \tag{16}$$

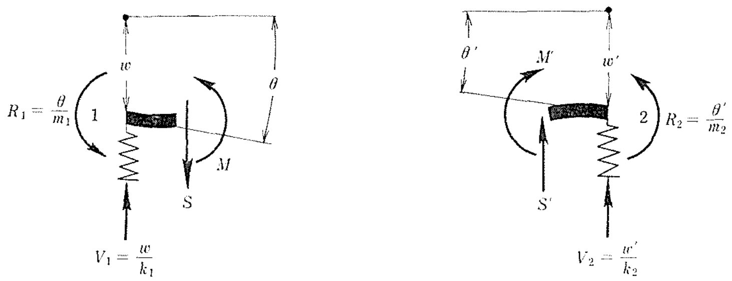
4. BOUNDARY CONDITIONS

To show the generalized boundary conditions of the beam, both extreme end supports are considered as elastic ones, that is to say, settlements and slopes at these supports are respectively proportional to corresponding reactions. Figs. 2a and 2b illustrate these conditions, and Eqs. 17 and 18 show their formulation, provided that letters k and m represent spring constants attached to these supports. Those conditions are expressed by the equations:

At the left end:
$$\begin{bmatrix} M \\ S \end{bmatrix} + \begin{bmatrix} \dfrac{\theta}{m_1} \\ -\dfrac{w}{k_1} \end{bmatrix} = 0. \tag{17}$$

At the right end:
$$\begin{bmatrix} M' \\ S' \end{bmatrix} + \begin{bmatrix} -\dfrac{\theta'}{m_2} \\ \dfrac{w'}{k_2} \end{bmatrix} = 0. \tag{18}$$

Here, the physical quantities at the right end of the beam ($\rho = 1$) are primed, while those at the left end ($\rho = 0$) are unprimed. This notation holds throughout the present paper.



(a) **Left Extremity** (b) **Right Extremity**

Fig. 2. Boundary Conditions.

In virtue of the key equation, Eq. 1, the above boundary conditions for beam analysis are reduced to the following matrix equations:

At the left end: $$\boldsymbol{BN} = 0. \tag{19}$$

At the right end: $$\boldsymbol{B}'\boldsymbol{N}' = \boldsymbol{B}'[\boldsymbol{N} + \boldsymbol{K}] = 0. \tag{20}$$

Here, $\boldsymbol{B}$, $\boldsymbol{B}'$ = the boundary matrices of 2-by-4 rectangular form, the values of which are summarized in Table 1 for respective beams, and $\boldsymbol{K}$ = the summation of load matrices acting on the span considered, i.e., the "load term" of the span considered.

Assuming due values of the spring constants, all kinds of boundary conditions can be represented by the boundary matrices in Table 1. Several examples are shown in Table 2.

Table 1. Boundary Matrices.

	Left End B	Right End B'
Ordinary Beams	$\begin{bmatrix} 0, & 1, & -\dfrac{2EIm_1}{L}, & 0 \\ 1, & 0, & 0, & \dfrac{6EIk_1}{L^3} \end{bmatrix}$	$\begin{bmatrix} 0, & 1, & 2+\dfrac{2EIm_2}{L}, & 3+\dfrac{6EIm_2}{L} \\ 1, & 1, & 1, & 1-\dfrac{6EIk_2}{L^3} \end{bmatrix}$
Beams on Elastic Foundation	$\begin{bmatrix} 1, & 1-\dfrac{2\beta EIm_1}{L}, & -1, & 1+\dfrac{2\beta EIm_1}{L} \\ -1+\dfrac{L^3}{2\beta^3 EIk_1}, & 1, & 1+\dfrac{L^3}{2\beta^3 EIk_1}, & 1 \end{bmatrix}$	$\begin{bmatrix} \phi_1-\left(1+\dfrac{2\beta EIm_2}{L}\right)\phi_2, & \left(1+\dfrac{2\beta EIm_2}{L}\right)\phi_1+\phi_2, & -\phi_3-\left(1-\dfrac{2\beta EIm_2}{L}\right)\phi_4, & \left(1-\dfrac{2\beta EIm_2}{L}\right)\phi_3-\phi_4 \\ \left(1+\dfrac{L^3}{2\beta^3 EIk_2}\right)\phi_1+\phi_2, & -\phi_1+\left(1+\dfrac{L^3}{2\beta^3 EIk_2}\right)\phi_2, & \left(-1+\dfrac{L^3}{2\beta^3 EIk_2}\right)\phi_3+\phi_4, & -\phi_3+\left(-1+\dfrac{L^3}{2\beta^3 EIk_2}\right)\phi_4 \end{bmatrix}$ $\phi_1=e^{\beta}\cos\beta,\quad \phi_2=e^{\beta}\sin\beta,\quad \phi_3=e^{-\beta}\cos\beta,\quad \phi_4=e^{-\beta}\sin\beta$
Beams with Axial Loads	$\begin{bmatrix} 0, & 1, & \dfrac{\alpha EIm_1}{L}, & 1 \\ 1, & 0, & 1, & -\dfrac{\alpha^3 EIk_1}{L^3} \end{bmatrix}$	$\begin{bmatrix} 0, & -1, & \sin\alpha+\dfrac{\alpha EIm_2}{L}\cos\alpha, & -\cos\alpha+\dfrac{\alpha EIm_2}{L}\sin\alpha \\ 1, & \alpha, & \cos\alpha-\dfrac{\alpha^3 EIk_2}{L^3}\sin\alpha, & \sin\alpha+\dfrac{\alpha^3 EIk_2}{L^3}\cos\alpha \end{bmatrix}$

