Subsidiary Operational Method for Continuous Beam-Columns

By
Norikazu YOSHIZAWA* and Bennosuke TANIMOTO**

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1. INTRODUCTION

The prevailing key equations to continuous beam-column analyses by the slope-deflection method are derived from the eigenmatrix of the operational method. The three-slope equation and the three-moment equation are also derived.

The eigenmatrix[1] is defined by the assemblage of integration constants of the general solution of differential equation for a beam-column member arranged in a 4-by-1 column matrix. Then, in the operational method, the problem is reduced to attack the eigenmatrix. Treating completely compatibility conditions of displacements and equilibrium conditions of forces at an intermediate support of a continuous system, a shift formula between two consecutive eigenmatrices is obtained. By the recurrent use of the shift formula, the eigenmatrices of the entire system can be expressed in terms of the eigenmatrix of a span, called the current-matrix. Finally, the current-matrix is determined by both extreme boundary conditions. Thus, the system can be solved systematically dispensing with simultaneous equations.

From such a standpoint, it may be said that the above key equations are the results obtained from the eigenmatrix after a preliminary treatment. The outlines are as follows:

Slope-deflection equations:

1. The eigenmatrix is specified by end deflections and end slopes of a member.
2. The end moments expressed by this specified eigenmatrix at once give the desired key equations.

Three-slope equation:

1. The eigenmatrix is specified by end deflections and end slopes of a

* Assistant, Department of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.
** Professor of Civil Engineering, Faculty of Engineering, Shinshu University, Nagano, Japan.

member.

2. Taking two consecutive spans, the slope continuity condition at the intermediate support is to be treated preliminarily.

3. The moment equilibrium condition at the intermediate support gives the desired key equation.

Three-moment equation:

1. The eigenmatrix is specified by end deflections and end moments of a member.

2. Taking two consecutive spans, the moment equilibrium condition at the intermediate support is to be treated preliminarily.

3. The slope continuity condition at the intermediate support gives the desired key equation.

The recurrent procedure in the operational method can be extended to the prevailing key equations. The application to the three moment equation will be shown in the subsequent discussions. The entire course of analysis is carried out with matrix algebra, and the recurrence formula for the set of two consecutive support moments will be derived, which can also avoid simultaneous equations. Although the analysis can be composed of the recurrent procedure, it will be not always most preferable, because of the incomplete classification of data in the key equations. In fact, as will be seen subsequently, the resulting recurrence formulas require a considerable amount of computations and in addition the elements in the operators become complicated. This analysis can be one of the operational procedures, but it would be, so to speak, a bad second. Numerical examples are added at the end of this paper.

2. BASIC CONCEPTS

The flexural behavior of a beam-column is governed under no lateral loads by the following differential equation:

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} = 0. \qquad (1)^{2)}$$

Here $w=$ the deflection, $x=$ the current abscissa, $P=$ the compressive axial load, and $EI=$ the flexural rigidity. The general solution of Eq. 1 is given by the form

$$w = \frac{L^3}{\alpha^3 EI}\lfloor 1 \quad \alpha\rho \quad \cos\alpha\rho \quad \sin\alpha\rho \rfloor \boldsymbol{N}, \qquad (2)$$

in which L = the member length, $\alpha = \sqrt{PL^2/EI}$, $\rho = x/L$, and $\mathbf{N} = \{A \quad B \quad C \quad D\}$ = the assemblage of integration constants, which is referred to as the "eigen-matrix" of the member.

The state vector of a beam-column member is defined by a 4-by-1 column matrix whose elements are composed of the deflection w, the slope θ, the bending moment M, and the shearing force S. Based on the deflection given by Eq. 2, the above state vector is expressed by the form

$$\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix} = \begin{bmatrix} \frac{L^3}{\alpha^3 EI} & 0 & 0 & 0 \\ 0 & \frac{L^2}{\alpha^2 EI} & 0 & 0 \\ 0 & 0 & -\frac{L}{\alpha} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \alpha\rho & \cos\alpha\rho & \sin\alpha\rho \\ 0 & 1 & -\sin\alpha\rho & \cos\alpha\rho \\ 0 & 0 & -\cos\alpha\rho & -\sin\alpha\rho \\ 0 & 0 & \sin\alpha\rho & -\cos\alpha\rho \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}, \tag{3}$$

or, symbolically,

$$\mathbf{W}(\alpha\rho) = \mathbf{DP}(\alpha\rho)\mathbf{N}. \tag{4}$$

When the beam-column is subjected to lateral external loads, such as the concentrated load, the arbitrary distributed load, and the concentrated moment, as shown in Fig. 1, the continuity condition between two state vectors at a loaded point are to be treated. Then, the external loads will be represented by the corresponding "load-matrix." Referring to Fig. 1, the state vectors at the cross-sections i and j are expressed as follows:

$$\mathbf{W}_i = \mathbf{DPN}_i = \mathbf{DPN}, \tag{5}$$

$$\mathbf{W}_j = \mathbf{DPN}_j = \mathbf{DP}(\mathbf{N} + \mathbf{K}_Q + \mathbf{K}_q + \mathbf{K}_m). \tag{6}$$

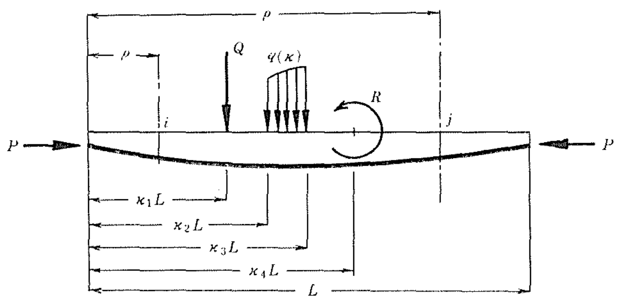


Fig. 1. Loading Conditions.

Here $\boldsymbol{K}_Q$, $\boldsymbol{K}_q$, and $\boldsymbol{K}_m$ are the load-matrices for concentrated load, arbitrary distributed load, and concentrated moment, respectively. They are given by the forms

$$\boldsymbol{K}_Q = Q\{-\alpha\kappa_1 \quad 1 \quad \sin\alpha\kappa_1 \quad -\cos\alpha\kappa_1\}, \tag{7}$$

$$\boldsymbol{K}_q = L\int_{\kappa_2}^{\kappa_3} q(\kappa)\{-\alpha\kappa \quad 1 \quad \sin\alpha\kappa \quad -\cos\alpha\kappa\}d\kappa, \tag{8}$$

$$\boldsymbol{K}_m = \frac{\alpha R}{L}\{1 \quad 0 \quad -\cos\alpha\kappa_4 \quad -\sin\alpha\kappa_4\}. \tag{9}$$

In the analysis of the continuous system, the state vector at a member end is to be interconnected with that of the adjacent member end. The eigenmatrices at both member ends are to be specified as follows:

$$\text{At } \rho = 0: \ \boldsymbol{N}_r = \{A \quad B \quad C \quad D\}_r = \text{normal eigenmatrix}. \tag{10}$$

$$\text{At } \rho = 1: \ \boldsymbol{N}'_r = \boldsymbol{N}_r + \boldsymbol{K}_r = \text{conjugate eigenmatrix}. \tag{11}$$

