Operational Method for Clapeyron's Theorem

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SYNOPSIS

Various methods for the analysis of continuous beams have been proposed and investigated. They are compelled to treating simultaneous equations involving many unknowns, which have been recognized to be unavoidably necessary in the analysis of solid mechanics.

This paper takes a set of bending moments at any two consecutive supports of a continuous beam as the matrix consisting 2-by-1 elements, which is referred to as the "eigenmatrix."[1]

Then, one of these eigenmatrices can be shifted from one span to its adjacent one by a certain shift operator matrix which is defined by the span length and the moment of inertia of cross-section. By such a shift operation, the bending moment at any support of the continuous beam can be obtained readily and systematically, dispensing with simultaneous equations.[2]

The influence from external loads, settlement of suppots, and temperature change is also expressible by a 2-by-1 shift operator matrix. For these factors, it will be necessary to make a simple modification to the fundamental operators.

This method will be of great advantage in the analysis of complicated continuous beams.

INTRODUCTION

The generalized Clapeyron's theorem is written in the following form for any consecutive two spans of the continuous beam illustrated in Fig. 1a.

$$M_{r-1}\frac{l_{r-1}}{I_{r-1}}+2M_r\left(\frac{l_{r-1}}{I_{r-1}}+\frac{l_r}{I_r}\right)+M_{r+1}\frac{l_r}{I_r}$$

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$$= -6\left(\frac{\mathfrak{B}_{r-1}}{I_{r-1}} + \frac{\mathfrak{U}_r}{I_r}\right) + 6E\left(\frac{w_r - w_{r-1}}{l_{r-1}} - \frac{w_{r+1} - w_r}{l_r}\right)$$

$$+ 3E\varepsilon\left(\frac{l_{r-1}}{h_{r-1}}\varDelta T_{r-1} + \frac{l_r}{h_r}\varDelta T_r\right). \qquad (1)^{4), 7)}$$

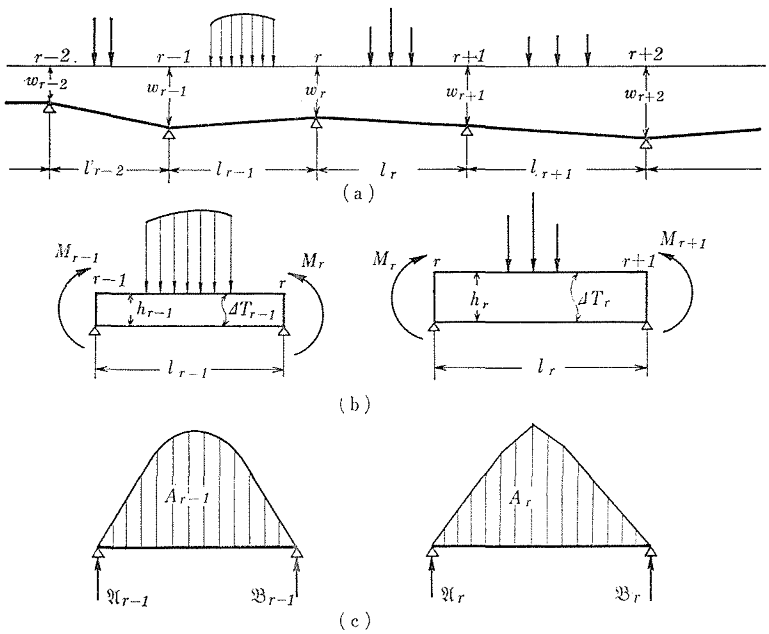

Fig. 1.

In the above equation, the right side represents the influence by the external load, the settlement of support, and the difference in temperature respectively. *)

Introducing the ratios

$$l'_r = \frac{I_c}{I_r} l_r, \qquad \alpha_r = \frac{l_r}{l_{r-1}}, \qquad k_r = \frac{l'_r}{l'_{r-1}}, \qquad (2)$$

Eq. 1 yields

$$M_{r-1} + 2M_r(1 + k_r) + M_{r+1}k_r = -6\left(\mathfrak{B}_{r-1}\frac{1}{l_{r-1}} + \mathfrak{U}_r\frac{1}{l_r}k_r\right)$$

$$-\frac{6EI_r}{l^2_r}k_r(\alpha_r w_{r-1} - \alpha_r w_r - w_r + w_{r+1}) + 3E\varepsilon\left(\frac{I_{r-1}}{h_{r-1}}\varDelta T_{r-1} + \frac{I_r}{h_r}\varDelta T_r k_r\right), \qquad (3)$$

and, for simplicity, using the symbols

$$\left.\begin{aligned} K'_{r-1} &= \frac{6\mathfrak{B}_{r-1}}{l_{r-1}}, \qquad K_r = \frac{6\mathfrak{U}_r}{l_r}, \\ \delta_r &= \frac{6EI_r}{l^2_r}k_r, \qquad Z_r = \frac{2I_r}{h_r}, {}^{**)} \qquad \beta = \frac{3E\varepsilon}{2}, \end{aligned}\right\} \tag{4}$$

the above equation becomes

$$\begin{aligned} M_{r-1} &+ 2M_r(1+k_r) + M_{r+1}k_r \\ &= -(K'_{r-1} + K_r k_r) - \delta_r(\alpha_r w_{r-1} - \alpha_r w_r - w_r + w_{r+1}) \\ &\qquad + \beta(Z_{r-1}\Delta T_{r-1} + Z_r \Delta T_r k_r). \end{aligned} \tag{5}$$

This is the fundamental equation adopted for use in the present paper.