Table 2. Spring Constants and Boundary Conditions.

	k_i	m_i
Elastic Supports	k_i	m_i
Fixed Ends	0	0
Simple Ends	0	∞
Free Ends	∞	∞

5. CONNECTION CONDITIONS

The connection conditions between two consecutive constituent members are satisfied by due treatment of compatibility and equilibrium conditions for both state vectors at the common ends of constituent members. For generalization, assuming the elastic proportionality at the common point, those conditions are given as follows:

$$-\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_{i-1}' + \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_i + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\theta}{m} \\ -\dfrac{w}{k} \end{bmatrix}_i = 0. \tag{21}$$

Here, k_i, m_i = the spring constants attached to the intermediate support i. Assuming the values of the spring constants, various connection conditions can be represented by the above equation; for instance, taking $k_i = m_i = \infty$, the above equation shows the connection condition at point of abrupt change in cross-section in the plate-girder bridges, and taking $k_i = 0$, and $m_i = \infty$, it would result in the connection condition at intermediate rigid support of the continuous beams.

On the other hand, for the continuous beams composed of only the combination of rigid supports and pin joints, it would be preferable from the philosopical and computational viewpoint to use the following procedure. That is to say, noticing the characteristics of the rigid support and the pin joint, the preliminary treatments are to be made for respective eigenmatrices of constituent members. Consequently, the order of them can be reduced to a 2-by-1 column vector. For instance, taking a constituent member whose left and right ends are connected with rigid support and pin joint, respectively, the eigenmatrix after the preliminary teatment becomes

$$\boldsymbol{N}_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix} \boldsymbol{A}_i + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{K}_i, \tag{22}$$

in which

$$\boldsymbol{A}_i = \{B \quad D\}, \tag{23}$$

and $\boldsymbol{K}_i$ = the load term of the span considered. The matrix $\boldsymbol{A}_i$ is called the "semi-eigenmatrix."

After such preliminary treatments, the connection condition can be satisfied by the following equations:[1),3),4)]

$$\text{For rigid supports:} \quad -\begin{bmatrix} \theta \\ M \end{bmatrix}'_{i-1} + \begin{bmatrix} \theta \\ M \end{bmatrix}_i + \begin{bmatrix} 0 \\ \frac{\theta}{m} \end{bmatrix}_i = 0. \tag{24}$$

$$\text{For pin joints:} \quad -\begin{bmatrix} w \\ S \end{bmatrix}'_{i-1} + \begin{bmatrix} w \\ S \end{bmatrix}_i - \begin{bmatrix} 0 \\ \frac{w}{k} \end{bmatrix}_i = 0. \tag{25}$$

In virtue of Eq. 1, Eq. 21 can be reduced to the following consolidated form:

$$\boldsymbol{C}_i\{\boldsymbol{N}'_{i-1} \quad \boldsymbol{N}_i\} = 0. \tag{26}$$

This is the desired connection equation at the intermediate support. $\boldsymbol{C}_i$ is the 4–by–8 rectangular matrix, and is called the "connection matrix," or briefly the "connector," whose values for various beams are summarized in Table 3.

In a similar manner, the connection matrices for semi-eigenmatrices can be obtained from Eqs. 24 and 25, the order of which corresponds with the semi-eigenmatrix. In this case, by the possible combinations of rigid support and pin joint, there will be seven kinds of connectors.[3)]

6. SHIFT OPERATORS

The connectors in Table 3 are given in the form

$$\boldsymbol{C}_i = \lfloor \boldsymbol{c}'_{i-1} \quad \boldsymbol{c}_i \rfloor. \tag{27}$$

Substituting this equation into Eq. 26, and modifying the resulting equation, we obtain a relationship between two consecutive eigenmatrices $\boldsymbol{N}_{i-1}$ and $\boldsymbol{N}_i$ as follows:

$$\boldsymbol{N}_i = -\boldsymbol{c}_i^{-1}\boldsymbol{c}'_{i-1}(\boldsymbol{N} + \boldsymbol{K})_{i-1} = \boldsymbol{S}_i(\boldsymbol{N} + \boldsymbol{K})_{i-1}, \tag{28}$$

Table 3. Connection Matrices.

