Operational Method for Structural Networks
First Report
Plane Systems with Rigid Connection

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1. SYNOPSIS

As a further development of the operational method for continuous beams, [1] the authors extended it to the analysis of the plane network systems such as plane rigid frames and grid frames. In the present paper, a structure is divided into several constituent units. [2] Defining the perfectly classified displacement and force vectors, and treating all physical conditions consisting of equilibrium and compatibility at nodal points common between two consecutive units, a recurrence formula is obtained. Then the analysis can be simplified in both philosophy and computation.

Several formulas are presented in generalized forms.

2. INTRODUCTION

Notation.—The symbols adopted for use in this paper are defined where they first appear and are listed alphabetically in the Appendix.

The physical behavior of a constituent member of a plane network system is governed by the following differential equations:

1. for the flexural behavior:

$$\frac{d^2w}{dx^2}=-\frac{M}{EI}, \text{ or } \frac{d^4w}{dx^4}=0, \tag{1}$$

2. for the extensional behavior:

$$\frac{du}{dx}=\frac{F}{EA}, \text{ or } \frac{d^2u}{dx^2}=0, \tag{2}$$

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3. for the torsional behavior:

$$\frac{d\phi}{dx}=\frac{T}{GJ}, \text{ or } \frac{d^2\phi}{dx^2}=0. \tag{3}$$

Here $w=$ the flexural deflection, $x=$ the current abscissa, $u=$ the axial displacement, $\phi=$ the angle of torsion, $F=$ the axial force, $T=$ the torsional moment, $EI=$ the flexural rigidity, $EA=$ the extensional rigidity, and GJ $=$ the torsional rigidity.

The behavior of each constituent member can be represented by combination of Eq. 2 or 3 with Eq. 1. Then the constituent member has six degrees of freedom composed of integration constants of basic differential equations. They are arranged in the 6-by-1 column vector and are designated as the "eigenmatrix" of the member.

The physical quantities of a constituent member can be completely represented by the eigenmatrix which should be determined so as to satisfy the given boundary conditions.

In accordance with the substantial difference in respective physical quantities, two kinds of state vectors, the displacement and force vectors, are to be defined. Each of them consists of a 3-by-1 column vector of physical quantities, or of the product form of a third-order diagonal matrix, a 3-by-6 abscissa matrix, and the 6-by-1 eigenmatrix. The assemblage of the two state vectors above has the complete correspondence to the member eigenmatrix.

The connection chart at a nodal point can be composed of treating all compatibility conditions of displacement vectors and the equilibrium conditions of force vectors.

By suitable selection of several members in a structural system, the constituent unit is defined. Treating the physical conditions at the nodal points between two consecutive units, the recurrence formula for these unit-eigenmatrices can be obtained, in which only an inverse of small size is necessary.

The first unit-eigenmatrix will be reduced to a half size column vector in virtue of the boundary conditions of the first unit, and the reduced column vector will become current to all the units or the entire structural system by the repeated use of the recurrence formula.

In the last step of analysis, the current-matrix can be determined by the boundary conditions at the last unit where an inverse of moderate size is required.

3. BASIC CONCEPTS

The plane network systems with rigid connection can be classified into two kinds: one being the rigid frames and the other the grid frames. Eqs. 1 and 2 are referred to the rigid frames, while Eqs. 1 and 3 to the grid frames. The behavior of a constituent member can be completed by those equations respectively. The physical quantities to be treated in the analysis of plane network systems are classified as shown in Table 1.

Table 1. Physical Quantities of Plane Network Systems.

	Displacement Factor			Force Factor		
Rigid Frames	u	w	θ	F	S	M
Grid Frames	ϕ	θ	w	M	T	S

Notation:

u = axial displacement, F = axial force,
w = flexural deflection, S = shearing force,
θ = angle of deflection, M = bending moment,
ϕ = angle of torsion, T = torsional moment.

The respective quantities in Table 1 can be given by the equations shown in Table 2.

Table 2. Basic Equations for Plane Network Systems.

$u=\frac{l}{EA}\lfloor 1 \;\; \rho \rfloor \boldsymbol{M}$	$F=\frac{EA}{l}\frac{du}{d\rho}$
$w=\frac{l^3}{6EI}\lfloor 1 \;\; \rho \;\; \rho^2 \;\; \rho^3 \rfloor \boldsymbol{N}$	$S=-\frac{EI}{l^3}\frac{d^3w}{d\rho^3}$
$\theta=\frac{1}{l}\frac{dw}{d\rho}$	$M=-\frac{EI}{l^2}\frac{d^2w}{d\rho^2}$
$\phi=\frac{l^2}{GJ}\lfloor 1 \;\; \rho \rfloor \boldsymbol{M}$	$T=\frac{GJ}{l}\frac{d\phi}{d\rho}$

Notation:

EI = flexural rigidity, GJ = torsional rigidity,
EA = extensional rigidity, l = member length,
ρ = dimensionless current abscissa $=\frac{x}{l}$,
$\boldsymbol{M}$ = integral constants arranged in 2-by-1 column vector,
$\boldsymbol{N}$ = integral constants arranged in 4-by-1 column vector.

Combining the column vector $\boldsymbol{M}$ with another vector $\boldsymbol{N}$, the following 6-by-1 eigenmatrix of the constituent member can be composed:

$$\boldsymbol{X} = \{\boldsymbol{M} \quad \boldsymbol{N}\} = \{a \quad b \quad c \quad d \quad e \quad f\}. \tag{4}$$

In virtue of substantial difference in physical characteristics, the physical quantities are classified into two kinds of vectors, the generalized displacement vector $\boldsymbol{U}(\rho)$, and the generalized force vector $\boldsymbol{V}(\rho)$, or simply the "displacement vector" and the "force vector."

They are defined as follows:

1. for the rigid frames: $\boldsymbol{U}(\rho) = \{u \quad w \quad \theta\}_\rho, \qquad \boldsymbol{V}(\rho) = \{F \quad S \quad M\}_\rho,$ (5)
2. for the grid frames: $\boldsymbol{U}(\rho) = \{\phi \quad \theta \quad w\}_\rho, \qquad \boldsymbol{V}(\rho) = \{M \quad T \quad S\}_\rho.$ (6)

These vectors can be decomposed into the product forms

$$\boldsymbol{U}(\rho) = \boldsymbol{D}\boldsymbol{P}(\rho)\boldsymbol{X}, \qquad \boldsymbol{V}(\rho) = \boldsymbol{\Delta}\boldsymbol{Q}(\rho)\boldsymbol{X}, \tag{7}$$

in which the respective matrices on the right sides are given as follows:

1. diagonal matrices:

(i) for the rigid frames:

$$\boldsymbol{D} = \mathrm{diag}\begin{bmatrix} \frac{l}{EA} & \overline{6EI} & \frac{l^2}{6EI} \end{bmatrix}, \qquad \boldsymbol{\Delta} = \mathrm{diag}\begin{bmatrix} 1 & -1 & -\frac{l}{3} \end{bmatrix}, \tag{8}$$

(ii) for the grid frames:

$$\boldsymbol{D} = \mathrm{diag}\begin{bmatrix} \frac{l^2}{GJ} & \frac{l^2}{6EI} & \frac{l^3}{6EI} \end{bmatrix}, \qquad \boldsymbol{\Delta} = \mathrm{diag}\begin{bmatrix} -\frac{l}{3} & l & -1 \end{bmatrix}, \tag{9}$$

2. abscissa matrices:

(i) for the rigid frames:

$$\boldsymbol{P}(\rho) = \begin{bmatrix} 1 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \rho & \rho^2 & \rho^3 \\ 0 & 0 & 0 & 1 & 2\rho & 3\rho^2 \end{bmatrix}, \qquad \boldsymbol{Q}(\rho) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3\rho \end{bmatrix}, \tag{10}$$

(ii) for the grid frames:

$$\boldsymbol{P}(\rho) = \begin{bmatrix} 1 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\rho & 3\rho^2 \\ 0 & 0 & 1 & \rho & \rho^2 & \rho^3 \end{bmatrix}, \qquad \boldsymbol{Q}(\rho) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3\rho \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

The complete state vector $\boldsymbol{W}(\rho)$ of the constituent member is given by

$$\boldsymbol{W}(\rho) = \begin{bmatrix} \boldsymbol{U}(\rho) \\ \boldsymbol{V}(\rho) \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{P}(\rho) \\ \boldsymbol{Q}(\rho) \end{bmatrix} \boldsymbol{X}. \tag{12}$$

The values of displacement and force vectors at both member ends will be referred to, and they are denoted by

$$\begin{bmatrix} \boldsymbol{U}(0) \\ \boldsymbol{V}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{P}(0) \\ \boldsymbol{Q}(0) \end{bmatrix} \boldsymbol{X}, \quad \text{or} \quad \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix} \boldsymbol{X}, \tag{13}$$

and

$$\begin{bmatrix} \boldsymbol{U}(l) \\ \boldsymbol{V}(l) \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{P}(l) \\ \boldsymbol{Q}(l) \end{bmatrix} \boldsymbol{X}', \quad \text{or} \quad \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}' = \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}' \boldsymbol{X}', \tag{14}$$

in which

$$\boldsymbol{X}' = \boldsymbol{X} + \boldsymbol{K}. \tag{15}$$

Here $\boldsymbol{K}$ represents the load term of the member which is given by the formula:

$$\boldsymbol{K} = \sum (\text{load-matrices on the member}). \tag{16}$$

4. LOAD-MATRICES

Definition.—The terminology "domain" is meant by a portion cut out from a member at loaded points. The subscript i is referred to the i-th intermediate domain. But the first extreme left domain will have no subscript nor superscript, and the last extreme right domain will be primed.