CONNECTION CONDITION

In order to derive the basic relation between bending moments at any consecutive supports of a continuous beam, we shall consider the case in which all the influences represented in the right side of Eq. 5 equal to zero. Taking out any set of three consecutive spans as shown in Fig. 2, the Clapeyron's theorem yields two equations

$$\left.\begin{aligned} M_{r-1} + 2M_r(1+k_r) + M_{r+1}k_r &= 0, \\ M_r + 2M_{r+1}(1+k_{r+1}) + M_{r+2}k_{r+1} &= 0. \end{aligned}\right\} \tag{6}$$

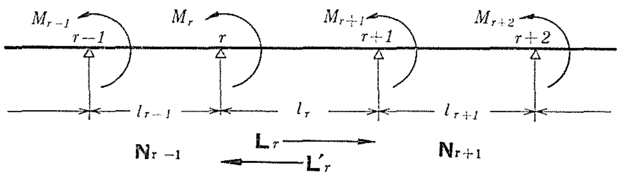

Fig. 2.

The above equations can be written in the matrix form

$$\begin{bmatrix} 1, & 2(1+k_r) \\ 0, & 1 \end{bmatrix} \begin{bmatrix} M_{r-1} \\ M_r \end{bmatrix} + \begin{bmatrix} k_r, & 0 \\ 2(1+k_{r+1}), & k_{r+1} \end{bmatrix} \begin{bmatrix} M_{r+1} \\ M_{r+2} \end{bmatrix} = 0, \tag{7}$$

which may also be written in the compact form

$$\mathbf{C}_r\{\mathbf{N}_{r-1}, \quad \mathbf{N}_{r+1}\} = 0, \tag{8}$$

*) The notations in this paper are indicated in Appendix III.

**) It is assumed here that the beam has a symmetrical cross-section with respect to its neutral axis.

in which

$$\mathbf{C}_r = \left[\begin{bmatrix} 1, & 2(1+k_r) \\ 0, & 1 \end{bmatrix}, \begin{bmatrix} k_r, & 0 \\ 2(1+k_{r+1}), & k_{r+1} \end{bmatrix}\right], \tag{9}$$

and

$$\mathbf{N}_{r-1} = \begin{bmatrix} M_{r-1} \\ M_r \end{bmatrix}, \qquad \mathbf{N}_{r+1} = \begin{bmatrix} M_{r+1} \\ M_{r+2} \end{bmatrix}. \tag{10}$$

Eq. 7 or 8 is the connection equation for the two physical quantities $\mathbf{N}_{r-1}$ and $\mathbf{N}_{r+1}$. The physical matrix of the form of Eqs. 10 will be referred to as the "eigenmatrix" of the span considered. Contrarily, Eq. 9 is one of the operational matrices, with which definite operations are to be performed on eigenmatrices.

SHIFT OPERATOR

Eq. 8 will yield the shift formulas

$$\mathbf{N}_{r+1} = \mathbf{L}_r \mathbf{N}_{r-1}, \qquad \mathbf{N}_{r-1} = \mathbf{L}'_r \mathbf{N}_{r+1}, \tag{11}$$

in which the shift operators $\mathbf{L}_r$ and $\mathbf{L}'_r$ represent

$$\mathbf{L}_r = -\frac{1}{k_r k_{r+1}} \begin{bmatrix} k_{r+1}, & 0 \\ -2(1+k_{r+1}), & k_r \end{bmatrix} \begin{bmatrix} 1, & 2(1+k_r) \\ 0, & 1 \end{bmatrix}$$

$$= \frac{1}{k_r k_{r+1}} \begin{bmatrix} -k_{r+1}, & -2(1+k_r)k_{r+1} \\ 2(1+k_{r+1}), & -k_r + 4(1+k_r)(1+k_{r+1}) \end{bmatrix}, \tag{12}$$

and

$$\mathbf{L}'_r = -\begin{bmatrix} 1, & -2(1+k_r) \\ 0, & 1 \end{bmatrix} \begin{bmatrix} k_r, & 0 \\ 2(1+k_{r+1}), & k_{r+1} \end{bmatrix}$$

$$= \begin{bmatrix} -k_r + 4(1+k_r)(1+k_{r+1}), & 2(1+k_r)k_{r+1} \\ -2(1+k_{r+1}), & -k_{r+1} \end{bmatrix}, \tag{13}$$

respectively. It can be verified that

$$\mathbf{L}_r \mathbf{L}'_r = \mathbf{E} \quad (\mathbf{E} \text{ being the 2-by-2 unit matrix}), \tag{14}$$

which proves, from the practical standpoint, the correctness of the computations.