Ordinary Beams	$\left[\begin{bmatrix} 1, & 1, & 1, & 1 \\ 0, & 1, & 2, & 3 \\ 0, & 0, & 1, & 3 \\ 0, & 0, & 0, & 1 \end{bmatrix}, \; -\begin{bmatrix} \frac{\gamma^3}{\varepsilon}, & 0, & 0, & 0 \\ 0, & \frac{\gamma^2}{\varepsilon}, & 0, & 0 \\ 0, & -\frac{1}{2}\gamma\mu, & \gamma, & 0 \\ \frac{\lambda}{6}, & 0, & 0, & 1 \end{bmatrix}_i\right]$
Beams on Elastic Foundation	$\left[\begin{bmatrix} \phi_1, & \phi_2, & \phi_3, & \phi_4 \\ \phi_1-\phi_2, & \phi_1+\phi_2, & -\phi_3-\phi_4, & \phi_3-\phi_4 \\ -\phi_2, & \phi_1, & \phi_4, & -\phi_3 \\ -\phi_1-\phi_2, & \phi_1-\phi_2, & \phi_3-\phi_4, & \phi_3+\phi_4 \end{bmatrix}_{i-1}, \; -\begin{bmatrix} \frac{\gamma^3}{\varepsilon\xi^3}, & 0, & \frac{\gamma^3}{\varepsilon\xi^3}, & 0 \\ \frac{\gamma^2}{\varepsilon\xi^2}, & \frac{\gamma^2}{\varepsilon\xi^2}, & -\frac{\gamma^2}{\varepsilon\xi^2}, & \frac{\gamma^2}{\varepsilon\xi^2} \\ -\frac{\gamma\mu}{2\beta\xi}, & \frac{\gamma}{\xi}\left(1-\frac{\mu}{2\beta}\right), & \frac{\gamma\mu}{2\beta\xi}, & -\frac{\gamma}{\xi}\left(1+\frac{\mu}{2\beta}\right) \\ \frac{\lambda}{2\beta^3}-1, & 1, & \frac{\lambda}{2\beta^3}+1, & 1 \end{bmatrix}_i\right]$
Beams with Axial Loads	$\left[\begin{bmatrix} 1, & \alpha, & \cos\alpha, & \sin\alpha \\ 0, & 1, & -\sin\alpha, & \cos\alpha \\ 0, & 0, & -\cos\alpha, & -\sin\alpha \\ 0, & 0, & \sin\alpha, & -\cos\alpha \end{bmatrix}_{i-1}, \; -\begin{bmatrix} \frac{\gamma^3}{\varepsilon\nu^3}, & 0, & \frac{\gamma^3}{\varepsilon\nu^3}, & 0 \\ 0, & \frac{\gamma^2}{\varepsilon\nu^2}, & 0, & \frac{\gamma^2}{\varepsilon\nu^2} \\ 0, & -\frac{\gamma\mu}{\alpha\nu}, & -\frac{\gamma}{\nu}, & -\frac{\gamma\mu}{\alpha\nu} \\ \frac{\lambda}{\alpha^3}, & 0, & \frac{\lambda}{\alpha^3}, & -1 \end{bmatrix}_i\right]$
	$\gamma_i = \frac{L_i}{L_{i-1}}, \quad \varepsilon_i = \frac{EI_i}{EI_{i-1}}, \quad \lambda_i = \left(\frac{L^3}{EIk}\right)_i, \quad \mu_i = \left(\frac{L}{EIm}\right)_i, \quad \nu_i = \frac{\alpha_i}{\alpha_{i-1}}, \quad \xi_i = \frac{\beta_i}{\beta_{i-1}},$

$$\boldsymbol{N}_{i-1} = -\boldsymbol{c}'_{i-1}{}^{-1}\boldsymbol{c}_i\boldsymbol{N}_i - \boldsymbol{K}_{i-1} = \boldsymbol{S}'_i\boldsymbol{N}_i - \boldsymbol{K}_{i-1}. \tag{29}$$