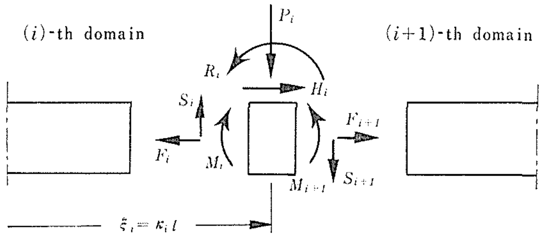


Fig. 1. Loaded Point of Rigid Frames.

4.1. Rigid Frames.

In Fig. 1 is shown a small portion of the i-th loaded point of a rigid frame. The physical conditions at this point are given as follows:

1. compatibility conditions: $\boldsymbol{U}_{i+1} - \boldsymbol{U}_i = 0,$ (17)
2. equilibrium conditions: $\boldsymbol{V}_{i+1} - \boldsymbol{V}_i = \boldsymbol{k}_i,$ (18)

in which $\boldsymbol{k}_i$ = the "load-vector" given by

$$\boldsymbol{k}_i = -\{H \quad P \quad R\}_i. \tag{19}$$

Using the complete state vector, Eqs. 17 and 18 are written in the assembled form:

$$\boldsymbol{W}_{i+1} - \boldsymbol{W}_i = \{\boldsymbol{0} \quad \boldsymbol{k}\}_i = \{0 \quad 0 \quad 0 \quad -H \quad -P \quad -R\}_i, \tag{20}$$

or

$$\text{diag}\begin{bmatrix} \frac{l}{EA} & \frac{l^3}{6EI} & \frac{l^2}{6EI} & 1 & -1 & -\frac{l}{3} \end{bmatrix}_i \begin{bmatrix} \boldsymbol{P}(\kappa) \\ \boldsymbol{Q}(\kappa) \end{bmatrix}_i [\boldsymbol{X}_{i+1} - \boldsymbol{X}_i]$$

$$= \{0 \quad 0 \quad 0 \quad -H \quad -P \quad -R\}_i. \tag{21}$$

From Eq. 21, we obtain the formula at the loaded point

$$\boldsymbol{X}_{i+1} = \boldsymbol{X}_i + \boldsymbol{K}_i. \tag{22}$$

Here $\boldsymbol{K}_i$ = the "load-matrix" at the i-th loaded point. Its derivation is given in the following:

$$\boldsymbol{K}_i = \boldsymbol{X}_{i+1} - \boldsymbol{X}_i = \begin{bmatrix} \boldsymbol{P}(\kappa) \\ \boldsymbol{Q}(\kappa) \end{bmatrix}_i^{-1} \text{diag}[\boldsymbol{D} \quad \varDelta]_i^{-1} \{0 \quad 0 \quad 0 \quad -H \quad -P \quad -R\}_i$$

$$= \begin{bmatrix} 1 & 0 & 0 & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\kappa & 0 & -\kappa^3 & \kappa^2 \\ 0 & 0 & 1 & 0 & 3\kappa^2 & -2\kappa \\ 0 & 0 & 0 & 0 & -3\kappa & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_i \begin{bmatrix} 0 \\ 0 \\ 0 \\ -H \\ P \\ \frac{3}{l}R \end{bmatrix}_i = \begin{bmatrix} H\kappa \\ -H \\ -P\kappa^3 + \frac{3}{l}R\kappa^2 \\ 3P\kappa^2 - \frac{6}{l}R\kappa \\ -3P\kappa + \frac{3}{l}R \\ P \end{bmatrix}_i. \tag{23}$$

4.2. Grid Frames.

Fig. 2 shows a small portion of a grid frame subjected to external loads. In this case, the following physical conditions must be satisfied:

1. compatibility conditions: $\boldsymbol{U}_{i+1} - \boldsymbol{U}_i = 0,$ (24)

2. equilibrium conditions: $\boldsymbol{V}_{i+1} - \boldsymbol{V}_i = \boldsymbol{k}_i,$ (25)

in which the load-vector is given by

$$\boldsymbol{k}_i = -\{R \quad Q \quad P\}_i. \tag{26}$$

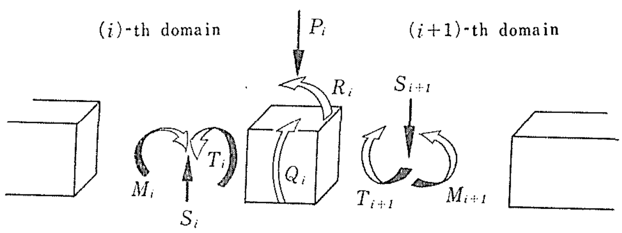


Fig. 2. Loaded Point of Grid Frams.

Using the complete state vector, Eqs. 24 and 25 are consolidated as follows:

$$\boldsymbol{W}_{i+1} - \boldsymbol{W}_i = \{\boldsymbol{0} \;\; \boldsymbol{k}\}_i = \{0 \quad 0 \quad 0 \quad -R \quad -Q \quad -P\}_i, \tag{27}$$

or

$$\mathrm{diag}\left[\frac{l^2}{GJ} \; \frac{l^2}{6EI} \; \frac{l^3}{6EI} \; -\frac{l}{3} \; l \; -1\right]_i \begin{bmatrix} 1 & \kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\kappa & 3\kappa^2 \\ 0 & 0 & 1 & \kappa & \kappa^2 & \kappa^3 \\ 0 & 0 & 0 & 0 & 1 & 3\kappa \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_i [\boldsymbol{X}_{i+1} - \boldsymbol{X}_i] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R \\ -Q \\ -P \end{bmatrix}_i. \tag{28}$$

Eq. 28 also results in Eq. 22. In this case, the load-matrix is given by

$$\boldsymbol{K}_i = \begin{bmatrix} \frac{1}{l}Q\kappa \\ -\frac{1}{l}Q \\ -P\kappa^3 + \frac{3}{l}R\kappa^2 \\ 3P\kappa^2 - \frac{6}{l}R\kappa \\ -3P\kappa + \frac{3}{l}R \\ P \end{bmatrix}_i . \tag{29}$$

5. CONNECTION CONDITIONS AT NODAL POINT

A nodal point at which four members intersect with an arbitrary angle is to be treated.

The symbols adopted for use at this point are shown in Table 3. In this table, the following definitions are given:

1. The subscript "h" denotes horizontal-like members, and "v" vertical-like members.
2. Each symbol is attached with prime (′) at end $\rho = 1$, while no superscript is added at end $\rho = 0$.
3. The physical quantity attached with vinculum (¯) denotes its projection into the orthogonal coordinates.