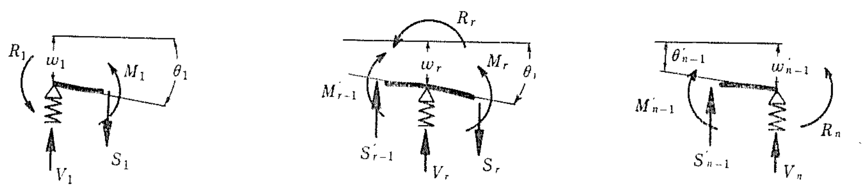
Here $\boldsymbol{K}_r$ = the "load term" of member r, which is given by the form

$$\boldsymbol{K}_r = \sum_{\kappa=0}^{1}(\boldsymbol{K}_Q + \boldsymbol{K}_q + \boldsymbol{K}_m)_r. \tag{12}$$

Figs. 2a, 2b, and 2c illustrate small portions at the extreme left end, the intermediate support, and the extreme right end of the continuous beam-column. Assuming the elastic proportionality, all the support conditions are given by the relationships

$$w = kV, \quad \theta = mR, \tag{13}$$

in which V = the support reaction, R = the support resisting moment, and k, m = the spring constants attached to the elastic support. Then the relationships between the state vectors (Figs. 2) are interconnected with the following equations:



(a) Left Boundary. (b) Intermediate Support. (c) Right Boundary.

Fig. 2. Support Conditions.

(a) Left boundary conditions (2 conditions)

$$\begin{bmatrix} M \\ S \end{bmatrix}_1 + \begin{bmatrix} \dfrac{\theta}{m} \\ -\dfrac{w}{k} \end{bmatrix}_1 = 0. \tag{14}$$

(b) Intermediate support conditions (4 conditions)

$$-\begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}'_{r-1} + \begin{bmatrix} w \\ \theta \\ M \\ S \end{bmatrix}_r + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\theta}{m} \\ -\dfrac{w}{k} \end{bmatrix}_r = 0. \tag{15}$$

(c) Right boundary conditions (2 conditions)

$$\begin{bmatrix} M \\ S \end{bmatrix}'_{n-1} + \begin{bmatrix} -\dfrac{\theta'_{n-1}}{m_n} \\ \dfrac{w'_{n-1}}{k_n} \end{bmatrix} = 0. \tag{16}$$

All the possible kinds of support conditions can be expressed by due values of k and m in the above equation. Several examples are shown in Table 1.

Table 1. Support Conditions.

	m_1	k_1	m_r	k_r	m_n	k_n
Elastic support	m_1	k_1	m_r	k_r	m_n	k_n
Rigid support	—	—	∞	0	—	—
Hinged end	∞	0	—	—	∞	0
Clamped end	0	0	—	—	0	0

In virtue of Eq. 3, Eq. 15 yields a shift formula between two eigenmatrices of members connected at an intermediate support of the continuous beam-column. By the recurrent use of the shift formula, the eigenmatrix of a span can become current to the entire spans. Finally, both the extreme boundary conditions given by Eqs. 14 and 16 are used for determination of the current eigenmatrix, and then the system can be solved completely. This is the orthodox procedure of the operational method.[3),4),5)] In the subsequent discussions, several key equations to prevailing methods will be commented from the operational view-point and the application of the operational procedure will be shown.

3. SLOPE-DEFLECTION EQUATIONS

The slope-deflection equation is derived from the key equation, Eq. 3, to the operational method as a preliminary treatment of the eigenmatrix.[5] That is to say, taking a member, the nodal displacements, the deflections and the slopes, at both member ends are to be imposed on the eigenmatrix. Then the end moments expressed by this eigenmatrix at once give the desired slope-deflection equations. The derivation is given in the following:

Fig. 3 shows the r-th constituent member of the continuous system, wherein the displacements of nodal points r and $r+1$, and the end forces of member r are illustrated. Referring to the figure, the compatibility conditions between member end displacements and the nodal displacements are written

$$\begin{bmatrix} w \\ \theta \end{bmatrix}_r = \begin{bmatrix} W \\ \Theta \end{bmatrix}_r, \quad \begin{bmatrix} w \\ \theta \end{bmatrix}'_r = \begin{bmatrix} W \\ \Theta \end{bmatrix}_{r+1}. \tag{17}$$

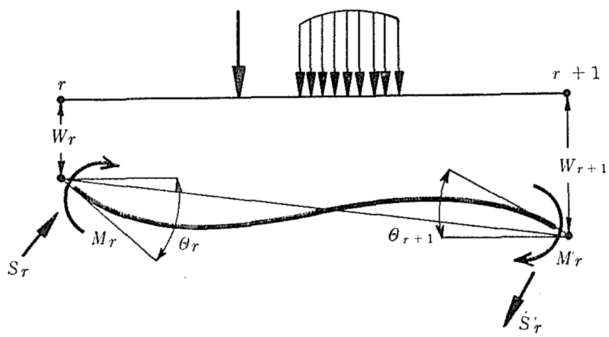


Fig. 3. Constituent Member and End Conditions.

First, in virtue of Eq. 3, Eq. 17a is written

$$\left(\frac{L^2}{\alpha^2 EI}\right)_r \begin{bmatrix} \frac{L}{\alpha} & 0 \\ 0 & 1 \end{bmatrix}_r \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}_r = \begin{bmatrix} W \\ \Theta \end{bmatrix}_r. \tag{18}$$

Then the eigenmatrix $\mathbf{N}_r$ becomes

$$\boldsymbol{N}_r = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}_r = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}_r + \left(\frac{\alpha^2 EI}{L^2}\right)_r \begin{bmatrix} \frac{\alpha}{L} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_r \begin{bmatrix} W \\ \Theta \end{bmatrix}_r . \tag{19}$$

Using Eqs. 3 and 19, Eq. 17b is written

$$\begin{bmatrix} \cos\alpha - 1, \sin\alpha - \alpha \\ -\sin\alpha, \ \cos\alpha - 1 \end{bmatrix}_r \begin{bmatrix} C \\ D \end{bmatrix}_r + \left(\frac{\alpha^2 EI}{L^2}\right)_r \begin{bmatrix} \frac{\alpha}{L} & \alpha \\ 0 & 1 \end{bmatrix}_r \begin{bmatrix} W \\ \Theta \end{bmatrix}_r$$
$$+ \begin{bmatrix} 1 & \alpha & \cos\alpha & \sin\alpha \\ 0 & 1 & -\sin\alpha & \cos\alpha \end{bmatrix}_r \boldsymbol{K}_r = \left(\frac{\alpha^2 EI}{L^2}\right)_r \begin{bmatrix} \frac{\alpha}{L} & 0 \\ 0 & 1 \end{bmatrix}_r \begin{bmatrix} W \\ \Theta \end{bmatrix}_{r+1} . \tag{20}$$