In virtue of Eq. 28, the eigenmatrix $\boldsymbol{N}_{i-1}$ in the left-hand span is shifted to $\boldsymbol{N}_i$ in the right-hand span. Therefore, the matrix $\boldsymbol{S}_i$ is called the "rightward shift operator," or briefly the "rightward shiftor." Similarly, the matrix $\boldsymbol{S}'_i$ in Eq. 29 is called the "leftward shiftor." These values for ordinary beams are given as follows:

$$\boldsymbol{S}_i = \begin{bmatrix} \frac{\varepsilon}{\gamma^3} & 0 & 0 & 0 \\ 0 & \frac{\varepsilon}{\gamma^2} & 0 & 0 \\ 0 & \frac{\varepsilon\mu}{2\gamma^2} & \frac{1}{\gamma} & 0 \\ -\frac{\varepsilon\lambda}{6\gamma^3} & 0 & 0 & 1 \end{bmatrix}_i \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\varepsilon}{\gamma^3} & \frac{\varepsilon}{\gamma^3} & \frac{\varepsilon}{\gamma^3} & \frac{\varepsilon}{\gamma^3} \\ 0 & \frac{\varepsilon}{\gamma^2} & \frac{2\varepsilon}{\gamma^2} & \frac{3\varepsilon}{\gamma^2} \\ 0 & \frac{\varepsilon\mu}{2\gamma^2} & \frac{1}{\gamma}\left(1+\frac{\varepsilon\mu}{\gamma}\right) & \frac{3}{\gamma}\left(1+\frac{\varepsilon\mu}{2\gamma}\right) \\ -\frac{\varepsilon\lambda}{6\gamma^3} & -\frac{\varepsilon\lambda}{6\gamma^3} & -\frac{\varepsilon\lambda}{6\gamma^3} & 1-\frac{\varepsilon\lambda}{6\gamma^3} \end{bmatrix}_i, \tag{30}$$

$$\boldsymbol{S}'_i = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\gamma^3}{\varepsilon} & 0 & 0 & 0 \\ 0 & \frac{\gamma^2}{\varepsilon} & 0 & 0 \\ 0 & -\frac{1}{2}\gamma\mu & \gamma & 0 \\ \frac{\lambda}{6} & 0 & 0 & 1 \end{bmatrix}_i$$

$$= \begin{bmatrix} \frac{\gamma^3}{\varepsilon} - \frac{\lambda}{6} & \gamma\left(\frac{\gamma}{\varepsilon} - \frac{\mu}{2}\right) & \gamma & -1 \\ \frac{\lambda}{2} & \gamma\left(\frac{\gamma}{\varepsilon} + \mu\right) & -2\gamma & 3 \\ -\frac{\lambda}{2} & -\frac{1}{2}\gamma\mu & \gamma & -3 \\ \frac{\lambda}{6} & 0 & 0 & 1 \end{bmatrix}_i. \tag{31}$$

In the case of beams on elastic foundation and beams with axial loads, the inverses of the matrices $\mathbf{c}'_{i-1}$ and $\mathbf{c}_i$ become complicated, so that numerical procedure is recommended.

In the analysis of continuous beams composed of only the combination of rigid supports and pin joints, the connection conditions given by Eqs. 24 and 25 are to be treated, from which the following shift formulas for consecutive semi-eigenmatrices $\mathbf{A}_{i-1}$ and $\mathbf{A}_i$ can be obtained:

$$\mathbf{A}_i = \mathbf{L}_i\mathbf{A}_{i-1} + \mathbf{P}_i\mathbf{K}_{i-1} + \mathbf{Q}_i\mathbf{K}_i, \tag{32}$$

$$\mathbf{A}_{i-1} = \mathbf{L}'_i\mathbf{A}_i + \mathbf{P}'_i\mathbf{K}_{i-1} + \mathbf{Q}'_i\mathbf{K}_i. \tag{33}$$

Here, $\mathbf{L}_i$ and $\mathbf{L}'_i$ = 2-by-2 shift operators for semi-eigenmatrices. On the other hand, the matrices $\mathbf{P}_i$, $\mathbf{P}'_i$, $\mathbf{Q}_i$, and $\mathbf{Q}'_i$ = 2-by-4 rectangular matrices called the "feed operators," or briefly the "feeders." They introduce the influence of corresponding load term into the shift formula. It can be mentioned that Eq. 28 is a special case in which the shiftor and feeder are the same.

7. SHIFT OPERATIONS

Eqs. 28, 29, 32, and 33 are the recurrence formulas for continuous beams. In virtue of the recurrent use of such fromulas, the eigenmatrix selected as standard become current to the entire system, and hence it is called the "current-matrix," which usually can be detemined by both extreme boundary conditions. This is the standard procedure of the operational method.