$\mathbf{L}_r$ is the rightward shift operator, or briefly the right shiftor, since Eq. 11a suggests that the physical quantity $\mathbf{N}_{r-1}$ can at once be shifted rightwards by premultiplication by the shiftor $\mathbf{L}_r$. By similar reasoning,

$\mathbf{L}'_r$ is the left shiftor.

$\mathbf{L}_r$ and $\mathbf{L}'_r$ are, in the broad sense of words, the "ratios" between the two physical quantities $\mathbf{N}_{r-1}$ and $\mathbf{N}_{r+1}$.

VARIOUS FORMS OF FUNDAMENTAL SOLUTION

As a simple application of the preceeding reduction, a continuous beam subjected to the external edge moments $\mathfrak{M}$ and $\mathfrak{M}'$, which are given quantities, is taken as shown in Fig. 3.*) It will be found necessary to classify the analysis of the continuous beam into several groups by the number of spans, i. e, by the odd and even numbers of the constituent spans.

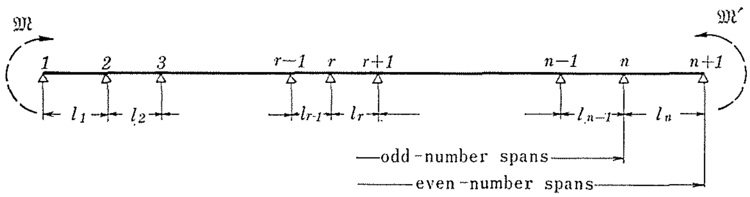

Fig. 3.

1. Continuous Beam with Odd-Number Spans.

When a continuous beam consists of odd-number spans, there will be four forms of solution, corresponding to the way of selection of the standard eigenmatrix $\mathbf{N}_r$.

Solution 1 (Fig. 4).

In this case, the eigenmatrices are taken to be

$$\mathbf{N}_1 = \begin{bmatrix} \mathfrak{M} \\ M_2 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} M_3 \\ M_4 \end{bmatrix}, \quad \mathbf{N}_5 = \begin{bmatrix} M_5 \\ M_6 \end{bmatrix}, \ \cdots, \ \mathbf{N}_{n-1} = \begin{bmatrix} M_{n-1} \\ \mathfrak{M}' \end{bmatrix}, \tag{15}$$

in which $\mathfrak{M}$ and $\mathfrak{M}'$ are the given external edge moments.

Taking $\mathbf{N}_1$ as standard, Eq. 11a yields the following shift formulas:

$$\mathbf{N}_3 = \mathbf{L}_2\mathbf{N}_1, \ \mathbf{N}_5 = \mathbf{L}_4\mathbf{N}_3 = \mathbf{L}_4\mathbf{L}_2\mathbf{N}_1, \ \cdots\cdots, \ \mathbf{N}_{n-1} = \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_4\mathbf{L}_2\mathbf{N}_1. \tag{16}$$

*) To denote the support number of a continuous beam, two letters r and n are used. The letter r represents any support number and can be any integer, while the letter n represents the support number at the extreme right or the next support and must be an even number:

$$n = 2i \ (i = 1, 2, 3, \cdots).$$

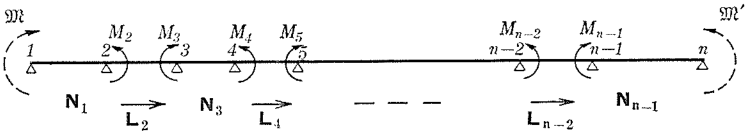

Fig. 4.

The shiftors $\mathbf{L}_2, \mathbf{L}_4, \cdots, \mathbf{L}_{n-2}$ are given by Eq. 12. Then substituting from Eqs. 15 into Eq. 16c,

$$\begin{bmatrix} M_{n-1} \\ \mathfrak{M}' \end{bmatrix} = \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_4\mathbf{L}_2 \begin{bmatrix} \mathfrak{M} \\ M_2 \end{bmatrix}, \tag{17}$$

which can be transformed to

$$\begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} M_{n-1} = \mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} + \mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} M_2, \tag{18}$$

from which it follows that

$$\begin{bmatrix} M_2 \\ M_{n-1} \end{bmatrix} = \left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right]^{-1} \times \left[-\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} \right]. \tag{19}$$

This is the desired final equation, and the present problem (Fig. 4) has been solved dispensing with simultaneous equations.

Solution 2 (Fig. 5).