From this equation, $\{C \quad D\}_r$ is obtained as follows:

$$\begin{bmatrix} C \\ D \end{bmatrix}_r = \left(\frac{\alpha^3 EI}{L^3} \frac{1}{2 - 2\cos\alpha - \alpha\sin\alpha}\right)_r \begin{bmatrix} 1 - \cos\alpha, & -1 + \cos\alpha \\ -\sin\alpha, & \sin\alpha \end{bmatrix}_r \begin{bmatrix} W_r \\ W_{r+1} \end{bmatrix}$$
$$+ \left(\frac{\alpha^2 EI}{L^2} \frac{1}{2 - 2\cos a - \alpha\sin\alpha}\right)_r \begin{bmatrix} \sin\alpha - \alpha\cos\alpha, & \alpha - \sin\alpha \\ 1 - \cos\alpha - \alpha\sin\alpha, & \cos\alpha - 1 \end{bmatrix}_r \begin{bmatrix} \Theta_r \\ \Theta_{r+1} \end{bmatrix}$$
$$+ \left(\frac{1}{2 - 2\cos\alpha - \alpha\sin\alpha}\right)_r \begin{bmatrix} 1 - \cos\alpha, & \sin\alpha - \alpha\cos\alpha, & \cos\alpha + \alpha\sin\alpha - 1, & \sin\alpha - \alpha\cos\alpha \\ -\sin\alpha, & 1 - \cos\alpha - \alpha\sin\alpha, & -\sin\alpha, & \cos\alpha - 1 \end{bmatrix}_r \boldsymbol{K}_r . \tag{21}$$

Consequently, the eigenmatrix $\boldsymbol{N}_r$ is represented by the end displacements as follows:

$$\boldsymbol{N}_r = \left(\frac{1}{2 - 2\cos\alpha - \alpha\sin\alpha}\right)_r \left[\frac{\alpha^3 EI}{L^3} \begin{bmatrix} 1 - \cos\alpha - \alpha\sin\alpha, & 1 - \cos\alpha \\ \sin\alpha, & -\sin\alpha \\ 1 - \cos\alpha, & -1 + \cos\alpha \\ -\sin\alpha, & \sin\alpha \end{bmatrix}, \right.$$

$$\frac{\alpha^2 EI}{L^2} \begin{bmatrix} -\sin\alpha + \alpha\cos\alpha, & -\alpha + \sin\alpha \\ 1 - \cos\alpha, & -\cos\alpha + 1 \\ \sin\alpha - \alpha\cos\alpha, & \alpha - \sin\alpha \\ 1 - \cos\alpha - \alpha\sin\alpha, & \cos\alpha - 1 \end{bmatrix},$$

$$\left[\begin{bmatrix} -1+\cos\alpha, & -\sin\alpha+\alpha\cos\alpha, & 1-\cos\alpha-\alpha\sin\alpha, & -\sin\alpha+\alpha\cos\alpha \\ \sin\alpha, & -1+\cos\alpha+\alpha\sin\alpha, & \sin\alpha, & -\cos\alpha+1 \\ 1-\cos\alpha, & \sin\alpha-\alpha\cos\alpha, & -1+\cos\alpha+\alpha\sin\alpha, & \sin\alpha-\alpha\cos\alpha \\ -\sin\alpha, & 1-\cos\alpha-\alpha\sin\alpha, & -\sin\alpha, & \cos\alpha-1 \end{bmatrix}\right]_r \begin{bmatrix} W_r \\ W_{r+1} \\ \Theta_r \\ \Theta_{r+1} \\ \boldsymbol{K}_r \end{bmatrix}. \tag{22}$$

Next, taking the direction of member end moments clockwise as shown in the figure, and using Eqs. 3 and 22, we obtain

$$\begin{bmatrix} M \\ M' \end{bmatrix}_r = \left(\frac{1}{2-2\cos\alpha-\alpha\sin\alpha}\right)_r \left[\left(\frac{\alpha^2 EI}{L^2}\right)_r \begin{bmatrix} 1-\cos\alpha, & -1+\cos\alpha \\ 1-\cos\alpha, & -1+\cos\alpha \end{bmatrix}_r \begin{bmatrix} W_r \\ W_{r+1} \end{bmatrix}\right.$$
$$+\left(\frac{\alpha EI}{L}\right)_r \begin{bmatrix} \sin\alpha-\alpha\cos\alpha, & \alpha-\sin\alpha \\ \alpha-\sin\alpha, & \sin\alpha-\alpha\cos\alpha \end{bmatrix}_r \begin{bmatrix} \Theta_r \\ \Theta_{r+1} \end{bmatrix}$$
$$\left.+\left(\frac{L}{\alpha}\right)_r \begin{bmatrix} 1-\cos\alpha, & \sin\alpha-\alpha\cos\alpha, & -1+\cos\alpha+\alpha\sin\alpha, & \sin\alpha-\alpha\cos\alpha \\ 1-\cos\alpha, & \alpha-\sin\alpha, & 1-\cos\alpha, & \alpha-\sin\alpha \end{bmatrix}_r \boldsymbol{K}_r\right]. \tag{23}$$

4. THREE-SLOPE EQUATION

The three-slope equation for continuous beam-columns is obtained from the following treatments:

(1) Preliminarily, at each intermediate support, the compatibility conditions for deflections and slopes between both spans are to be considered. After this treatment, taking the r-th support, all the physical quantities are expressed in terms of the support settlement W_r and the support slope Θ_r. The former, W_r, is in general taken as the given quantity, while the latter, Θ_r, as the unknown quantity.

(2) Due support settlements and slopes are to be imposed on the eigenmatrix of each member, the resulting form of which is the same as Eq. 22.

(3) Using this eigenmatrix, and in virtue of Eq. 3, the equilibrium condition of bending moments at the r-th support is to be treated, which results in the following three-slope equation:

$$\lfloor (fa)_{r-1},\ (fb)_{r-1}+(fb)_r,\ (fa)_r \rfloor \{\Theta_{r-1}\ \ \Theta_r\ \ \Theta_{r+1}\}$$
$$= \lfloor -(ec)_{r-1},\ (ec)_{r-1}-(ec)_r,\ (ec)_r \rfloor \{W_{r-1}\ \ W_r\ \ W_{r+1}\}$$
$$+ g_{r-1}\lfloor -c,\ -a,\ -c,\ -a \rfloor_{r-1}\boldsymbol{K}_{r-1} + g_r\lfloor -c,\ -b,\ c-d,\ -b\rfloor_r \boldsymbol{K}_r. \tag{24}$$

Here, the following symbols have been adopted for use:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_r = \left(\frac{1}{2-2\cos\alpha-\alpha\sin\alpha}\right)_r \begin{bmatrix} \alpha & -1 & 0 \\ 0 & 1 & -\alpha \\ 1 & 0 & -1 \\ 0 & \alpha & 0 \end{bmatrix}_r \begin{bmatrix} 1 \\ \sin\alpha \\ \cos\alpha \end{bmatrix}_r . \tag{25}$$

$$\begin{bmatrix} e \\ f \end{bmatrix}_r = \left(\frac{\alpha EI}{L}\right)_r \begin{bmatrix} \frac{\alpha}{L} \\ 1 \end{bmatrix}_r . \tag{26}$$

$$g_r = \left(\frac{L}{\alpha}\right)_r . \tag{27}$$

5. THREE-MOMENT EQUATION

In the three-moment equation, the support settlements W's are taken as given quantities, while the support moments $\mathfrak{M}$'s are as unknown quantities. Referring to Fig. 4, the former are the results obtained from the compatibility condition of member end deflections, while the latter are those from the equilibrium condition between the member end moments at respective supports.

Fig. 4. Consecutive Spans of Continuous Beam-Column.