The projection formula for physical quantities into the global coordinates is given by

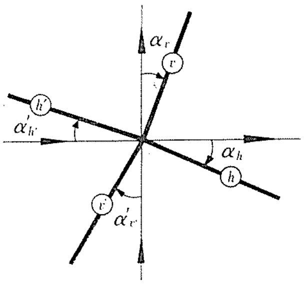
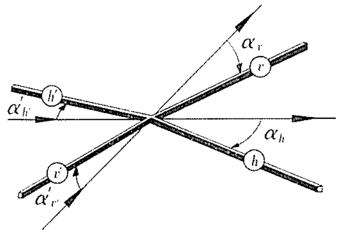
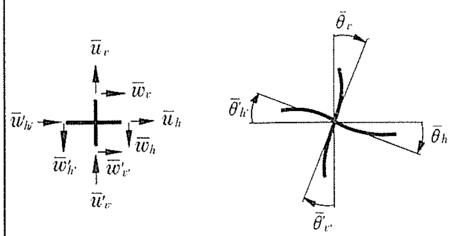
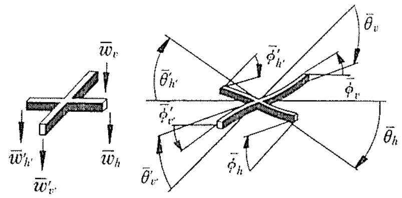
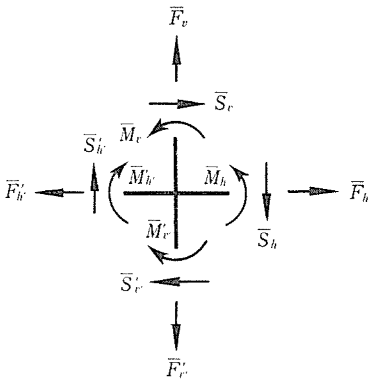
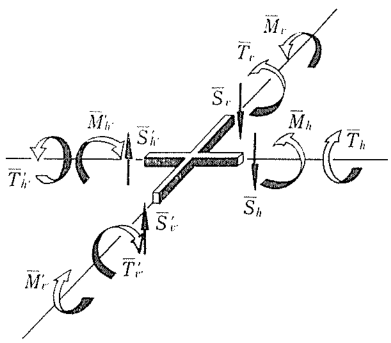
$$\overline{\boldsymbol{U}} = \boldsymbol{pU}, \qquad \overline{\boldsymbol{V}} = \boldsymbol{pV}, \qquad \overline{\boldsymbol{W}} = \mathrm{diag}\,[\,\boldsymbol{p} \quad \boldsymbol{p}\,]\boldsymbol{W}, \tag{30}$$

in which $\boldsymbol{p}$ = the projection matrix or briefly the "projector" given by

$$\boldsymbol{p} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{31}$$

The inverse of Eq. 31 can be readily given by its transpose. In Eq. 30, the following quantities should be used:

Table 3. Symbols at Nodal Point.

Rigid Frames	Grid Frames
Member symbol, abscissa, and angle	
	
Displacements on global coordinate	
	
Forces on global coordinate	
	

1. for the rigid frames:

$$\bar{\mathbf{U}} = \{\bar{u} \quad \bar{w} \quad \bar{\theta}\}, \qquad \mathbf{U} = \{u \quad w \quad \theta\}, \tag{32}$$

$$\bar{\mathbf{V}} = \{\bar{F} \quad \bar{S} \quad \bar{M}\}, \qquad \mathbf{V} = \{F \quad S \quad M\}, \tag{33}$$

2. for the grid frames:

$$\bar{\mathbf{U}} = \{\bar{\phi} \quad \bar{\theta} \quad \bar{w}\}, \qquad \mathbf{U} = \{\phi \quad \theta \quad w\}, \tag{34}$$

$$\bar{\mathbf{V}} = \{\bar{M} \quad \bar{T} \quad \bar{S}\}, \qquad \mathbf{V} = \{M \quad T \quad S\}. \tag{35}$$

The compatibility conditions of the displacement vectors at the nodal point become

$$\bar{\mathbf{U}}'_{h'} = \bar{\mathbf{U}}_h = \mathbf{R}\bar{\mathbf{U}}'_{v'} = \mathbf{R}\bar{\mathbf{U}}_v, \tag{36}$$

in which $\mathbf{R}$ = the "rearrangement matrix" given by

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

The equilibrium conditions of the force vectors at the nodal point become

$$\bar{\mathbf{V}}'_{h'} - \bar{\mathbf{V}}_h = -\mathbf{R}[\bar{\mathbf{V}}'_{v'} - \bar{\mathbf{V}}_v]. \tag{38}$$

Substitution from Eqs. 30 into Eqs. 36 and 38 yields

$$\mathbf{p}\mathbf{U}'_{h'} = \mathbf{p}\mathbf{U}_h = \mathbf{R}\mathbf{p}\mathbf{U}'_{v'} = \mathbf{R}\mathbf{p}\mathbf{U}_v, \tag{39}$$

$$\lfloor \mathbf{p} \quad -\mathbf{p} \rfloor \{\mathbf{V}'_{h'} \quad \mathbf{V}_h\} = -\mathbf{R}\lfloor \mathbf{p} \quad -\mathbf{p} \rfloor \{\mathbf{V}'_{v'} \quad \mathbf{V}_v\}. \tag{40}$$

From Eq. 39,

$$\begin{bmatrix} \mathbf{U}'_{h'} \\ \mathbf{U}_h \\ \mathbf{U}'_{v'} \\ \mathbf{U}_v \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{h'h}\mathbf{U}_h \\ \mathbf{L}_{hh'}\mathbf{U}'_{h'} \\ \mathbf{L}_{v'h'}\mathbf{U}'_{h'} \\ \mathbf{L}_{vh'}\mathbf{U}'_{h'} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{h'v'}\mathbf{U}'_{v'} \\ \mathbf{L}_{hv'}\mathbf{U}'_{v'} \\ \mathbf{L}_{v'h}\mathbf{U}_h \\ \mathbf{L}_{vh}\mathbf{U}_h \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{h'v}\mathbf{U}_v \\ \mathbf{L}_{hv}\mathbf{U}_v \\ \mathbf{L}_{v'v}\mathbf{U}_v \\ \mathbf{L}_{vv'}\mathbf{U}'_{v'} \end{bmatrix}. \tag{41}$$

From Eq. 40,

$$\begin{bmatrix} \mathbf{V}'_{h'} \\ \mathbf{V}_h \\ \mathbf{V}'_{v'} \\ \mathbf{V}_v \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{L}_{h'h} & -\mathbf{L}_{h'v'} & \mathbf{L}_{h'v} \\ \mathbf{L}_{hh'} & 0 & \mathbf{L}_{hv'} & -\mathbf{L}_{hv} \\ -\mathbf{L}_{v'h'} & \mathbf{L}_{v'h} & 0 & \mathbf{L}_{v'v} \\ \mathbf{L}_{vh'} & -\mathbf{L}_{vh} & \mathbf{L}_{vv'} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}'_{h'} \\ \mathbf{V}_h \\ \mathbf{V}'_{v'} \\ \mathbf{V}_v \end{bmatrix}. \tag{42}$$

In the above equations $\mathbf{L}$ = the "relative operators." They are given in consolidated form as follows:

1. for the rectilinearly arranged members:

write

$$\begin{bmatrix} i \\ i' \end{bmatrix} = \begin{bmatrix} h \\ h' \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} v \\ v' \end{bmatrix}, \tag{43}$$

and then

$$\mathbf{L}_{i'i} = \mathbf{L}_{ii'}^{-1}, \tag{44}$$

$$\mathbf{L}_{i'i} = \begin{bmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \delta = \alpha^i - \alpha_{i'}, \tag{45}$$

and

$$\mathbf{L}_{ii'} = \begin{bmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \delta = \alpha_i - \alpha_{i'}, \tag{46}$$

2. for the orthogonally arranged members:

write

$$\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} h \\ v \end{bmatrix}, \quad \begin{bmatrix} h \\ v' \end{bmatrix}, \quad \begin{bmatrix} h' \\ v \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} h' \\ v' \end{bmatrix}, \tag{47}$$

and then

$$\mathbf{L}_{ij} = \mathbf{L}_{ji}^{-1}, \tag{48}$$

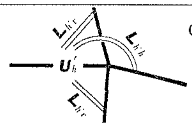
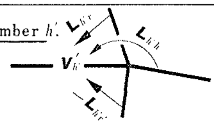
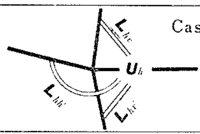
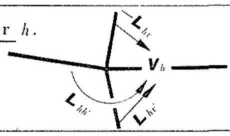
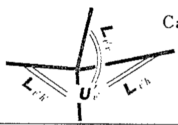
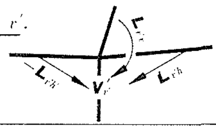
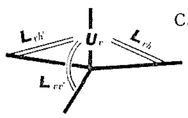
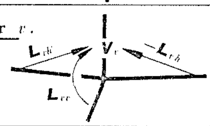
$$\mathbf{L}_{ij} = \begin{bmatrix} \sin\delta & \cos\delta & 0 \\ -\cos\delta & \sin\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \delta = \alpha_j - \alpha_i, \tag{49}$$

and

$$\mathbf{L}_{ji} = \begin{bmatrix} \sin\delta & -\cos\delta & 0 \\ \cos\delta & \sin\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \delta = \alpha_j - \alpha_i. \tag{50}$$

From Eqs. 41 and 42, a connection chart at a nodal point can be composed as shown in Table 4.