The eigenmatrices are taken to be

$$\mathbf{N}_2 = \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}, \quad \mathbf{N}_4 = \begin{bmatrix} M_4 \\ M_5 \end{bmatrix}, \cdots, \mathbf{N}_{n-2} = \begin{bmatrix} M_{n-2} \\ M_{n-1} \end{bmatrix}, \tag{20}$$

and then the rightward shift operation yields

$$\mathbf{N}_{n-2} = \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_5\mathbf{L}_3\mathbf{N}_2. \tag{21}$$

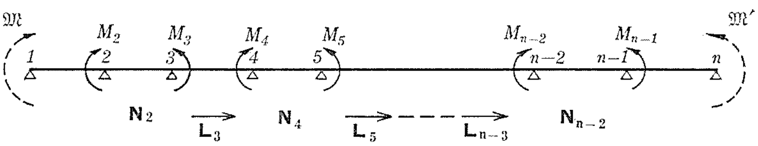

Fig. 5.

To take boundary conditions into account, the Clapeyron's theorem for the extreme right and the extreme left consecutive two spans is written as

$$\left.\begin{aligned} \mathfrak{M} + 2M_2(1+k_2) + M_3k_2 &= 0, \\ M_{n-2} + 2M_{n-1}(1+k_{n-1}) + \mathfrak{M}'k_{n-1} &= 0, \end{aligned}\right\} \tag{22}$$

which can be written in the matrix forms

$$\left.\begin{aligned} \mathfrak{M} + \lfloor 2(1+k_2), k_2 \rfloor \mathbf{N}_2 &= 0, \\ \lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{N}_{n-2} + \mathfrak{M}'k_{n-1} &= 0. \end{aligned}\right\} \tag{23}$$

Eqs. 23 are the necessary boundary equations which are expressed in terms of the given external moments $\mathfrak{M}$ and $\mathfrak{M}'$.

Substituting Eq. 21 into Eq. 23b, and referring to Eq. 23a, the final solution is obtained as follows :

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = -\begin{bmatrix} 2(1+k_2), & k_2 \\ \lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{M} \\ \mathfrak{M}'k_{n-1} \end{bmatrix}. \tag{24}$$

Solution 3 (Fig. 6).

Using the following method, the eigenmatrix at any odd numbered span is determined directly. For example, taking $\mathbf{N}_5$ as standard, the right and leftward shift operations from $\mathbf{N}_5$ yield

$$\left.\begin{aligned} \mathbf{N}_1 &= \begin{bmatrix} \mathfrak{M} \\ M_2 \end{bmatrix} = \mathbf{L}'_2\mathbf{L}'_4\mathbf{N}_5, \\ \mathbf{N}_{n-1} &= \begin{bmatrix} M_{n-1} \\ \mathfrak{M}' \end{bmatrix} = \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{N}_5. \end{aligned}\right\} \tag{25}^{*)}$$

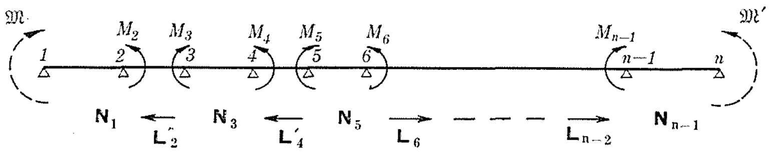

Fig. 6.

Writing the above equations together,

$$\begin{bmatrix} \begin{bmatrix} \mathfrak{M} \\ M_2 \end{bmatrix} \\ \begin{bmatrix} M_{n-1} \\ \mathfrak{M}' \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathfrak{M} \\ 0 \\ 0 \\ \mathfrak{M}' \end{bmatrix} + \begin{bmatrix} 0, & 0 \\ 1, & 0 \\ 0, & 1 \\ 0, & 0 \end{bmatrix} \begin{bmatrix} M_2 \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{L}'_2\mathbf{L}'_4 \\ \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_6 \end{bmatrix} \mathbf{N}_5, \tag{26}$$

from which

$$\begin{bmatrix} M_5 \\ M_6 \\ M_2 \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} -\mathbf{L}'_2\mathbf{L}'_4, & \begin{matrix} 0, & 0 \\ 1, & 0 \end{matrix} \\ -\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_6, & \begin{matrix} 0, & 1 \\ 0, & 0 \end{matrix} \end{bmatrix}^{-1} \begin{bmatrix} -\mathfrak{M} \\ 0 \\ 0 \\ -\mathfrak{M}' \end{bmatrix}. \tag{27}$$

This is the desired final solution. Eq. 27 requires to compute a 4-by-4 inverse, although this is of a simple form.

Solution 4 (Fig. 7).