Taking the r-th span, the deflections and moments at supports then are defined by

$$\begin{bmatrix} w \\ M \end{bmatrix}_r = \begin{bmatrix} W \\ \mathfrak{M} \end{bmatrix}_r , \qquad \begin{bmatrix} w \\ M \end{bmatrix}'_r = \begin{bmatrix} W \\ \mathfrak{M} \end{bmatrix}_{r+1} . \tag{28}$$

Treating Eq. 28a together with Eq. 3, the eigenmatrix $\boldsymbol{N}_r$ is degraded to the form of semi-eigenmatrix $\{B \quad D\}_r$ as follows:

$$\boldsymbol{N}_r = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}_r + \begin{bmatrix} \frac{\alpha^3 EI}{L^3} & -\frac{\alpha}{L} \\ 0 & 0 \\ 0 & \frac{\alpha}{L} \\ 0 & 0 \end{bmatrix}_r \begin{bmatrix} W \\ \mathfrak{M} \end{bmatrix}_r . \tag{29}$$

Then in virtue of Eq. 28b, the semi-eigenmatrix is expressed in the form

$$\begin{bmatrix} B \\ D \end{bmatrix}_r = \begin{bmatrix} \frac{\alpha^2 EI}{L^3} & -\frac{1}{L} \\ 0 & \frac{\alpha}{L}\csc\alpha \end{bmatrix}_r \begin{bmatrix} W \\ \mathfrak{M} \end{bmatrix}_{r+1} + \begin{bmatrix} -\frac{\alpha^2 EI}{L^3} & \frac{1}{L} \\ 0 & -\frac{\alpha}{L}\cot\alpha \end{bmatrix}_r \begin{bmatrix} W \\ \mathfrak{M} \end{bmatrix}_r + \begin{bmatrix} -\frac{1}{\alpha} & -1 & 0 & 0 \\ 0 & 0 & -\cot\alpha & -1 \end{bmatrix}_r \boldsymbol{K}_r. \tag{30}$$

Thus, the eigenmatrix $\boldsymbol{N}_r$ of the r-th span can be expressed as follows:

$$\boldsymbol{N}_r = \frac{1}{L_r}\begin{bmatrix} -\alpha & 0 \\ 1 & -1 \\ \alpha & 0 \\ -\alpha\cot\alpha & \alpha\csc\alpha \end{bmatrix}_r \begin{bmatrix} \mathfrak{M}_r \\ \mathfrak{M}_{r+1} \end{bmatrix} + \left(\frac{\alpha^2 EI}{L^3}\right)_r \begin{bmatrix} \alpha & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_r \begin{bmatrix} W_r \\ W_{r+1} \end{bmatrix} + \frac{1}{\alpha_r}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\cot\alpha & -\alpha \end{bmatrix}_r \boldsymbol{K}_r. \tag{31}$$

Next, at the r-th support of the continuous system, the continuity condition of slope angle between both spans are to be treated. Referring to Eqs. 3 and 31, this condition can be written in the following form, which will yield the desired three-moment equation.

$$\lfloor \beta_{r-1},\ \gamma_{r-1}+\gamma_r,\ \beta_r \rfloor \{\mathfrak{M}_{r-1}\ \ \mathfrak{M}_r\ \ \mathfrak{M}_{r+1}\} = \lfloor \delta_{r-1},\ -\delta_{r-1}-\delta_r,\ \delta_r \rfloor \{W_{r-1}\ \ W_r\ \ W_{r+1}\}$$
$$+\left(\frac{L^2}{\alpha^3 EI}\right)_{r-1} \lfloor 1\ \ 0\ \ \alpha\csc\alpha\ \ 0 \rfloor_{r-1} \boldsymbol{K}_{r-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_r \lfloor -1\ \ -\alpha\ \ -\alpha\cot\alpha\ \ -\alpha \rfloor_r \boldsymbol{K}_r. \tag{32}$$

Here, the following symbols have been adopted for use:

$$\begin{bmatrix} \beta \\ \gamma \end{bmatrix}_r = \left(\frac{L}{\alpha^2 EI}\right)_r \begin{bmatrix} 1-\alpha\csc\alpha \\ \alpha\cot\alpha - 1 \end{bmatrix}_r. \tag{33}$$

$$\delta_r = \frac{1}{L_r}. \tag{34}$$

6. OPERATIONAL PROCEDURE FOR THREE-MOMENT EQUATION

6.1. Recurrence Formula.

Taking two sets of three consecutive supports, $r-1$, r, and $r+1$, and r, $r+1$, and $r+2$, the three-moment equations can be put into the following matrix equation:

$$\begin{bmatrix}\beta_{r-1}, & \gamma_{r-1}+\gamma_r \\ 0, & \beta_r\end{bmatrix}\begin{bmatrix}\mathfrak{M}_{r-1} \\ \mathfrak{M}_r\end{bmatrix}+\begin{bmatrix}\beta_r, & 0 \\ \gamma_r+\gamma_{r+1}, & \beta_{r+1}\end{bmatrix}\begin{bmatrix}\mathfrak{M}_{r+1} \\ \mathfrak{M}_{r+2}\end{bmatrix}$$

$$=\begin{bmatrix}\delta_{r-1}, & -\delta_{r-1}-\delta_r \\ 0, & \delta_r\end{bmatrix}\begin{bmatrix}W_{r-1} \\ W_r\end{bmatrix}+\begin{bmatrix}\delta_r, & 0 \\ -\delta_r-\delta_{r+1}, & \delta_{r+1}\end{bmatrix}\begin{bmatrix}W_{r+1} \\ W_{r+2}\end{bmatrix}$$

$$+\left(\frac{L^2}{\alpha^3 EI}\right)_{r-1}\begin{bmatrix}1 & 0 & \alpha\csc\alpha & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}_{r-1}\boldsymbol{K}_{r-1}$$

$$+\left(\frac{L^2}{\alpha^3 EI}\right)_{r}\begin{bmatrix}-1 & -\alpha & -\alpha\cot\alpha & -\alpha \\ 1 & 0 & \alpha\csc\alpha & 0\end{bmatrix}_{r}\boldsymbol{K}_{r}$$

$$+\left(\frac{L^2}{\alpha^3 EI}\right)_{r+1}\begin{bmatrix}0 & 0 & 0 & 0 \\ -1 & -\alpha & -\alpha\cot\alpha & -\alpha\end{bmatrix}_{r+1}\boldsymbol{K}_{r+1}, \qquad (35)$$

or

$$-\boldsymbol{d}_{r-1}\boldsymbol{M}_{r-1}+\boldsymbol{c}_{r+1}\boldsymbol{M}_{r+1}=\boldsymbol{u}_{r-1}\boldsymbol{W}_{r-1}+\boldsymbol{v}_{r+1}\boldsymbol{W}_{r+1}+\boldsymbol{p}_{r-1}\boldsymbol{K}_{r-1}+\boldsymbol{q}_r\boldsymbol{K}_r+\boldsymbol{r}_{r+1}\boldsymbol{K}_{r+1}, \qquad (36)$$

providing

$$\boldsymbol{M}_{r-1}=\begin{bmatrix}\mathfrak{M}_{r-1} \\ \mathfrak{M}_r\end{bmatrix},\quad \boldsymbol{M}_{r+1}=\begin{bmatrix}\mathfrak{M}_{r+1} \\ \mathfrak{M}_{r+2}\end{bmatrix},\quad \boldsymbol{W}_{r-1}=\begin{bmatrix}W_{r-1} \\ W_r\end{bmatrix},\quad \boldsymbol{W}_{r+1}=\begin{bmatrix}W_{r+1} \\ W_{r+2}\end{bmatrix}. \qquad (37)$$

Here $\boldsymbol{M}_{r-1}$ and $\boldsymbol{M}_{r+1}$ are the support moment matrices which are taken as unknown quantities, and then, henceforth, they will be called the eigenmatrices in this analysis, while $\boldsymbol{W}_{r-1}$ and $\boldsymbol{W}_{r+1}$ are the support settlement matrices which are considered as given quantities.