In the continuous beams whose shift operations can be made by only the recurrent use of Eq. 28 or 29, taking the extreme left eigenmatrix as standard, and shifting it rightwards, the solution of the system is given in the form

$$\mathbf{N}_1 = -\begin{bmatrix} \mathbf{B} \\ \mathbf{B}'\mathbf{S}_n\mathbf{S}_{n-1}\cdots\mathbf{S}_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \mathbf{B}'[\mathbf{S}_n\mathbf{S}_{n-1}\cdots\mathbf{S}_2\mathbf{K}_1 + \cdots + \mathbf{S}_n\mathbf{S}_{n-1}\mathbf{K}_{n-2} + \mathbf{S}_n\mathbf{K}_{n-1} + \mathbf{K}_n] \end{bmatrix}. \tag{34}$$

Here an inverse matrix of fourth order appears.

In the systems composed of only the combination of rigid supports and pin joints, the form of final solution becomes a little complicated, but the size of inverse in the final equation becomes second order.

Practically, the computation design can be made effectively by the aid of the "shifting chart." In Fig. 3 is shown an example of the chart. In this figure, the numerals in the symbols -○-, ○, □, and ◇ represent the initial order of eigenmatrix in each span, the degraded order of eigenmatrix by the treatment of given conditions, the order of boundary conditions which can be treated independently in each span, and the order of connection conditions between two consecutive spans, respectively. The symbol $\longrightarrow$ denotes the direction of shift operation.[4)]

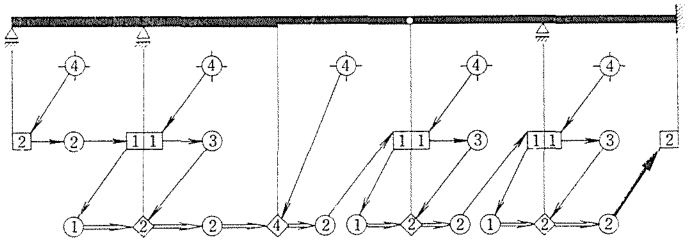


Fig. 3. Shifting Chart.

8. GEOMETRY MATRIX

After the determination of the current-matrix, using the recurrence formula again, the eigenmatrices in the entire spans can be evaluated. In this case, the solution should be written in the form separating a physical matrix from the load terms, i.e., in the case of n-span continuous beam, the entire solution is consolidated as follows:

$$\{\boldsymbol{N}\}_n = [\boldsymbol{G}]\{\boldsymbol{K}\}_n. \tag{35}$$

Here

$$\{\boldsymbol{N}\}_n = \{\boldsymbol{N}_1 \quad \boldsymbol{N}_2 \quad \boldsymbol{N}_3 \quad \cdots \quad \boldsymbol{N}_n\}, \tag{36}$$

$$[\boldsymbol{G}] = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} & \cdots & \boldsymbol{G}_{1n} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \cdots & \cdots & \cdots \\ \boldsymbol{G}_{31} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{G}_{n1} & \cdots & \cdots & \cdots & \boldsymbol{G}_{nn} \end{bmatrix}, \tag{37}$$

$$\{\boldsymbol{K}\}_n = \{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3 \quad \cdots \quad \boldsymbol{K}_n\}. \tag{38}$$

The matrix $[\boldsymbol{G}]$ is a $4n$-by-$4n$ square one, which can be obtained from only the geometrical quantities, I, L, of respective members, and the modulus of elasticity, E, which is known value for given structural materials, and therefore, this matrix is called the "geometry matrix." Thus the geometry matrix can be obtained independently of the loading conditions. In other words, by virtue of the geometry matrix, the eigenmatrix of the system considered can at once be obtained for arbitrary loading conditions. Therefore, using the geometry matrix, the influence of external loads on various physical quantities can be obtained numerically, so that all the influence lines can be formulated.

9. APPLICATION

Example 1. Fig. 4 shows a single span ordinary beam supported elastically at both ends. The boundary conditions of this beam are given by Eqs. 19 and 20. Writing these equations together, and rearranging the resulting equation a little, we obtain the solution of this system in the form

$$\boldsymbol{N} = -\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B}' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \boldsymbol{B}' \end{bmatrix} \boldsymbol{K} = \boldsymbol{G}\boldsymbol{K}. \tag{39}$$