Taking $\mathbf{N}_6$ as standard, the right and leftward shift operations yield

$$\mathbf{N}_{n-2} = \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_9\mathbf{L}_7\mathbf{N}_6, \quad \mathbf{N}_2 = \mathbf{L}'_3\mathbf{L}'_5\mathbf{N}_6. \tag{28}$$

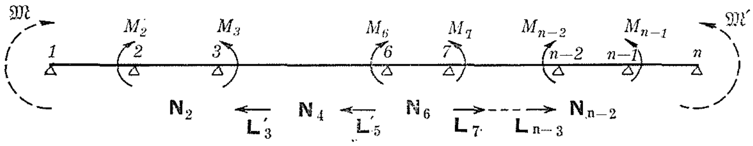

Fig. 7.

Substituting Eqs. 28 into Eqs. 23 yields

$$\left.\begin{aligned} \lfloor 2(1+k_2), \quad k_2 \rfloor \mathbf{L}'_3\mathbf{L}'_5\mathbf{N}_6 &= -\mathfrak{M}, \\ \lfloor 1, \quad 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_7\mathbf{N}_6 &= -\mathfrak{M}'k_{n-1}. \end{aligned}\right\} \tag{29}$$

Writing these equations together, the following final equation is obtained:

$$\begin{bmatrix} M_6 \\ M_7 \end{bmatrix} = -\begin{bmatrix} \lfloor 2(1+k_2), \quad k_2 \rfloor \mathbf{L}'_3\mathbf{L}'_5 \\ \lfloor 1, \; 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_7 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{M} \\ \mathfrak{M}'k_{n-1} \end{bmatrix}. \tag{30}$$

2. Continuous Beam with Even-Number Spans

As described in the preceding article, there are also the following cases of solution regarding the selection of standard eigenmatrix:

(a) $\mathbf{N}_1$ for the first span,

(b) $\mathbf{N}_2$ for the second span,

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*) There are the following relations between the right and left shiftors :

$$\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_4\mathbf{L}_2 = [\mathbf{L}'_2\mathbf{L}'_4\cdots\mathbf{L}'_{n-4}\mathbf{L}'_{n-2}]^{-1},$$
$$[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_4\mathbf{L}_2]^{-1} = \mathbf{L}'_2\mathbf{L}'_4\cdots\mathbf{L}'_{n-4}\mathbf{L}'_{n-2}.$$

Combining these relations with Eqs. 25, we can get the same result as solution 1.

(c) $\mathbf{N}_{2r-1}$ for an arbitrary odd numbered span, and

(d) $\mathbf{N}_{2r}$ for an arbitrary even numbered span.

Solution 5 (Fig. 8).

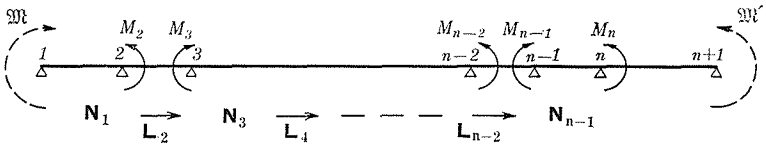


Fig. 8.

Taking $\mathbf{N}_1$ as standard, the rightward shift operation yields

$$\mathbf{N}_{n-1} = \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_4\mathbf{L}_2\mathbf{N}_1. \tag{31}$$

The right boundary equation is written to be

$$\lfloor 1,\ 2(1+k_n)\rfloor\, \mathbf{N}_{n-1} = -\,\mathfrak{M}'k_n. \tag{32}$$

Substituting Eq. 31 into the above equation,

$$\lfloor 1,\ 2(1+k_n)\rfloor\, \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\left[\begin{bmatrix}\mathfrak{M}\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} M_2\right] = -\,\mathfrak{M}'k_n, \tag{33}$$

from which the following final equation is obtained:

$$M_2 = -\left[\lfloor 1,\ 2(1+k_n)\rfloor\, \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}0\\ 1\end{bmatrix}\right]^{-1}$$

$$\times\left[\lfloor 1,\ 2(1+k_n)\rfloor\, \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}\mathfrak{M}\\ 0\end{bmatrix} + \mathfrak{M}'k_n\right]. \tag{34}$$

Solution 6 (Fig. 9).

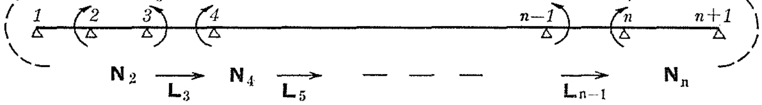


Fig. 9.

In this case, the standard eigenmatrix is taken as $\mathbf{N}_2$, and the rightward shift operation yields

$$\mathbf{N}_n = \mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5\mathbf{L}_3\mathbf{N}_2, \tag{35}$$

which is transformed to

$$\begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} M_n = \mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5\mathbf{L}_3\mathbf{N}_2. \tag{36}$$

The left boundary equation is written to be

$$\mathfrak{M} + \lfloor 2(1+k_2),\ k_2 \rfloor \mathbf{N}_2 = 0. \tag{37}$$

Writing Eqs. 36 and 37 together,

$$\begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}' \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} M_n + \begin{bmatrix} 2(1+k_2), & k_2 \\ & \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3 & \end{bmatrix} \mathbf{N}_2 = 0, \tag{38}$$

from which the final equation becomes

$$\begin{bmatrix} \mathbf{N}_2 \\ \\ M_n \end{bmatrix} = \begin{bmatrix} M_2 \\ M_3 \\ M_n \end{bmatrix} = - \begin{bmatrix} 2(1+k_2), & k_2, & 0 \\ & & 1 \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3 & & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}' \end{bmatrix}. \tag{39}$$

Solution 7 (Fig. 10).