The matrix $\boldsymbol{c}_{r+1}$ is square and nonsingular, and then the eigenmatrix $\boldsymbol{M}_{r+1}$ is expressed as follows:

$$\boldsymbol{M}_{r+1}=\overset{-1}{\underset{r+1}{\boldsymbol{c}}}\Big[\boldsymbol{d}_{r-1}\boldsymbol{M}_{r-1}+\lfloor\boldsymbol{u}_{r-1}\quad\boldsymbol{v}_{r+1}\rfloor\{\boldsymbol{W}_{r-1}\quad\boldsymbol{W}_{r+1}\}$$

$$+\lfloor\boldsymbol{p}_{r-1}\quad\boldsymbol{q}_r\quad\boldsymbol{r}_{r+1}\rfloor\{\boldsymbol{K}_{r-1}\quad\boldsymbol{K}_r\quad\boldsymbol{K}_{r+1}\}\Big], \qquad (38)$$

or

$$\boldsymbol{M}_{r+1} = \boldsymbol{D}_{r+1}\boldsymbol{M}_{r-1} + \lfloor \boldsymbol{U}_{r-1} \quad \boldsymbol{V}_{r+1} \rfloor \{\boldsymbol{W}_{r-1} \quad \boldsymbol{W}_{r+1}\} + \lfloor \boldsymbol{P}_{r-1} \quad \boldsymbol{Q}_r \quad \boldsymbol{R}_{r+1} \rfloor \{\boldsymbol{K}_{r-1} \quad \boldsymbol{K}_r \quad \boldsymbol{K}_{r+1}\}. \tag{39}$$

This is the desired recurrence formula between two consecutive eigenmatrices of the continuous beam-column.

In applying the recurrence formula to the analysis of continuous systems, the following notes are to be given:

1. The order of supports is counted from the extreme left support.
2. The subscript "i" denotes only even numbers ($i = 2, 4, 6, \cdots$).
3. The eigenmatrix of the system is defined from the extreme left end as follows:

$$\boldsymbol{M}_1 = \begin{bmatrix} \mathfrak{M}_1 \\ \mathfrak{M}_2 \end{bmatrix}, \quad \boldsymbol{M}_3 = \begin{bmatrix} \mathfrak{M}_3 \\ \mathfrak{M}_4 \end{bmatrix}, \quad \cdots, \quad \boldsymbol{M}_{i+1} = \begin{bmatrix} \mathfrak{M}_{i+1} \\ \mathfrak{M}_{i+2} \end{bmatrix}. \tag{40}$$

Then the recurrent use of Eq. 39 yields the following equation:

$$\begin{aligned} \boldsymbol{M}_{i+1} &= \boldsymbol{D}_{i+1}\boldsymbol{L}_{i-1}\boldsymbol{M}_1 \\ &+ \lfloor \boldsymbol{D}_{i+1}\boldsymbol{S}_{i-1,1}, \ \boldsymbol{D}_{i+1}\boldsymbol{S}_{i-1,3}, \ \cdots, \ \boldsymbol{D}_{i+1}\boldsymbol{S}_{i-1,i-1} + \boldsymbol{U}_{i-1}, \ \boldsymbol{V}_{i+1} \rfloor \{\boldsymbol{W}_1 \quad \boldsymbol{W}_3 \quad \cdots \quad \boldsymbol{W}_{i-1} \quad \boldsymbol{W}_{i+1}\} \\ &+ \lfloor \boldsymbol{D}_{i+1}\boldsymbol{T}_{i-1,1}, \ \boldsymbol{D}_{i+1}\boldsymbol{T}_{i-1,2}, \ \cdots, \ \boldsymbol{D}_{i+1}\boldsymbol{T}_{i-1,i-1} + \boldsymbol{P}_{i-1}, \ \boldsymbol{Q}_i, \ \boldsymbol{R}_{i+1} \rfloor \{\boldsymbol{K}_1 \ \boldsymbol{K}_2 \ \cdots \ \boldsymbol{K}_{i-1} \ \boldsymbol{K}_i \ \boldsymbol{K}_{i+1}\} \\ &= \boldsymbol{L}_{i+1}\boldsymbol{M}_1 + \lfloor \boldsymbol{S}_1 \quad \boldsymbol{S}_3 \quad \cdots \quad \boldsymbol{S}_{i-1} \quad \boldsymbol{S}_{i+1} \rfloor_{i+1} \{\boldsymbol{W}_1 \quad \boldsymbol{W}_3 \quad \cdots \quad \boldsymbol{W}_{i-1} \quad \boldsymbol{W}_{i+1}\} \\ &+ \lfloor \boldsymbol{T}_1 \quad \boldsymbol{T}_2 \quad \cdots \quad \boldsymbol{T}_{i-1} \quad \boldsymbol{T}_i \quad \boldsymbol{T}_{i+1} \rfloor_{i+1} \{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \cdots \quad \boldsymbol{K}_{i-1} \quad \boldsymbol{K}_i \quad \boldsymbol{K}_{i+1}\} \\ &= \boldsymbol{L}_{i+1}\boldsymbol{M}_1 + \lfloor \boldsymbol{S} \rfloor_{i+1} \{\boldsymbol{W}\}_{i+1} + \lfloor \boldsymbol{T} \rfloor_{i+1} \{\boldsymbol{K}\}_{i+1}, \end{aligned} \tag{41}$$

providing

$$\left. \begin{aligned} \boldsymbol{L}_3 &= \boldsymbol{D}_3, \\ \lfloor \boldsymbol{S} \rfloor_3 &= \lfloor \boldsymbol{U}_1 \quad \boldsymbol{V}_3 \rfloor, \\ \lfloor \boldsymbol{T} \rfloor_3 &= \lfloor \boldsymbol{P}_1 \quad \boldsymbol{Q}_2 \quad \boldsymbol{R}_3 \rfloor. \end{aligned} \right\} \tag{42}$$

Thus the eigenmatrix $\boldsymbol{M}_1$ becomes current to the entire system and then it may be called the current-matrix.

6.2. Left Boundary Conditions.

The boundary conditions at the extreme left end of the continuous structure are treated as follows.