$\boldsymbol{G}$ is the geometry matrix and is obtained in this case as follows:

$$\boldsymbol{G} = \frac{1}{D}\begin{bmatrix}
\frac{12EI}{L^3}k_1 + \frac{12(EI)^2}{L^4}k_1(m_1+m_2), & \frac{6EI}{L^3}k_1 + \frac{12(EI)^2}{L^4}k_1m_2, & -\frac{12(EI)^2}{L^4}k_1m_1 - \frac{24(EI)^3}{L^5}k_1m_1m_2, & -\frac{6EI}{L^3}k_1 - \frac{24(EI)^2}{L^4}k_1(m_1+m_2) - \frac{72(EI)^3}{L^5}k_1m_1m_2 - \frac{72(EI)^2}{L^6}k_1k_2 - \frac{72(EI)^3}{L^7}k_1k_2(m_1+m_2) \\
\frac{6EI}{L}m_1 + \frac{12(EI)^2}{L^2}m_1m_2, & \frac{4EI}{L}m_1 + \frac{12(EI)^2}{L^2}m_1m_2 + \frac{12(EI)^2}{L^4}k_1m_1 + \frac{12(EI)^2}{L^4}k_2m_1, & \frac{2EI}{L}m_1 + \frac{8(EI)^2}{L^2}m_1m_2 + \frac{24(EI)^2}{L^4}m_1(k_1+k_2) + \frac{24(EI)^3}{L^5}m_1m_2(k_1+k_2), & \frac{36(EI)^2}{L^4}k_1m_1 + \frac{72(EI)^3}{L^5}k_1m_1m_2 \\
3 + \frac{6EI}{L}m_2, & 2 + \frac{6EI}{L}m_2 + \frac{6EI}{L^3}k_1 + \frac{6EI}{L^3}k_2, & 1 + \frac{4EI}{L}m_2 + \frac{12EI}{L^3}(k_1+k_2) + \frac{12(EI)^2}{L^4}m_2(k_1+k_2), & \frac{18EI}{L^3}k_1 + \frac{36(EI)^2}{L^4}k_1m_2 \\
-2 - \frac{2EI}{L}(m_1+m_2), & -1 - \frac{2EI}{L}m_2, & \frac{2EI}{L}m_1 + \frac{4(EI)^2}{L^2}m_1m_2, & 1 + \frac{4EI}{L}(m_1+m_2) + \frac{12(EI)^2}{L^2}m_1m_2 + \frac{12EI}{L^3}k_2 + \frac{12(EI)^2}{L^4}k_2(m_1+m_2)
\end{bmatrix}, \tag{40}$$

in which

$$D = -1 - \frac{4EI}{L}(m_1 + m_2)$$

$$- \left\{ \frac{12EI}{L^3} + \frac{12(EI)^2}{L^4}(m_1 + m_2) \right\}(k_1 + k_2) - \frac{12(EI)^2}{L^2} m_1 m_2. \qquad (41)$$

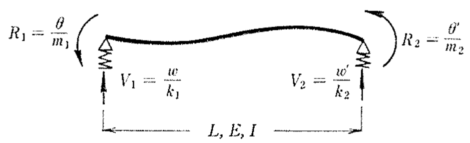


Fig. 4. Single Span Beam on Elastic Supports.

Selecting due values of the spring constants as shown in Table 2, the geometry matrices for possible cases of ordinary beams can be summarized in Table 4.

Example 2. Neglecting the axial elongation of members, the rigid frame shown in Fig. 5 can be analyzed applying the basic equations for ordinary beams.

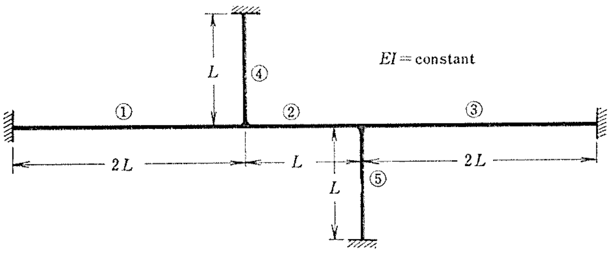


Fig. 5. Rigid Frame.

First, at both ends of each member, the boundary conditions, which can be given independently, are to be treated, and the order of each eigenmatrix is degraded from 4-by-1 to 2-by-1 or 1-by-1.

Next, the compatibility condition at connection point of respective members, which is the continuity of slope angles, are to be treated; consequently, the semi-eigenmatrix of the member ② becomes current to the entire system.

Lastly, the current-matrix can be evaluated by the moment equilibrium conditions at two connection points.