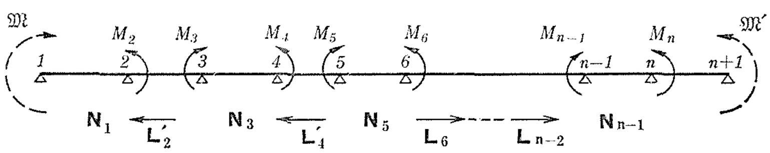


Fig. 10.

For example, taking $\mathbf{N}_5$ as standard, the right and leftward shift operations from $\mathbf{N}_5$ yield

$$\mathbf{N}_{n-1} = \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{N}_5, \qquad \mathbf{N}_1 = \mathbf{L}'_2\mathbf{L}'_4\mathbf{N}_5. \tag{40}$$

The right boundary equation is

$$\lfloor 1,\ 2(1+k_n) \rfloor \mathbf{N}_{n-1} + \mathfrak{M}' k_n = 0. \tag{41}$$

Substituting Eq. 40a into the above equation,

$$\lfloor 1,\ 2(1+k_n)\rfloor \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{N}_5 + \mathfrak{M}'k_n = 0. \tag{42}$$

Eq. 40b is transformed to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} M_2 - \mathbf{L}'_2\mathbf{L}'_4\mathbf{N}_5 + \begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} = 0. \tag{43}$$

Writing Eqs. 42 and 43 together,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} M_2 + \begin{bmatrix} -\mathbf{L}'_2\mathbf{L}'_4 \\ \lfloor 1,\ 2(1+k_n)\rfloor \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_6 \end{bmatrix} \mathbf{N}_5 + \begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}'k_n \end{bmatrix} = 0, \tag{44}$$

from which the final equation becomes

$$\begin{bmatrix} M_2 \\ M_5 \\ M_6 \end{bmatrix} = - \begin{bmatrix} 0, & -\mathbf{L}'_2\mathbf{L}'_4 \\ 1, & \\ 0, & \lfloor 1,\ 2(1+k_n)\rfloor \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_6 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}'k_n \end{bmatrix}. \tag{45}$$

Solution 8 (Fig. 11).

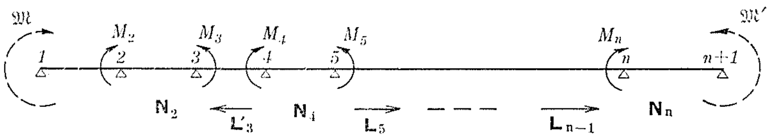


Fig. 11.

Taking $\mathbf{N}_4$ as standard, the right and leftward shift operations yield

$$\mathbf{N}_n = \mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5\mathbf{N}_4, \qquad \mathbf{N}_2 = \mathbf{L}'_3\mathbf{N}_4. \tag{46}$$

The left boundary equation is written to be

$$\mathfrak{M} + \lfloor 2(1+k_2),\ k_2\rfloor \mathbf{N}_2 = 0. \tag{47}$$

Using the same procedure as described in the preceding solution 3, Eqs. 46a and 47 may be written in the form

$$\begin{bmatrix} \lfloor 2(1+k_2),\ k_2\rfloor \mathbf{L}'_3 \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5 \end{bmatrix} \mathbf{N}_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} M_n + \begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}' \end{bmatrix} = 0, \tag{48}$$

from which, the final equation becomes

$$\begin{bmatrix} M_4 \\ M_5 \\ M_n \end{bmatrix} = - \begin{bmatrix} \lfloor 2(1+k_2), \ k_2 \rfloor \mathbf{L}'_3, & 0 \\ & 1 \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5, & \\ & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{M} \\ 0 \\ \mathfrak{M}' \end{bmatrix}. \tag{49}$$

3. Continuous Beam with Clamped End.

The clamped end of a continuous beam (Fig. 12a, and 12b) may be considered to be equivalent to the adjacent imaginary span which has an infinitely great amount of moment of inertia as shown in Fig. 12c, and 12d.[5)]

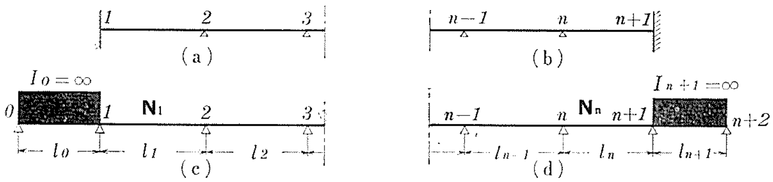


Fig. 12.