Table 4. Connection Chart.

Dislacement vector $\boldsymbol{U}$		Force vector $\boldsymbol{V}$
	Case 1. Connection with member h'.	
	Case 2. Connection with member h.	
	Case 3. Connection with member v'.	
	Case 4. Connection with member v.	

For the orthogonal systems, the preceding equations reduce to the following:

$$\boldsymbol{L}_{i'i} = \boldsymbol{L}_{ii'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{E}, \tag{51}$$

$$\boldsymbol{L}_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{R}, \quad \boldsymbol{L}_{ji} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{R}^{-1}, \tag{52}$$

$$\begin{bmatrix} \boldsymbol{U}'_{h'} \\ \boldsymbol{U}_h \\ \boldsymbol{U}'_{v'} \\ \boldsymbol{U}_v \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_h \\ \boldsymbol{U}'_{h'} \\ \boldsymbol{R}^{-1}\boldsymbol{U}'_{h'} \\ \boldsymbol{R}^{-1}\boldsymbol{U}'_{h'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{U}'_{v'} \\ \boldsymbol{R}\boldsymbol{U}'_{v'} \\ \boldsymbol{R}^{-1}\boldsymbol{U}_h \\ \boldsymbol{R}^{-1}\boldsymbol{U}_h \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}\boldsymbol{U}_v \\ \boldsymbol{R}\boldsymbol{U}_v \\ \boldsymbol{U}_v \\ \boldsymbol{U}'_{v'} \end{bmatrix}, \tag{53}$$

and

$$\begin{bmatrix} \mathbf{V}'_{h'} \\ \mathbf{V}_h \\ \mathbf{V}'_{v'} \\ \mathbf{V}_v \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{E} & -\mathbf{R} & \mathbf{R} \\ \mathbf{E} & 0 & \mathbf{R} & -\mathbf{R} \\ -\mathbf{R}^{-1} & \mathbf{R}^{-1} & 0 & \mathbf{E} \\ \mathbf{R}^{-1} & -\mathbf{R}^{-1} & \mathbf{E} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}'_{h'} \\ \mathbf{V}_h \\ \mathbf{V}'_{v'} \\ \mathbf{V}_v \end{bmatrix}. \tag{54}$$

6. BOUNDARY CONDITIONS

The generalized boundary conditions[3)] for rigid and grid frames are illustrated in Figs. 3 and 4, where the displacement and force vectors are related with each other by the following equations:

1. for the rigid frames:

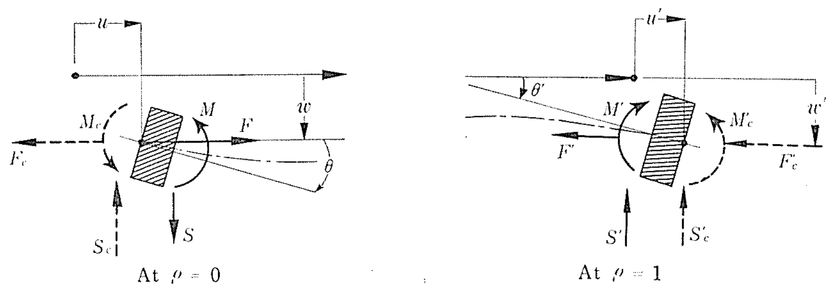


Fig. 3. Boundary Conditions for Rigid Frames.

$$\mathbf{U} = \mathrm{diag}[f \quad k \quad m] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{V}, \tag{55}$$

$$\mathbf{U}' = \mathrm{diag}[f \quad k \quad m] \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{V}', \tag{56}$$

2. for the grid frames:

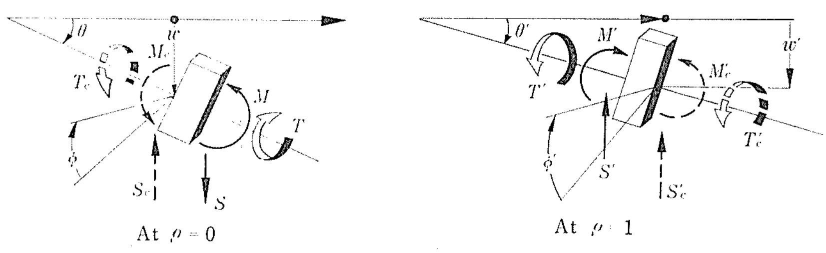


Fig. 4. Boundary Conditions for Grid Frames.

$$\boldsymbol{U} = \text{diag}[\,n \quad m \quad k\,]\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{V}, \tag{57}$$

$$\boldsymbol{U}' = \text{diag}[\,n \quad m \quad k\,]\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{V}'. \tag{58}$$

Here f, k, m, and n = constants between the displacement and force vectors.

By virtue of Eqs. 5 through 11, the above boundary conditions will result in the following consolidated equations:

$$\boldsymbol{BX} = 0 \quad \text{at } \rho = 0, \tag{59}$$

$$\boldsymbol{B'X'} = 0 \quad \text{at } \rho = 1. \tag{60}$$

Here $\boldsymbol{B}$, $\boldsymbol{B}'$ = the "boundary matrices" at $\rho = 0$ and $\rho = 1$, which are given respectively by

$$\boldsymbol{B} = \begin{bmatrix} 1 & -\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 1 & -\mu & 0 \end{bmatrix}, \tag{61}$$

$$\boldsymbol{B}' = \begin{bmatrix} 1 & 1+\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1-\lambda \\ 0 & 0 & 0 & 1 & 2+\mu & 3+3\mu \end{bmatrix}. \tag{62}$$

In the above boundary matrices, the following ratios have been introduced:

$$\lambda = \frac{6EI}{l^3}k, \qquad \mu = \frac{2EI}{l}m, \tag{63}$$

and

$$\left.\begin{aligned} \nu &= \frac{EA}{l}f \quad \text{for the rigid frames,} \\ \nu &= \frac{GJ}{l}n \quad \text{for the grid frames.} \end{aligned}\right\} \tag{64}$$

All the possible boundary conditions will be expressible by assigning due values for the above elastic constants. A simple example thereof is shown in Table 5.

Table 5. Support Constants.

	f, n, or ν	k or λ	m or μ
Elastic support	f, n, ν	k, λ	m, μ
Fixed end	0	0	0
Free end	∞	∞	∞
Simple support	0	0	∞

7. APPLICATION

As a simple application of the preceding developments, an illustrative example will be referred to the portal frame shown in Fig. 5.

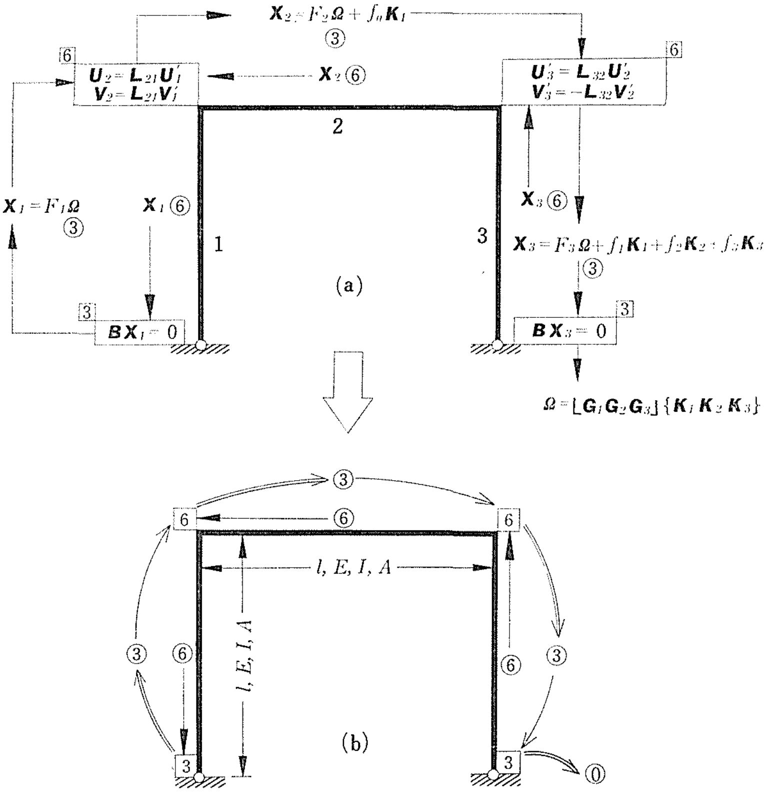


Fig. 5. Operation Chart for Portal Frame.