Table 4. Geometry Matrices for Single Span Ordinary Beams

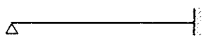
Systems	Geometry Matrices.
 Fix-Fix	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$
 Fix-Simple	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$
 Fix-Free	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
 Simple-Fix	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -2 \end{bmatrix}$
 Simple-Simple	$\frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$
 Free-Fix	$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Table 5. Geometry Matrix for Rigid Frame.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2.687	−1.896	−1.000	0	0.125	0.094	0.052	0	−0.062	−0.042	−0.021	0	0.156	0.104	0.052	0	−0.031	−0.010	0	0
1.687	0.896	0	−1.000	−0.125	−0.094	−0.052	0	0.062	0.042	0.021	0	−0.156	−0.104	−0.052	0	0.031	0.010	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1.250	−0.417	0	0	−0.500	−0.375	−0.208	0	0.250	0.167	0.083	0	−0.625	−0.417	−0.208	0	0.125	0.042	0	0
2.250	0.750	0	0	−1.500	−1.125	−0.625	0	0.750	0.500	0.250	0	1.125	0.750	0.375	0	0.375	0.125	0	0
−1.000	−0.333	0	0	1.000	0.500	−0.167	−1.000	−1.000	−0.667	−0.333	0	−0.500	−0.333	−0.167	0	−0.500	−0.167	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.062	0.021	0	0	−0.125	−0.031	0.010	0	−0.312	−0.208	−0.104	0	0.031	0.021	0.010	0	−0.156	−0.052	0	0
−0.124	−0.042	0	0	0.250	0.062	−0.020	0	−2.375	−1.583	−0.792	0	−0.062	−0.042	−0.020	0	0.312	0.104	0	0
0.062	0.021	0	0	−0.125	−0.031	0.010	0	1.687	0.791	−0.104	−1.0	0.031	0.021	0.010	0	−0.156	−0.052	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1.250	−0.417	0	0	−0.500	−0.375	−0.208	0	0.250	0.167	0.083	0	−0.625	−0.417	−0.208	0	0.125	0.042	0	0
2.500	0.834	0	0	1.000	0.750	0.416	0	−0.500	−0.334	−0.166	0	−1.750	−1.167	−0.583	0	−0.250	−0.084	0	0
−1.250	−0.417	0	0	−0.500	−0.375	−0.208	0	0.250	0.167	0.083	0	1.375	0.583	−0.208	−1.000	0.125	0.042	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−0.250	−0.083	0	0	0.500	0.125	−0.042	0	1.250	0.833	0.417	0	−0.125	−0.083	−0.042	0	−2.375	−1.792	−1.000	0
0.250	0.083	0	0	−0.500	−0.125	0.042	0	−1.250	−0.833	−0.417	0	0.125	0.083	0.042	0	1.375	0.792	0	−1.000

Thus, the solution of the system can be obtained in a form similar to Eq. 35. The geometry matrix in this case is given in Table 5.

Example 3. Referring to Eq. 34, the eigenvalue problem of continuous long-columns can be reduced to find the axial load satisfying the equation

$$\begin{vmatrix} \boldsymbol{B} \\ \boldsymbol{B}'\boldsymbol{S}_n\boldsymbol{S}_{n-1}\cdots\boldsymbol{S}_2 \end{vmatrix} = 0. \tag{42}$$

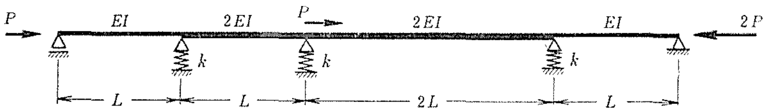


Fig. 6. Continuous Long Column on Elastic Supports.

Fig. 6 shows a continuous long column on elastic intermediate supports whose spring constants are commonly given by k. The relationship between the value of spring constant and the critical load is shown in Table 6, from which it can be mentioned that, as diminishing the value of k, the critical load obtained approaches to the case of continuous long column on rigid supports. This procedure will be recommended to the electronic computer operation.

Table 6. Critical Loads ($\times EI/L^2$)

k	Critical Load
1/1 000	4.250
1/10 000	4.265
1/100 000	4.266
Rigid Support	4.266

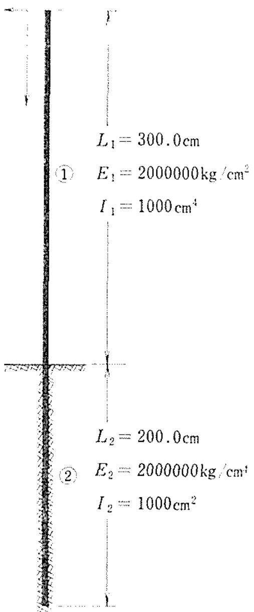


Fig. 7. Dolphin.

Example 4. Fig. 7 shows a simple dolphin built in an elastic foundation. The part without the foundation will be treated as ordinary beams, while that inbedded in the foundation must be treated as beams on elastic foundation assuming the elastic proportionality between the beam deflection and corresponding reactive force. Then such a system will be reduced to the connection of the ordinary beam ① with the beam in elastic foundation ②.