To derive the basic equation of the clamped end, assume that there are no external load, no settlement of support, and no difference in temperature in this system.

Clamped at the Left End.

By Eq. 1, the Clapeyron's theorem for the span l_1 and the imaginary span l_0 in Fig. 12c yields

$$M_0 \frac{l_0}{\infty} + 2M_1\left(\frac{l_0}{\infty} + \frac{l_1}{I_1}\right) + M_2 \frac{l_1}{I_1} = 0, \tag{50}$$

from which it at once follows that

$$M_2 = -2M_1, \tag{51}$$

and the eigenmatrix for the span l_1 becomes

$$\mathbf{N}_1 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} M_1. \tag{52}$$

Two unknown elements are reduced to one, and the problem can be solved.[*)]

*) Writing down the three moment equation for the spans l_1 and l_2, the support moment M_3 can be represented by the clamped end moment M_1. Thus, in the case of a continuous beam with clamped end, we can express the entire support moment as a function of the clamped end moment.

Clamped at the Right End.

By the same procedure as in the preceding derivation, the eigenmatrix $\mathbf{N}_n$ in Fig. 12d is represented by

$$\mathbf{N}_n = \begin{bmatrix} -2 \\ 1 \end{bmatrix} M_{n+1}. \tag{53}$$

EXTERNAL LATERAL LOAD

When a continuous beam subjected to external lateral loads, the Clapeyron's theorem (Eq. 5) takes the following form for any two consecutive spans in a continuous beam:

$$\left.\begin{aligned} M_{r+1} + 2M_r(1+k_r) + M_{r+1}k_r &= -(K'_{r-1} + K_r k_r), \\ M_r + 2M_{r+1}(1+k_{r+1}) + M_{r+2}k_{r+1} &= -(K'_r + K_{r+1}k_{r+1}), \end{aligned}\right\} \tag{54}$$

in which

$$\left.\begin{aligned} K'_{r-1} &= \frac{6}{l_{r-1}} \mathfrak{B}_{r-1}, & K_r &= \frac{6}{l_r} \mathfrak{U}_r, \\ K'_r &= \frac{6}{l_r} \mathfrak{B}_r, & K_{r+1} &= \frac{6}{l_{r+1}} \mathfrak{U}_{r+1}. \end{aligned}\right\} \tag{55)**}$$

Eqs. 54 can be rearranged to the matrix form

$$\begin{bmatrix} 1, & 2(1+k_r) \\ 0, & 1 \end{bmatrix} \begin{bmatrix} M_{r-1} \\ M_r \end{bmatrix} + \begin{bmatrix} k_r, & 0 \\ 2(1+k_{r+1}), & k_{r+1} \end{bmatrix} \begin{bmatrix} M_{r+1} \\ M_{r+2} \end{bmatrix} = -\begin{bmatrix} K'_{r-1} + K_r k_r \\ K'_r + K_{r+1}k_{r+1} \end{bmatrix}, \tag{56}$$

which can also be written in the compact form

$$\mathbf{C}_r \{\mathbf{N}_{r-1}, \mathbf{N}_{r+1}\} = -\mathbf{K}_r, \tag{57}$$

in which, $\mathbf{C}_r$, $\mathbf{N}_{r-1}$, and $\mathbf{N}_{r+1}$ have been defined by Eqs. 9 and 10.

$\mathbf{K}_r$ is a 2-by-1 matrix which is determined by loading conditions as follows:

$$\mathbf{K}_r = \begin{bmatrix} K'_{r-1} + K_r k_r \\ K'_r + K_{r+1}k_{r+1} \end{bmatrix}. \tag{58}$$

Eq. 57 yields the shift formulas

**) The values in Eqs. 55 are defined as the "Load Term" in this paper, and are collected in Appendix I for typical loading conditions.

$$\left.\begin{aligned} \mathbf{N}_{r+1} &= \mathbf{L}_r\mathbf{N}_{r-1} + \mathbf{P}_r, \\ \mathbf{N}_{r-1} &= \mathbf{L}'_r\mathbf{N}_{r+1} + \mathbf{P}'_r, \end{aligned}\right\} \tag{59}$$

in which the shiftors $\mathbf{L}_r$ and $\mathbf{L}'_r$ have been given by Eqs. 12 and 13.

$\mathbf{P}_r$ and $\mathbf{P}'_r$ are the shiftors which represent the influence of the external lateral load, and are defined by

$$\left.\begin{aligned} \mathbf{P}_r &= \frac{1}{k_rk_{r+1}}\begin{bmatrix} -(K'_{r-1}+K_rk_r)\,k_{r+1} \\ 2(K'_{r-1}+K_rk_r)(1+k_{r+1}) - (K'_r+K_{r+1}k_{r+1})\,k_r \end{bmatrix}, \\ \mathbf{P}'_r &= \begin{bmatrix} 2(K'_r+K_{r+1}k_{r+1})\,(1+k_r) - (K'_{r-1}+K_rk_r) \\ -(K'_r+K_{r+1}k_{r+1}) \end{bmatrix}. \end{aligned}\right\} \tag{60}$$

1. Continuous Beam with Odd-Number Spans Subjected to Lateral Load.

When a continuous beam consists of odd-number spans, there are also four cases of solution as described in the previous article.