For the simple support, the support moment vanishes, and then the boundary condition is given by taking the first eigenmatrix $\boldsymbol{M}_1$ as follows:

$$\boldsymbol{M}_1 = \begin{bmatrix} 0 \\ \mathfrak{M}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathfrak{M}_2, \quad \text{or} \quad \boldsymbol{M}_1 = \boldsymbol{B}\mathfrak{M}_2. \tag{43}$$

For the fixed end, the slope angle is zero, and then the boundary condition is given by taking the terms in Eq. 36 (in this case $r = 2$) as follows:

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathfrak{M}_2, \qquad \text{or} \qquad \boldsymbol{M}_1 = \boldsymbol{B}\mathfrak{M}_2. \tag{44}$$

$$-\boldsymbol{d}_1 = \begin{bmatrix} -\dfrac{\beta_1{}^2}{\gamma_1}, & \gamma_1 + \gamma_2 \\ 0\,, & \beta_2 \end{bmatrix}. \tag{45}$$

$$\boldsymbol{u}_1 = \begin{bmatrix} \delta_1\left(1 + \dfrac{\beta_1}{\gamma_1}\right), & -\delta_1\left(1 + \dfrac{\beta_1}{\gamma_1}\right) - \delta_2 \\ 0\,, & \delta_2 \end{bmatrix}. \tag{46}$$

$$\boldsymbol{p}_1 = \left(\frac{L^2}{\alpha^3 EI}\right)_1 \begin{bmatrix} 1 + \dfrac{\beta}{\gamma}, & \dfrac{\alpha\beta}{\gamma}, & \alpha \csc \alpha\left(1 + \dfrac{\beta}{\gamma}\cos \alpha\right), & \dfrac{\alpha\beta}{\gamma} \\ 0\,, & 0\,, & 0\,, & 0 \end{bmatrix}_1. \tag{47}$$

After treating the extreme left boundary condition above, and considering Eq. 41, all the support moments are expressed by $\mathfrak{M}_2$ as follows:

$$\boldsymbol{M}_{i+1} = \boldsymbol{L}_{i+1}\boldsymbol{B}\mathfrak{M}_2 + \lfloor \boldsymbol{S} \rfloor_{i+1}\{\boldsymbol{W}\}_{i+1} + \lfloor \boldsymbol{T} \rfloor_{i+1}\{\boldsymbol{K}\}_{i+1}. \tag{48}$$

6.3. Right Boundary Conditions.

The simple support condition is merely given by taking the extreme right support moment equal to zero, while, for the fixed end, the slope angle vanishes, and then treating this condition, the following relation is obtained providing n represents the extreme right support number.

$$\begin{aligned} \mathfrak{M}_n = & -\frac{\beta_{n-1}}{\gamma_{n-1}}\mathfrak{M}_{n-1} + \frac{\delta_{n-1}}{\gamma_{n-1}}\lfloor 1 \quad -1 \rfloor\{W_{n-1} \quad W_n\} \\ & + \frac{1}{\gamma_{n-1}}\left(\frac{L^2}{\alpha^3 EI}\right)_{n-1}\lfloor 1 \quad 0 \quad \alpha \csc \alpha \quad 0 \rfloor_{n-1}\boldsymbol{K}_{n-1}. \end{aligned} \tag{49}$$

6.4. Final Treatment.

The eigenmatrix is composed of two consecutive support moments beginning with the extreme left support of the continuous system. After treating the extreme left boundary condition and considering the recurrence formula, the second support moment $\mathfrak{M}_2$ becomes current to the entire system. This current element can be evaluated by due treatment of the extreme right boundary condition, in which case there occurs the following four cases regarding the beam configuration. Here n denotes the largest even support number of the system.

1. Simple support (support number $= n$).
2. Simple support (support number $= n+1$).
3. Fixed end (support number $= n$).
4. Fixed end (support number $= n+1$).

Respective cases are treated in the following.

6.4.1. Simple support (support number $= n$).

In virtue of Eq. 48, the eigenmatrix of the last unit is represented as follows:

$$\boldsymbol{M}_{n-1} = \begin{bmatrix} \mathfrak{M}_{n-1} \\ \mathfrak{M}_n \end{bmatrix} = \boldsymbol{L}_{n-1}\boldsymbol{B}\mathfrak{M}_2 + \lfloor \boldsymbol{S} \rfloor_{n-1}\{\boldsymbol{W}\}_{n-1} + \lfloor \boldsymbol{T} \rfloor_{n-1}\{\boldsymbol{K}\}_{n-1}. \tag{50}$$

The support condition is in this case given by

$$\lfloor 0 \quad 1 \rfloor \boldsymbol{M}_{n-1} = 0, \qquad \text{or} \qquad \boldsymbol{B}'\boldsymbol{M}_{n-1} = 0. \tag{51}$$

Eqs. 50 and 51 yield the final equation

$$\mathfrak{M}_2 = -[\boldsymbol{B}'\boldsymbol{L}_{n-1}\boldsymbol{B}]^{-1}\boldsymbol{B}'\Big[\lfloor \boldsymbol{S} \rfloor_{n-1}\{\boldsymbol{W}\}_{n-1} + \lfloor \boldsymbol{T} \rfloor_{n-1}\{\boldsymbol{K}\}_{n-1}\Big]. \tag{52}$$

6.4.2. Simple support (support number $= n+1$).

In this case, the eigenmatrix $\boldsymbol{M}_{n-1}$ is given by the same form as Eq. 50. Considering the extreme right end condition, $\mathfrak{M}_{n+1} = 0$, the three moment equation for the last three supports is written

$$\lfloor \beta_{n-1}, \ \gamma_{n-1} + \gamma_n \rfloor \boldsymbol{M}_{n-1} = \lfloor \delta_{n-1}, \ -\delta_{n-1} - \delta_n \rfloor \boldsymbol{W}_{n-1} + \delta_n W_{n+1}$$

$$+ \left(\frac{L^2}{\alpha^3 EI}\right)_{n-1} \lfloor 1 \quad 0 \quad \alpha\csc\alpha \quad 0 \rfloor_{n-1} \boldsymbol{K}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_n \lfloor -1 \quad -\alpha \quad -\alpha\cot\alpha \quad -\alpha \rfloor_n \boldsymbol{K}_n. \tag{53}$$

Substitution from Eq. 50 into Eq. 53 yields

$$\boldsymbol{B}'\boldsymbol{L}_{n-1}\boldsymbol{B}\mathfrak{M}_2 + \lfloor \boldsymbol{S} \rfloor'_{n+1}\{\boldsymbol{W}\}'_{n+1} + \lfloor \boldsymbol{T} \rfloor'_n\{\boldsymbol{K}\}_n = 0, \tag{54}$$

in which

$$\boldsymbol{B}' = \lfloor \beta_{n-1}, \ \gamma_{n-1} + \gamma_n \rfloor, \tag{55}$$

$$\lfloor \boldsymbol{S} \rfloor'_{n+1}\{\boldsymbol{W}\}'_{n+1} = \boldsymbol{B}'\lfloor \boldsymbol{S} \rfloor_{n-1}\{\boldsymbol{W}\}_{n-1} + \lfloor -\delta_{n-1}, \ \delta_{n-1} + \delta_n \rfloor \boldsymbol{W}_{n-1} - \delta_n W_{n+1}, \tag{56}$$

$$\lfloor \boldsymbol{T} \rfloor'_n \{\boldsymbol{K}\}_n = \boldsymbol{B}' \lfloor \boldsymbol{T} \rfloor_{n-1} \{\boldsymbol{K}\}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_{n-1} \lfloor -1 \quad 0 \quad -\alpha \csc\alpha \quad 0 \rfloor_{n-1} \boldsymbol{K}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_n \lfloor 1 \quad \alpha \quad \alpha\cot\alpha \quad \alpha \rfloor_n \boldsymbol{K}_n. \tag{57}$$