First, the boundary conditions, at the top and bottom ends of the system, the bending moment and shearing force vanish, are to be treated, and then the eigenmatrices of respective members are reduced to 2-by-1.

Next, connecting all physical quantities of both members at connection point, the solution of the system can be obtained.

Using the values shown in Fig. 7, the results obtained are as follows:

$$\{\boldsymbol{N}_o \quad \boldsymbol{N}_e\} = \boldsymbol{G}\{\boldsymbol{K}_o \quad \boldsymbol{K}_e\}. \tag{43}$$

Here $k = 10\ \mathrm{kg/cm^2}$ has been used for the modulus of foundation. Also $\boldsymbol{N}_o$ and $\boldsymbol{N}_e$ = the eigenmatrices for beams ① and ②, $\boldsymbol{K}_o$ = the load term for ordinary beams (Eq. 14), and $\boldsymbol{K}_e$ = the load term for beams on elastic foundation (Eq. 15), but except for the special case in which several external forces act on the beam part in the foundation, this matrix is given by

$$\boldsymbol{K}_e = \{0 \quad 0 \quad 0 \quad 0\}. \tag{44}$$

The geometry matrix in this system is evaluated as follows:

$$\boldsymbol{G} = \begin{bmatrix} -1.000 & 0 & 4.204 & 14.655 & -2.629 & 9.375 & 1.572 & -1.402 \\ 0 & -1.000 & -4.491 & -12.612 & 1.195 & -7.270 & -1.195 & 1.109 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.014 & 0.080 & -1.040 & 0.063 & 0.040 & 0.017 \\ 0 & 0 & -0.363 & -1.407 & 0.317 & -1.928 & -0.317 & 0.294 \\ 0 & 0 & 1.335 & 5.678 & -1.674 & 3.919 & 0.674 & -0.571 \\ 0 & 0 & -0.958 & -3.191 & 0.317 & -1.928 & -0.317 & 0.294 \end{bmatrix}. \tag{45}$$

10. CONCLUSIONS

The operational method for bending problems of various continuous beams is presented in this paper.

This method is based on the systematic treatment of the eigenmatrix which is the column assemblage of integration constants of the general solution for governing differential equation. The boundary and connection conditions are given by simplified matrix formulas, and then the analysis can be carried out readily and systematically. It should be noted here that the pure and complete classification in data treatment results in the readiness in philosophy and computation.

Four typical examples are appended at the latter part of this paper, in which the geometry matrices have been evaluated in three examples. These matrices will be particularly useful for structural design.

The prevailing key equations to known methods in structural analysis, such as the three-moment method, the slope-deflection method, etc., may be derived from the approach equation of the present method, and their characteristics can be commented.[5] Applying the philosophy of the operational method to these key equations, recursive procedures for respective prevailing methods may be composed.[6]

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APPENDIX.—NOTATION

The following symbols have been adopted for use in this paper:

$\mathbf{A}$ = semi-eigenmatrix, see Eq. 23;
$\mathbf{B}, \mathbf{B}'$ = boundary matrices, see Table 1;
$\mathbf{C}$ = connection matrix, see Table 3;
$\mathbf{D}$ = coefficient matrix, see Eqs. 3, 4, and 6;
E = Young's modulus;
$\mathbf{G}$ = geometry matrix, see Eq. 37;
I = moment of inertia;
k = modulus of foundation;
k = elastic support constant;
$\mathbf{K}$ = load matrix, see Eqs. 14, 15, and 16;

L = beam length;
$\mathbf{L}, \mathbf{L}'$ = shift operators, see Eqs. 32 and 33;
M = bending moment;
m = elastic support constant;
$\mathbf{N}$ = eigenmatrix, see Eq. 12;
$\mathbf{P}(\rho)$ = abscissa matrix, see Eqs. 8, 9, and 11;
$\mathbf{P}, \mathbf{P}'$ = feed operators, see Eqs. 32, and 33;
Q = axial compressive force;
$\mathbf{Q}, \mathbf{Q}'$ = feed operators, see Eqs. 32 and 33;
S = shearing force;
$\mathbf{S}, \mathbf{S}'$ = shift operators, see Eqs. 28 and 29;
w = deflection;
$\mathbf{W}(\rho)$ = state vector, see Eq. 2;
x = current abscissa;
α = $\sqrt{QL^2/EI}$, see Eq. 7;
β = $\sqrt[4]{kL^4/4EI}$, see Eq. 5;
θ = slope angle;
κ = non-dimensional load abscissa;
ξ = load abscissa;
ρ = non-dimensional current abscissa;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.