7.1. Operation Chart.

A computational procedure of the portal frame is illustrated in Fig. 5a. Starting from the lower end of the left vertical member, the operation will be carried out. The working out of the current-matrix $\boldsymbol{\Omega}$, its currency, and its determination are shown symbolically in Figs. 5a and 5b. In both figures are also shown the degradation of the order of unknown eigenmatrices. Numbers in small circles denote the order of member-eigenmatrices, while those in small squares represent the number of physical conditions.

Respective members are numbered as shown in Fig. 5a. Their state vectors are given by the following consolidated forms:

$$\mathbf{U}(\rho)_j = \mathbf{D}_j\mathbf{P}(\rho)\mathbf{X}(\rho)_j, \tag{65}$$

$$\mathbf{V}(\rho)_j = \boldsymbol{\Delta}_j\mathbf{Q}(\rho)\mathbf{X}(\rho)_j, \tag{66}$$

$$\mathbf{D}_j = \mathrm{diag}\begin{bmatrix} \frac{l}{EA} & \frac{l^3}{6EI} & \frac{l^2}{6EI} \end{bmatrix}_j, \tag{67}$$

$$\boldsymbol{\Delta}_j = \mathrm{diag}\begin{bmatrix} 1 & -1 & -\frac{l}{3} \end{bmatrix}_j, \tag{68}$$

take at $\rho = 0$:

$$\mathbf{P}(0) = \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{69}$$

$$\mathbf{Q}(0) = \mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{70}$$

$$\mathbf{X}(0)_j = \mathbf{X}_j = \{a \quad b \quad c \quad d \quad e \quad f\}_j, \tag{71}$$

take at $\rho = 1$:

$$\mathbf{P}(1) = \mathbf{P}' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \tag{72}$$

$$\boldsymbol{Q}(1)=\boldsymbol{Q}'=\begin{bmatrix}0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&3\end{bmatrix}, \tag{73}$$

$$\boldsymbol{X}(1)_j=\boldsymbol{X}'_j=\boldsymbol{X}_j+\boldsymbol{K}_j, \tag{74}$$

providing

$$j=1,\ 2,\ 3. \tag{75}$$

7.2. Operators.

In the present example, the following operators are introduced:

(i) relative operators at the upper nodal points:

$$\boldsymbol{L}_{21}=\boldsymbol{R}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}, \tag{76}$$

$$\boldsymbol{L}_{32}=\boldsymbol{R}^{-1}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}, \tag{77}$$

(ii) boundary matrix for the simple support:

$$\boldsymbol{B}=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\end{bmatrix}. \tag{78}$$

7.3. Operations.

1. Support conditions at left lower end.

The boundary conditions at the left lower end are given by

$$\boldsymbol{B}\boldsymbol{X}_1=0. \tag{79}$$

Then $\boldsymbol{X}_1$ becomes

$$\boldsymbol{X}_1=\boldsymbol{S}_1\boldsymbol{\Omega}, \tag{80}$$

in which

$$\boldsymbol{S}_1=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}, \qquad \boldsymbol{\Omega}=\begin{bmatrix}b\\d\\f\end{bmatrix}. \tag{81}$$

Here $\boldsymbol{\Omega}$ = the current-matrix to be shifted to other members.

2. Connection conditions at left node.

At the left nodal point, the connection conditons are given by

$$\boldsymbol{U}_2 = \boldsymbol{L}_{21}\boldsymbol{U}'_1, \quad \boldsymbol{V}_2 = \boldsymbol{L}_{21}\boldsymbol{V}'_1, \tag{82}$$

or

$$\boldsymbol{D}_2\boldsymbol{P}\boldsymbol{X}_2 = \boldsymbol{R}\boldsymbol{D}_1\boldsymbol{P}'[\boldsymbol{X} + \boldsymbol{K}]_1, \tag{83}$$

$$\boldsymbol{\varDelta}_2\boldsymbol{Q}\boldsymbol{X}_2 = \boldsymbol{R}\boldsymbol{\varDelta}_1\boldsymbol{Q}'[\boldsymbol{X} + \boldsymbol{K}]_1. \tag{84}$$

Putting Eqs. 83 and 84 together yields

$$\boldsymbol{X}_2 = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \boldsymbol{\varDelta} \end{bmatrix}_2^{-1} \begin{bmatrix} \boldsymbol{R} & 0 \\ 0 & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \boldsymbol{\varDelta} \end{bmatrix}_1 \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}' [\boldsymbol{X} + \boldsymbol{K}]_1, \tag{85}$$

or

$$\boldsymbol{X}_2 = \boldsymbol{S}_2[\boldsymbol{X} + \boldsymbol{K}]_1 = \boldsymbol{S}_2\boldsymbol{S}_1\boldsymbol{\Omega} + \boldsymbol{S}_2\boldsymbol{K}_1. \tag{86}$$

The physical properties of the three members are assumed for simplicity to be the same, and then

$$\boldsymbol{S}_2 = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \boldsymbol{\varDelta} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{R} & 0 \\ 0 & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \boldsymbol{\varDelta} \end{bmatrix} \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{Q} \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & 0 & \eta & \eta & \eta & \eta \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -\frac{1}{\eta} & -\frac{1}{\eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{87}$$

and

$$\boldsymbol{S}_2\boldsymbol{S}_1 = \begin{bmatrix} 0 & \eta & \eta \\ 0 & 0 & -1 \\ -\frac{1}{\eta} & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}, \tag{88}$$

in which

$$\eta = \frac{Al^2}{6I}. \tag{89}$$

Here $\mathbf{S}_2$ is designated as the shift operator or briefly the "shiftor" at this nodal point, and hence $\mathbf{S}_2\mathbf{S}_1$ may be taken as the shiftor for the current-matrix $\boldsymbol{\Omega}$.

3. Connection conditions at right node.

At the right nodal point, the following connection conditions are to be given:

$$\mathbf{U}'_3 = \mathbf{R}^{-1}\mathbf{U}'_2, \qquad \mathbf{V}'_3 = -\mathbf{R}^{-1}\mathbf{V}'_2. \tag{90}$$

These equations readily yield

$$\mathbf{X}_3 + \mathbf{K}_3 = \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}'^{-1} \begin{bmatrix} \mathbf{D} & 0 \\ 0 & \boldsymbol{\Delta} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}^{-1} & 0 \\ 0 & -\mathbf{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{D} & 0 \\ 0 & \boldsymbol{\Delta} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}' [\mathbf{X} + \mathbf{K}]_2, \tag{91}$$

or

$$\mathbf{X}_3 = \mathbf{S}_3\mathbf{X}_2 + \mathbf{S}_3\mathbf{K}_2 - \mathbf{K}_3, \tag{92}$$

in which the shiftor $\mathbf{S}_3$ is given by

$$\mathbf{S}_3 = \begin{bmatrix} 0 & 0 & -\eta & -\eta & -\eta & 1-\eta \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \frac{1}{\eta} & \frac{1}{\eta}-1 & 0 & -1 & -3 & -6 \\ 0 & 3 & 0 & 1 & 4 & 9 \\ 0 & -3 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{93}$$

4. Support conditions at right lower end.

The boundary conditions at the right lower end are given by

$$\mathbf{B}\mathbf{X}_3 = 0. \tag{94}$$

Substituting from Eq. 92 to Eq. 94, the current-matrix $\boldsymbol{\Omega}$ is obtained as follows:

$$\boldsymbol{\Omega} = -[\mathbf{B}\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1]^{-1}\mathbf{B}[\mathbf{S}_3\mathbf{S}_2 \quad \mathbf{S}_3 \quad -\mathbf{E}] \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \\ \mathbf{K}_3 \end{bmatrix}, \tag{95}$$

or, taking $\mathbf{B} = \mathbf{S}_4$,

$$\boldsymbol{\Omega} = -[\mathbf{S}_4\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1]^{-1}[\mathbf{S}_4\mathbf{S}_3\mathbf{S}_2 \quad \mathbf{S}_4\mathbf{S}_3 \quad -\mathbf{S}_4]\begin{bmatrix}\boldsymbol{K}_1\\ \boldsymbol{K}_2\\ \boldsymbol{K}_3\end{bmatrix}. \tag{96}$$

The inverse in this case becomes

$$[\boldsymbol{B}\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1]^{-1} = [\mathbf{S}_4\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1]^{-1} = \begin{bmatrix}2-\eta & -\eta & -6\eta\\ -6 & 0 & -10-\dfrac{1}{\eta}\\ -3 & 0 & 0\end{bmatrix}^{-1}$$

$$= \frac{1}{3(10\eta+1)}\begin{bmatrix}0 & 0 & -10\eta-1\\ -3\left(10+\dfrac{1}{\eta}\right) & 18\eta & -26\eta-19-\dfrac{2}{\eta}\\ 0 & -3\eta & 6\eta\end{bmatrix}. \tag{97}$$