From Eq. 54, the final equation is obtained as follows:

$$\mathfrak{M}_2 = -[\boldsymbol{B}'\boldsymbol{L}_{n-1}\boldsymbol{B}]^{-1}\Big[\lfloor \boldsymbol{S} \rfloor'_{n+1}\{\boldsymbol{W}\}'_{n+1} + \lfloor \boldsymbol{T} \rfloor'_n \{\boldsymbol{K}\}_n\Big]. \tag{58}$$

6.4.3. Fixed end (support number $= n$).

The last eigenmatrix $\boldsymbol{M}_{n-1}$ is given by the form of Eq. 50. Then the fixed end condition is given by the equation

$$\boldsymbol{B}'\boldsymbol{L}_{n-1}\boldsymbol{B}\mathfrak{M}_2 + \lfloor \boldsymbol{S} \rfloor'_{n-1}\{\boldsymbol{W}\}_{n-1} + \lfloor \boldsymbol{T} \rfloor'_{n-1}\{\boldsymbol{K}\}_{n-1} = 0, \tag{59}$$

in which

$$\boldsymbol{B}' = \lfloor \beta \quad \gamma \rfloor_{n-1}, \tag{60}$$

$$\lfloor \boldsymbol{S} \rfloor'_{n-1}\{\boldsymbol{W}\}_{n-1} = \boldsymbol{B}'\lfloor \boldsymbol{S} \rfloor_{n-1}\{\boldsymbol{W}\}_{n-1} + \delta_{n-1}\lfloor -1 \quad 1 \rfloor \boldsymbol{W}_{n-1}, \tag{61}$$

$$\lfloor \boldsymbol{T} \rfloor'_{n-1}\{\boldsymbol{K}\}_{n-1} = \boldsymbol{B}'\lfloor \boldsymbol{T} \rfloor_{n-1}\{\boldsymbol{K}\}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_{n-1} \lfloor -1 \quad 0 \quad -\alpha\csc\alpha \quad 0 \rfloor_{n-1}\boldsymbol{K}_{n-1}. \tag{62}$$

From Eq. 59, $\mathfrak{M}_2$ is obtained as follows:

$$\mathfrak{M}_2 = -[\boldsymbol{B}'\boldsymbol{L}_{n-1}\boldsymbol{B}]^{-1}\Big[\lfloor \boldsymbol{S} \rfloor'_{n-1}\{\boldsymbol{W}\}_{n-1} + \lfloor \boldsymbol{T} \rfloor'_{n-1}\{\boldsymbol{K}\}_{n-1}\Big]. \tag{63}$$

6.4.4. Fixed end (support number $= n+1$).

The eigenmatrix $\boldsymbol{M}_{n-1}$ can be expressed by Eq. 50. Taking the moment relation given by Eq. 49 into account, the extreme right three moment equation is written in the form

$$\begin{aligned}
&\lfloor \beta_{n-1},\ \gamma_{n-1} + \gamma_n - \frac{\beta_n^2}{\gamma_n} \rfloor \boldsymbol{M}_{n-1} \\
&= \left\lfloor \delta_{n-1},\ -\delta_{n-1} - \delta_n\left(1 + \frac{\beta_n}{\gamma_n}\right),\ \delta_n\left(1 + \frac{\beta_n}{\gamma_n}\right) \right\rfloor \{W_{n-1} \quad W_n \quad W_{n+1}\} \\
&\quad + \left(\frac{L^2}{\alpha^3 EI}\right)_{n-1} \lfloor 1 \quad 0 \quad \alpha\csc\alpha \quad 0 \rfloor_{n-1}\boldsymbol{K}_{n-1} \\
&\quad + \left(\frac{L^2}{\alpha^3 EI}\right)_n \left\lfloor -\left(1 + \frac{\beta}{\gamma}\right),\ -\alpha,\ -\alpha\left(\frac{\beta}{\gamma}\csc\alpha + \cot\alpha\right),\ -\alpha \right\rfloor_n \boldsymbol{K}_n.
\end{aligned} \tag{64}$$

Substitution from Eq. 50 into the above equation yields

$$\mathbf{B}'\mathbf{L}_{n-1}\mathbf{B}\mathfrak{M}_2 + \lfloor \mathbf{S} \rfloor'_{n+1}\{\mathbf{W}\}'_{n+1} + \lfloor \mathbf{T} \rfloor'_n\{\mathbf{K}\}_n = 0, \tag{65}$$

in which

$$\mathbf{B}' = \lfloor \beta_{n-1}, \ \gamma_{n-1} + \gamma_n - \frac{\beta_n^2}{\gamma_n} \rfloor, \tag{66}$$

$$\lfloor \mathbf{S} \rfloor'_{n+1}\{\mathbf{W}\}'_{n+1} = \mathbf{B}'\lfloor \mathbf{S} \rfloor_{n-1}\{\mathbf{W}\}_{n-1} + \left\lfloor -\delta_{n-1}, \ \delta_{n-1} + \delta_n\left(1 + \frac{\beta_n}{\gamma_n}\right), -\delta_n\left(1 + \frac{\beta_n}{\gamma_n}\right) \right\rfloor \{W_{n-1} \ \ W_n \ \ W_{n+1}\}, \tag{67}$$

$$\lfloor \mathbf{T} \rfloor'_n\{\mathbf{K}\}_n = \mathbf{B}'\lfloor \mathbf{T} \rfloor_{n-1}\{\mathbf{K}\}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_{n-1} \lfloor -1 \quad 0 \quad -\alpha \csc\alpha \quad 0 \rfloor_{n-1}\mathbf{K}_{n-1} + \left(\frac{L^2}{\alpha^3 EI}\right)_n \left\lfloor 1 + \frac{\beta}{\gamma}, \ \alpha, \ \alpha\left(\frac{\beta}{\gamma}\csc\alpha + \cot\alpha\right), \ \alpha \right\rfloor_n \mathbf{K}_n. \tag{68}$$

From Eq. 65, the final equation for determining the current element $\mathfrak{M}_2$ is obtained as follows:

$$\mathfrak{M}_2 = -[\mathbf{B}'\mathbf{L}_{n-1}\mathbf{B}]^{-1}\left[\lfloor \mathbf{S} \rfloor'_{n+1}\{\mathbf{W}\}'_{n+1} + \lfloor \mathbf{T} \rfloor'_n\{\mathbf{K}\}_n \right]. \tag{69}$$

6.5. Critical Load.

The critical value of compressive force in the continuous long-column is obtained from the condition that the determinant of the right side of Eq. 52, 58, 63, or 69 must vanish, i. e.,

$$|\mathbf{B}'\mathbf{L}_{n-1}\mathbf{B}| = 0. \tag{70}$$

7. NUMERICAL EXAMPLES

7.1. Example 1.

Fig. 5 shows a bar with length L and flexural rigidity EI, which is subjecetd to the compressive force P at both extreme ends. In this example, the bar is laid on several simple supports arranged with equal intervals. For each case, the

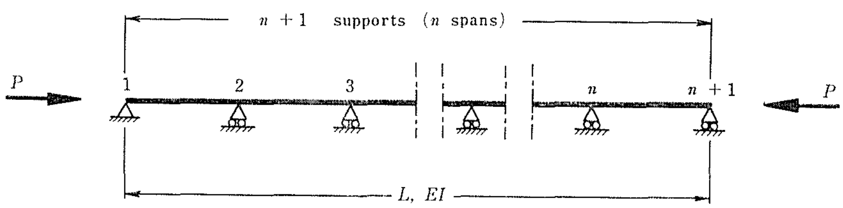
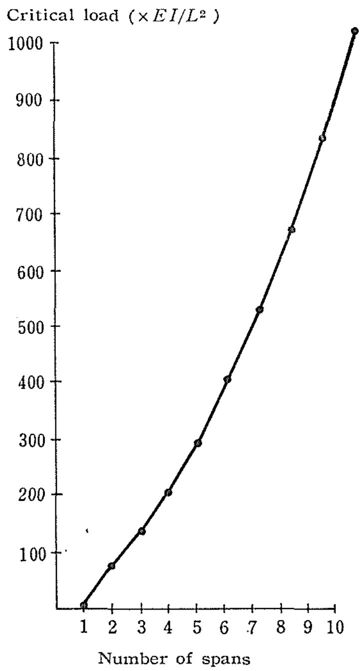


Fig. 5. Long-Column on Simple Supports.

critical value of the compressive force of the bar is evaluated and summarized in Table 2. With the increase of the number of intermediate supports, the critical value increases very rapidly, which is also shown graphically.

Table 2. Critical Load.

Number of spans	Number of supports	Critical load (× EI/L^2)
1	2	9.87 [6)]
2	3	80.8 [6)]
3	4	134
4	5	204
5	6	294
6	7	403
7	8	532
8	9	680
9	10	848
10	11	1035



7.2. Example 2.[7)]

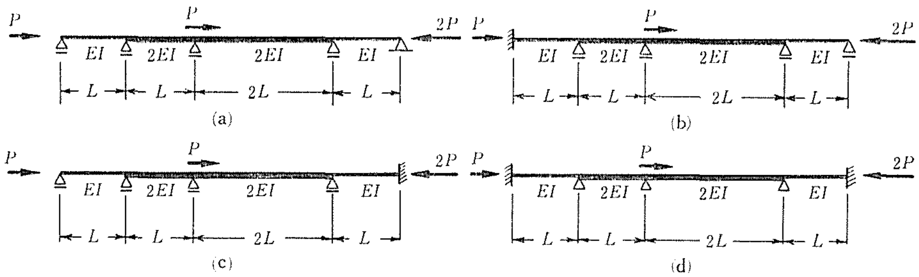


Fig. 6. Four Span Continuous Long-Columns.

In Figs. 6 are illustrated four span continuous long-columns of various end support conditions. The geometrical dimensions and the loading conditions are also shown in the figure. The critical loads for respective cases are evaluated and summarized in Table 3, wherein the first, second, and third eigenvalues are given, of which only the smallest compressive forces are of practical importance.

Table 3. Critical Load ($\times EI/L^2$).

	Case (a)	Case (b)	Case (c)	Case (d)
I	4.266	4.286	5.052	5.087
II	8.556	8.641	10.933	11.197
III	12.943	13.471	15.364	17.563

8. CONCLUSIONS

In conclusion, the following notes are given:

1. The composition of prevailing bending theories for continuous beam-columns can be commented from the viewpoint of the operational method, which begins with Eq. 3.

2. The key equations to respective theories are obtained from due preliminary treatments of the eigenmatrix. The form of such key equations leads to a complexity in matrix elements. The prevailing key equations to the slope-deflection method confine themselves the scope of their utilization because of the preliminary treatments.

3. The operational procedure can also be extended to other theories. An application to the three-moment theory has for example been shown, in which the analysis can be carried out systematically by matrix algebra dispensing with simultaneous equations.

4. By the recurrent use of the shift formulas, all the support moments of the continuous beam-column can be expressed by an arbitrary set of two consecutive support moments, which is referred to as the current-matrix.

5. In the statical equilibrium problems, both extreme boundary conditions are used for determination of the current-matrix. On the other hand, to find the critical load of the continuous column, the eigenvalue equation is that the determinant constructed by the coefficient matrix of the final equation vanishes.

6. The recurrent procedure shown in this paper is a subsidiary operational methods. Comparing it with the orthodox operational method, it requires considerable amount of computations, besides the elements in the operators become complicated.

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APPENDIX. —NOTATION

The following symbols are adopted for use in this paper:

A, B, C, D = elements in eigenmatrix, see Eqs. 2 and 3;

a, b, c, d, e, f, g = constants, see Eqs. 25, 26, and 27;

$\mathbf{B}$ = boundary matrix at extreme left end, see Eqs. 43 and 44;

$\mathbf{B}'$ = boundary matrix at extreme right end, see Eqs. 51, 55, 60 and 66;

$\mathbf{D}_r$ = 2-by-2 shift operator, see Eq. 39;

E = Young's modulus;

I = moment of inertia;

i = even integer representing the support or span order;

$\mathbf{K}$ = load-matrix, see Eqs. 7, 8, and 9;

$\{\mathbf{K}\}_r$ = load-matrix assembalge, see Eq. 41;

L = span length;

$\mathbf{L}_r$ = integrated shift operator for support moment, see Eq. 41

M = bending moment;

$\mathfrak{M}$ = external concentrated moment in load-matrix, or support moment in three-moment method;

$\mathbf{M}_r = \{\mathfrak{M}_r \ \ \mathfrak{M}_{r+1}\}$ = eigenmatrix in the three-moment method, see Eq. 37;

n = even integer;

P = axial compressive force;

$\mathbf{P}_r$ = 2-by-4 matrix for load-matrix, see Eq. 39;

Q = intensity of concentrated load;

$\mathbf{Q}_r$ = 2-by-4 matrix for load-matrix, see Eq. 39;

q = intensity of distributed load;
$\mathbf{R}_r$ = 2-by-4 matrix for load-matrix, see Eq. 39;
r = integer representing the support or span order;
S = shearing force;
$\lfloor \mathbf{S} \rfloor_r$ = integrated feed operator for support settlement, see Eq. 41;
$\lfloor \mathbf{S} \rfloor'_r$ = integrated feed operator of one row, see Eqs. 56, 61 and 67;
$\lfloor \mathbf{T}_r \rfloor$ = integrated feed operator for load-matrix, see Eq. 41;
$\lfloor \mathbf{T} \rfloor'_r$ = integrated feed operator of one row, see Eqs. 57, 62 and 68;
$\mathbf{U}_r$ = 2-by-2 matrix for support settlement, see Eq. 39;
$\mathbf{V}_r$ = 2-by-2 matrix for support settlement, see Eq. 39;
W = support settlement;
$\mathbf{W}_r = \{W_r \quad W_{r+1}\}$ = support settlement matrix, see Eqs. 37;
$\{\mathbf{W}\}_r$ = support settlement assemblage, see Eq. 41;
w = deflection;
$\alpha = \sqrt{PL^2/EI}$;
$\beta = (L/\alpha^2 EI)(1 - \csc\alpha)$;
$\gamma = (L/\alpha^2 EI)(\alpha\cot\alpha - 1)$;
$\delta = 1/L$;
Θ = support slope;
θ = slope angle;
κ = non-dimensional load abscissa;
ρ = non-dimensional current abscissa;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.