Then the solution of the current-matrix $\boldsymbol{\Omega}$ is given as follows:

$$\boldsymbol{\Omega} = \frac{1}{3(10\eta+1)}\begin{bmatrix}0 & 0 & 0 & 0 & 10\eta+1 & 0\\ 3\left(10+\dfrac{1}{\eta}\right) & 0 & -18\eta & 0 & 26\eta+19+\dfrac{2}{\eta} & 0\\ 0 & 0 & 3\eta & 0 & -6\eta & 0\end{bmatrix}$$

$$\times\left[\begin{bmatrix}1 & 2-\eta & 0 & -\eta & -3\eta & -6\eta\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & -6 & 1 & 0 & -4 & -10-\dfrac{1}{\eta}\\ 0 & 9 & 0 & 1 & 6 & 12\\ 0 & -3 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 0 & -\eta & -\eta & -\eta & 1-\eta\\ 0 & 0 & 0 & 0 & 0 & -1\\ \dfrac{1}{\eta} & \dfrac{1}{\eta}-1 & 0 & -1 & -3 & -6\\ 0 & 3 & 0 & 1 & 4 & 9\\ 0 & -3 & 0 & 0 & -1 & -3\\ 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}-1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{bmatrix}\right]\begin{bmatrix}\boldsymbol{K}_1\\ \boldsymbol{K}_2\\ \boldsymbol{K}_3\end{bmatrix} = \frac{1}{3(10\eta+1)}\left[\left[\begin{matrix}0\\ 3\left(10+\dfrac{1}{\eta}\right)\\ 0\end{matrix}\right.\right.$$

$$
\left.\begin{matrix}
-3(10\eta+1) & 0 & 0 & -10\eta-1 & 0 \\
0 & -18\eta & -3(10\eta+1) & -44\eta-28-\dfrac{2}{\eta} & 0 \\
0 & 3\eta & 0 & -6\eta & -3(10\eta+1)
\end{matrix}\right],
$$

$$
\begin{bmatrix}
0 & -3(10\eta+1) & 0 & 0 & -10\eta-1 & -3(10\eta+1) \\
-18 & -60\eta-75-\dfrac{6}{\eta} & -3(10\eta+1) & -12\eta-3 & -2\eta-22-\dfrac{2}{\eta} & -3\left(10+\dfrac{1}{\eta}\right) \\
3 & 15\eta+3 & 0 & -3\eta & -3\eta & 0
\end{bmatrix},
$$

$$
\left.\begin{bmatrix}
0 & 0 & 0 & 0 & -10\eta-1 & 0 \\
-3\left(10+\dfrac{1}{\eta}\right) & 0 & 18\eta & 0 & -26\eta-19-\dfrac{2}{\eta} & 0 \\
0 & 0 & -3\eta & 0 & 6\eta & 0
\end{bmatrix}\right]
\begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \\ \boldsymbol{K}_3 \end{bmatrix}, \tag{98}
$$

or

$$
\boldsymbol{\Omega} = \lfloor \boldsymbol{G}_1 \quad \boldsymbol{G}_2 \quad \boldsymbol{G}_3 \rfloor \{\boldsymbol{K}_1 \quad \boldsymbol{K}_2 \quad \boldsymbol{K}_3\}. \tag{99}
$$

In Eq. 99, the matrices $\boldsymbol{G}_j$ $(j=1,2,3)$ correspond to the respective load-matrices $\boldsymbol{K}_j$, and therefore the current-matrix $\boldsymbol{\Omega}$ can be evaluated for arbitrary loading conditions. Thus, the present system can be solved. Note that the $\boldsymbol{G}_j$matrix is determined only from geometrical and physical properties of members. They are designated as the "geometry matrix." Substituting Eq. 98 into Eqs. 80, 86, and 92, and rearranging a little, the complete eigenmatrix of the present portal frame is obtained in the following form:

$$
\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \boldsymbol{X}_3 \end{bmatrix} =
\begin{bmatrix}
\boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{G}_{13} \\
\boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \boldsymbol{G}_{23} \\
\boldsymbol{G}_{31} & \boldsymbol{G}_{32} & \boldsymbol{G}_{33}
\end{bmatrix}
\begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \\ \boldsymbol{K}_3 \end{bmatrix}, \tag{100}
$$

or

$$
[\boldsymbol{X}] = [\boldsymbol{G}]\{\boldsymbol{K}\}. \tag{101}
$$

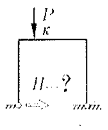
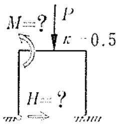
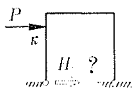
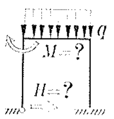
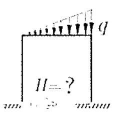
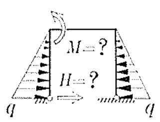
Here, the first factor on the right side of Eq. 100 or 101 is the complete geometry matrix of the system considered, and is given in Table 6.

Using the complete geometry matrix, the physical quantities for given loading conditions can at once be obtained by Eqs. 7, 22, 23, 29, and 100. Some typical examples are illustrated in Table 7, where the solutions by the operational method and by other prevailing methods are shown. In every solution by the latter methods, the effect of the axial displacement of member is neglected as is ordinarily the case, so that it corresponds to the solution by the operational method as a particular case in which $\eta=\infty$.

Table 6. Geometry Matrix of Portal Frame $\left(\times \dfrac{1}{3(10\eta+1)}\right)$.

	K_1	K_2	K_3
G_{1j}	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3(10\eta+1), & 0 & 0 & -10\eta-1, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3\left(10+\frac{1}{\eta}\right), & 0 & -18\eta, & -3(10\eta+1), & -44\eta-28-\frac{2}{\eta}, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\eta, & 0 & -6\eta, & -3(10\eta+1) \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3(10\eta+1), & 0 & 0 & -10\eta-1, & -3(10\eta+1) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -18, & -60\eta-75-\frac{6}{\eta}, & -3(10\eta+1), & -12\eta-3, & -2\eta-22-\frac{2}{\eta}, & -3\left(10+\frac{1}{\eta}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3, & 15\eta+3, & 0 & -3\eta, & -3\eta, & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10\eta-1, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3\left(10+\frac{1}{\eta}\right), & 0 & 18\eta, & 0 & -26\eta-19-\frac{2}{\eta}, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3\eta, & 0 & 6\eta, & 0 \end{bmatrix}$
G_{2j}	$\begin{bmatrix} 3(10\eta+1), & 0 & 3\eta(5\eta+1), & 0 & -20\eta^2-25\eta-2, & 0 \\ 0 & 0 & -3\eta, & 0 & 6\eta, & 0 \\ -3\left(10+\frac{1}{\eta}\right), & 0 & 0 & 0 & 10+\frac{1}{\eta}, & 0 \\ 3\left(10+\frac{1}{\eta}\right), & 0 & -9\eta, & 0 & -2\eta-22-\frac{2}{\eta}, & 0 \\ 0 & 0 & 9\eta, & 0 & 12\eta+3, & 0 \\ 0 & 0 & 0 & 0 & -10\eta-1, & 0 \end{bmatrix}$	$\begin{bmatrix} -15\eta, & -45\eta^2-72\eta-6, & -3\eta(10\eta+1), & -3\eta(5\eta+1), & -5\eta^2-22\eta-2, & -3(10\eta+1) \\ -3, & -3(5\eta+1), & 0 & 3\eta, & 3\eta, & 0 \\ 0 & 3\left(10+\frac{1}{\eta}\right), & 0 & 0 & 10+\frac{1}{\eta}, & 3\left(10+\frac{1}{\eta}\right) \\ -9, & -15\eta-66-\frac{6}{\eta}, & -3(10\eta+1), & -21\eta-3, & -11\eta-22-\frac{2}{\eta}, & -3\left(10+\frac{1}{\eta}\right) \\ 9, & 45\eta+9, & 0 & -9\eta, & -9\eta, & 0 \\ 0 & -3(10\eta+1), & 0 & 0 & -10\eta-1, & -3(10\eta+1) \end{bmatrix}$	$\begin{bmatrix} -3(10\eta+1), & 0 & 15\eta^2, & 0 & -20\eta^2-19-2, & 0 \\ 0 & 0 & 3\eta, & 0 & -6\eta, & 0 \\ 0 & 0 & 0 & 0 & 10+\frac{1}{\eta}, & 0 \\ -3\left(10+\frac{1}{\eta}\right), & 0 & 9\eta, & 0 & -8\eta-19-\frac{2}{\eta}, & 0 \\ 0 & 0 & -9\eta, & 0 & 18\eta, & 0 \\ 0 & 0 & 0 & 0 & -10\eta-1, & 0 \end{bmatrix}$
G_{3j}	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10\eta+1, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3\left(10+\frac{1}{\eta}\right), & 0 & 18\eta, & 0 & -26\eta-19-\frac{2}{\eta}, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3\eta, & 0 & 6\eta, & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3(10\eta+1), & 0 & 0 & 10\eta+1, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 18, & -60\eta-57-\frac{6}{\eta}, & -3(10\eta+1), & -18\eta, & -8\eta-19-\frac{2}{\eta}, & -3\left(10+\frac{1}{\eta}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3, & 15\eta, & 0 & 3\eta, & 3\eta, & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3(10\eta+1), & 0 & 0 & 10\eta+1, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3\left(10+\frac{1}{\eta}\right), & 0 & -18\eta, & -3(10\eta+1), & -44\eta-28-\frac{2}{\eta}, & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\eta, & 0 & -6\eta, & -3(10\eta+1) \end{bmatrix}$

Table 7. Examples of Portal Frame.

Loading conditions and quantities to be required	Solution by operational method	Solution by other methods		
		Solution	Method used	Author
	$H=\frac{3\eta}{10\eta+1}P(\kappa-\kappa^2)$	$H=\frac{3}{10}P(\kappa-\kappa^2)$ $(\eta=\infty)$	Area-moment method	Timoshenko 4)
			Slope-deflection method	Yuki 5)
	$H=\frac{3}{4}\frac{\eta}{10\eta+1}P$	$H=\frac{3}{40}P$ $(\eta=\infty)$	Strain-energy method	Szabó 6)
	$M=-\frac{3}{4}\frac{\eta}{10\eta+1}Pl$	$M=-\frac{3}{40}Pl$ $(\eta=\infty)$	Area-moment method	Timoshenko 4)
	$H=-P$ $\times\frac{3+10\eta-6\eta\kappa+\eta\kappa^3}{10\eta+1}$	$H=-P$ $\times\frac{10-6\kappa+\kappa^3}{10}$ $(\eta=\infty)$	Slope-deflection method	Mogami 7)
	$H=\frac{\eta}{2(10\eta+1)}ql$	$H=\frac{ql}{20}$ $(\eta=\infty)$	Slope-deflection method	Yuki 5)
	$M=-\frac{\eta}{2(10\eta+1)}ql^2$	$M=-\frac{ql^2}{20}$ $(\eta=\infty)$	Strain-energy method	Hayashi 8)
	$H=\frac{\eta}{4(10\eta+1)}ql$	$H=\frac{ql}{40}$ $(\eta=\infty)$	Area-moment method	Timoshenko 4)
	$H=-\frac{31\eta+5}{10(10\eta+1)}ql$	$H=-\frac{31}{100}ql$ $(\eta=\infty)$	Area-moment method	Timoshenko 4)
	$M=-\frac{7\eta-5}{30(10\eta+1)}ql^2$	$M=-\frac{7}{300}ql^2$ $(\eta=\infty)$	Area-moment method	Timoshenko 4)

8. ADDITIONAL NOTES

In addition, the following notes are given:

1. Numbers of physical conditions at a nodal point are classified in Table 8.

Table 8. Nodal Conditions.

Kind of Node	Equilibrium condition	Continuity condition	Connection condition
L-type Node 	3	3	6
T-type Node 	3	$3+3$	9
Cross Node 	3	$3+3+3$	12

2. If the external restraint conditions are given at a nodal point, the corresponding physical conditions will be introduced in place of the given conditions. In consequence of such an interchange of conditions the total number of physical conditions at a nodal point always remains constant. Several illustrative examples are given below.

(i) For free L-type node of a rigid frame (Fig. 6):

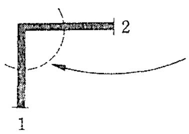


Fig. 6.

$$\left\{\begin{aligned} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}_1' &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}_2, \\ \begin{bmatrix} F \\ S \\ M \end{bmatrix}_1' &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ S \\ M \end{bmatrix}_2. \end{aligned}\right\} \tag{102}$$

(ii) For simply supported L-type node of a rigid frame. (Fig. 7):

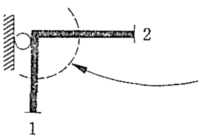


Fig. 7.

$$\left\{\begin{aligned} \begin{bmatrix} u \\ \theta \end{bmatrix}_1' &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \end{bmatrix}_2, \\ \begin{bmatrix} F \\ M \end{bmatrix}_1' &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S \\ M \end{bmatrix}_2, \\ w'_1 = 0,& \qquad u_2 = 0. \end{aligned}\right\} \tag{103}$$

(iii) For elastically supported L-type node of a rigid frame (Fig. 8):

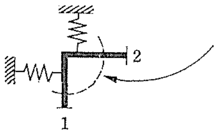


Fig. 8.

$$\begin{bmatrix} u \\ w \\ \theta \end{bmatrix}'_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ \theta \end{bmatrix}_2$$
$$= \begin{bmatrix} -f & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} F \\ S \\ M \end{bmatrix}'_1 + \begin{bmatrix} 0 & f & 0 \\ k & 0 & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{bmatrix} F \\ S \\ M \end{bmatrix}_2 . \qquad (104)$$

(iv) For free T-type node of a grid frame (Fig. 9):

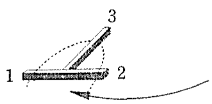


Fig. 9.

$$\left\{ \begin{aligned} &\begin{bmatrix} \phi \\ \theta \\ w \end{bmatrix}'_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ w \end{bmatrix}_3, \\ &\begin{bmatrix} \phi \\ \theta \\ w \end{bmatrix}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ w \end{bmatrix}_3, \\ &\begin{bmatrix} M \\ T \\ S \end{bmatrix}'_1 - \begin{bmatrix} M \\ T \\ S \end{bmatrix}_2 - \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M \\ T \\ S \end{bmatrix}_3 = 0. \end{aligned} \right\} \qquad (105)$$

(v) For elastically supported T-type node of a grid frame (Fig. 10):

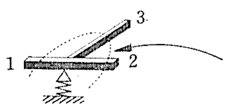


Fig. 10.

$$\left\{ \begin{aligned} &\begin{bmatrix} \phi \\ \theta \end{bmatrix}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix}_3, \\ &\begin{bmatrix} \phi \\ \theta \end{bmatrix}'_1 = \begin{bmatrix} \phi \\ \theta \end{bmatrix}_2, \\ &\begin{bmatrix} M \\ T \end{bmatrix}'_1 - \begin{bmatrix} M \\ T \end{bmatrix}_2 - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} M \\ T \end{bmatrix}_3 = 0, \\ &w'_1 = w_2 = w_4 = k(-S'_1 + S_2 + S_4). \end{aligned} \right\} \qquad (106)$$

3. The number of unknown elements in a given structural system is equal to the number of physical conditions. For example, in the case of the m-span by n-story frame shown in Fig. 11, it follows that

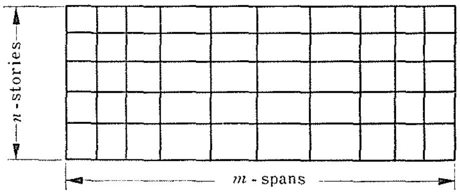


Fig. 11. Plane Frame.

$$\text{Number of constituent member} = (2mn + m + n), \tag{107}$$

$$\text{Number of cross-type node} = (m-1)(n-1), \tag{108}$$

$$\text{Number of T-type node} = 2(m+n-2), \tag{109}$$

$$\text{Number of L-type node} = 4. \tag{110}$$

Then,

$$\text{Number of unknown elements} = 6(2mn + m + n) = 12mn + 6(m+n), \tag{111}$$

$$\text{Number of connection conditions} \\ = 12(m-1)(n-1) + 9 \times 2(m+n-2) + 6 \times 4 = 12mn + 6(m+n). \tag{112}$$

4. At a member end independent of any other members, three boundary conditions are to be prescribed. Then the eigenmatrix of the member is degraded to the third order.

5. Two kinds of compatibility conditions at both ends of a member can completely correspond to the eigenmatrix of the member. Its verification is given as follows:

The compatibility conditions at both ends are written

$$\boldsymbol{DPX} = \boldsymbol{F}, \tag{113}$$

$$\boldsymbol{DP'X'} = \boldsymbol{DP'}[\boldsymbol{X} + \boldsymbol{K}] = \boldsymbol{H}, \tag{114}$$

in which $\boldsymbol{F}$ and $\boldsymbol{H}$ are considered as displacement quantities of some adjacent member. Writing Eqs. 113 and 114 together

$$\begin{bmatrix} \boldsymbol{DP} \\ \boldsymbol{DP'} \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 0 \\ \boldsymbol{DP'} \end{bmatrix} \boldsymbol{K} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{H} \end{bmatrix}. \tag{115}$$

Then the eigenmatrix can be represented by

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{P'} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{D} & 0 \\ 0 & \boldsymbol{D} \end{bmatrix}^{-1} \left[\begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{H} \end{bmatrix} - \begin{bmatrix} 0 \\ \boldsymbol{DP'} \end{bmatrix} \boldsymbol{K} \right]. \tag{116}$$

The above inverses are nonsingular as shown below, and hence the above statement has been verified.

(i) for the rigid frames:

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{P}' \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & 3 & -1 \\ 0 & 2 & 1 & 0 & -2 & 1 \end{bmatrix}, \tag{117}$$

$$\begin{bmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{bmatrix}^{-1} = \mathrm{diag}\left[\frac{EA}{l} \quad \frac{6EI}{l^3} \quad \frac{6EI}{l^2} \quad \frac{EA}{l} \quad \frac{6EI}{l^3} \quad \frac{6EI}{l^2}\right], \tag{118}$$

(ii) for the grid frames:

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{P}' \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 & -1 & 3 \\ 0 & 1 & 2 & 0 & 1 & -2 \end{bmatrix}, \tag{119}$$

$$\begin{bmatrix} \mathbf{D} & 0 \\ 0 & \mathbf{D} \end{bmatrix}^{-1} = \mathrm{diag}\left[\frac{GJ}{l^2} \quad \frac{6EI}{l^2} \quad \frac{6EI}{l^3} \quad \frac{GJ}{l^2} \quad \frac{6EI}{l^2} \quad \frac{6EI}{l^3}\right]. \tag{120}$$

6. The combined abscissa matrix $\{\mathbf{P}(x) \quad \mathbf{Q}(x)\}$, as well as the diagonal

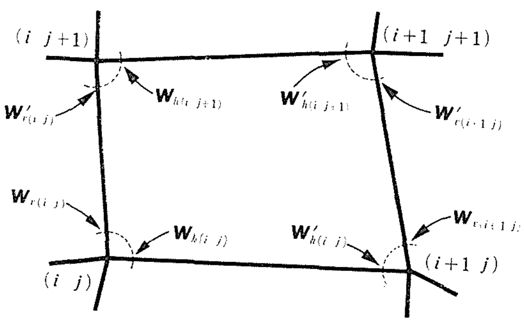


Fig. 12. Symbols for Complete State Vector at Member Ends.

matrix diag [$\boldsymbol{D}$ $\boldsymbol{\Delta}$], is square and nonsingular, and hence the assembled state vector $\boldsymbol{W}(x)$ completely corresponds to the mathematical generality of the eigenmatrix $\boldsymbol{X}$.

7. Conventional symbols for analysis of structure are given in Fig. 12.

9. CONCLUSIONS

In conclusion, the following notes are given:

1. In the present procedure, a network is taken as an assemblage of contistuent units.
2. A unit is characterized by its eigenmatrix, called the unit eigenmatrix.
3. The eigenmatrix of a member consists of integration constants of differential equations governing member behaviors.
4. Two consecutive units produce a recurrence formula for the unit eigenmatrix.
5. The boundary conditions at the left end of the network result in a definite degradation of the first unit eigenmatrix, which is referred to as the current-matrix.
6. The repeated use of the recurrence formula permits the currency of the current-matrix to all the units or the entire network.
7. The boundary conditions at the right end enables to determine the value of the current-matrix.
8. In statical problems, the geometry matrix can be obtained independently of the external loading conditions, which can save time and labor. An illustrative example is given for the portal frame.
9. Both of plane rigid frames and grid frames can be reduced to the same matrix analysis.

REFERENCES

1) N. YOSHIZAWA and B. TANIMOTO, "Operational Method for Continuous Beams, Second Report, Generalized Continuous Beams," Journal of the Faculty of Engineering, Shinshu University, Nagano, Japan, Vol. 20, June, 1966, pp. 31–60.

2) B. TANIMOTO, "Operational Method for Pin-Jointed Trusses," Proceedings of the ASCE, Structural Division, June, 1966, pp. 179–198.

3) B. TANIMOTO, "Some Improvements on Proposed Eigen-Matrix Method," Proceedings of the ASCE, Structural Division, Feb., 1966, pp. 101–119.

4) S. TIMOSHENKO, "Strength of Materials," Part I, Third Edition, D. VAN NOSTRAND Co., Inc., 1955, pp. 190–198.

5) T. YUKI, "Applied Mechanics for Bridges" (in Japanese), Sixth Edition, Rikoh Book Co. Ltd., 1953, pp. 24–31.

6) I. SZABÓ, "Einführung in die Technische Mechanik," Sechste neubearbeitete und erweiterte Auflage, SPRINGER-VERLAG OHG., 1963, pp. 463–464.

7) T. MOGAMI, "Applied Mechanics" (in Japanese), Last Vol., Third Edition, KANEHARA Publishing Co., Ltd., 1955, pp. 153–170.

8) T. HAYASHI, "Theory of Light Structures and its Applications" (in Japanese), First Vol., First Edition, JUSE Publishing Co., Ltd., 1966, pp. 44–45.

APPENDIX.—NOTATION

The following symbols are used in this paper:

A = cross-sectional area;

$\mathbf{B}$, $\mathbf{B}'$ = boundary matrices;

$\mathbf{D}$ = diagonal matrix, see Eqs. 8 and 9;

EI = flexural rigidity;

$\mathbf{E}$ = unit matrix;

F = axial force;

F_c = resisting axial force for elastic support;

GJ = torsional rigidity;

$\mathbf{G}$ = geometry matrix;

H = external axial force;

h, h' = symbols representing the horizontal-like member;

$\mathbf{K}$ = load term of a member see Eq. 16;

$\mathbf{k}$ = external load matrix;

$\mathbf{L}$ = relative operator;

l = member length;

M = bending moment;

M_c = resisting moment for elastic support;

$\mathbf{M}$ = 2-by-1 eigenmatrix;

$\mathbf{N}$ = 4-by-1 eigenmatrix;

P = external lateral load;

$\mathbf{P}(\rho)$ = 3-by-6 abscissa matrix see Eqs. 10 and 11;

$\mathbf{p}$ = projection matrix, or briefly projector;

Q = external torsional moment;

$\mathbf{Q}(\rho)$ = 3-by-6 abscissa matrix, see Eqs. 10 and 11;

R = external bending moment;

$\mathbf{R}$ = rearrangement matrix;

S = shearing force;
S_c = resisting lateral force of elastic support;
$\mathbf{S}_i$ = shift operator;
T = torsional moment;
u = axial displacement;
$\mathbf{U}(\rho)$ = displacement vector at ρ;
$\mathbf{U}$ = displacement vector at $\rho = 0$;
$\mathbf{U}'$ = displacement vector at $\rho = 1$;
$\mathbf{V}(\rho)$ = force vector at ρ;
$\mathbf{V}$ = force vector at $\rho = 0$;
$\mathbf{V}'$ = force vector at $\rho = 1$;
v, v' = symbols representing vertical-like members;
$\mathbf{W}(\rho)$ = state vector at ρ;
$\mathbf{W}$ = state vector at $\rho = 0$;
$\mathbf{W}'$ = state vector at $\rho = 1$;
w = flexural deflection;
$\mathbf{X}$ = 6-by-1 eigenmatrix, $\mathbf{X} = \{\mathbf{M} \ \ \mathbf{N}\}$;
α = direction angle of member taken clockwise from the standard axis;
Δ = diagonal matrix, see Eqs. 8 and 9;
δ = difference angle, see Eqs. 45, 46, 49, and 50;
θ = angle of deflection;
κ = non-dimensional load abscissa;
λ, μ, ν = constants attatched to elastic support;
ρ = non-dimensional current abscissa;
ϕ = angle of torsion;
$\boldsymbol{\Omega}$ = current-matrix;
$-$(vinculum) = symbol representing the transposed matrix, or physical quantities projected to its standard co-ordinates;
$\lfloor \quad \rfloor$ = row vector; and
$\{ \quad \}$ = column vector.