Solution 1 (Fig. 13).

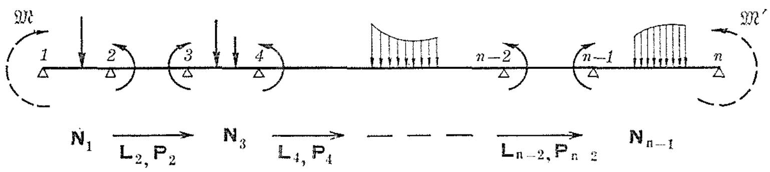


Fig. 13.

Taking $\mathbf{N}_1$ as standard, the rightward shift operation yields

$$\begin{aligned} \mathbf{N}_{n-1} &= \mathbf{L}_{n-2}\mathbf{L}_{n-4}\mathbf{L}_{n-6}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{L}_4\mathbf{L}_2\mathbf{N}_1 + \mathbf{L}_{n-2}\mathbf{L}_{n-4}\mathbf{L}_{n-6}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{L}_4\mathbf{P}_2 \\ &\quad + \mathbf{L}_{n-2}\mathbf{L}_{n-4}\mathbf{L}_{n-6}\cdots\mathbf{L}_8\mathbf{L}_6\mathbf{P}_4 + \cdots + \mathbf{L}_{n-2}\mathbf{P}_{n-4} + \mathbf{P}_{n-2} \\ &= \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\mathbf{N}_1 + \mathbf{L}_{n-2}\Big[\mathbf{L}_{n-4}\{\cdots\mathbf{L}_6(\mathbf{L}_4\mathbf{P}_2+\mathbf{P}_4) \\ &\qquad + \mathbf{P}_6\cdots\} + \mathbf{P}_{n-4}\Big] + \mathbf{P}_{n-2}. \end{aligned} \tag{61}$$

The above equation can be transformed to

$$\begin{aligned} \begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} M_{n-1} &= \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\left[\begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} M_2\right] \\ &\quad + \mathbf{L}_{n-2}\Big[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots) + \mathbf{P}_{n-4}\Big] + \mathbf{P}_{n-2}, \end{aligned} \tag{62}$$

from which it follows that

$$\begin{bmatrix} M_2 \\ M_{n-1} \end{bmatrix} = \left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right]^{-1} \left[-\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} - \left[\mathbf{L}_{n-2} \left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots) + \mathbf{P}_{n-4} \right] + \mathbf{P}_{n-2} \right] \right]. \tag{63}$$

This is the final equation for the present case. Comparing this with Eq. 19, it may be seen that these equations are quite similar, the only difference being in the additional term due to loading condition postmultiplied by Eq. 63.

Solution 2 (Fig. 14).

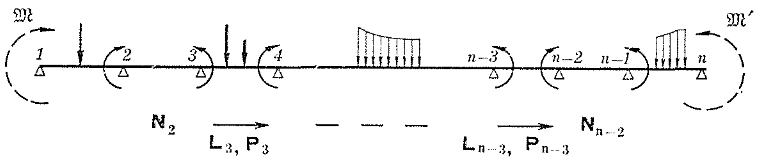


Fig. 14.

The standard eigenmatrix is in this case taken to be $\mathbf{N}_2$, and then the rightward shift operation yields

$$\begin{aligned} \mathbf{N}_{n-2} &= \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_5\mathbf{L}_3\mathbf{N}_2 \\ &\quad + \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_5\mathbf{P}_3 + \cdots + \mathbf{L}_{n-3}\mathbf{P}_{n-5} + \mathbf{P}_{n-3} \\ &= \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_5\mathbf{L}_3\mathbf{N}_2 \\ &\qquad + \mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_3\cdots) + \mathbf{P}_{n-5}] + \mathbf{P}_{n-3}. \end{aligned} \tag{64}$$

The right and left boundary equations are written to be

$$\left.\begin{aligned} \lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{N}_{n-2} &= -\mathfrak{M}'k_{n-1} - (K'_{n-2} + K_{n-1}k_{n-1}), \\ \lfloor 2(1+k_2),\ k_2 \rfloor \mathbf{N}_2 &= -\mathfrak{M} - (K'_1 + K_2k_2). \end{aligned}\right\} \tag{65}$$

Referring to Eq. 64, and using the same procedure as described in the derivation of Eq. 24, the final equation in this case is obtained in the form

$$\mathbf{N}_2 = \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = -\begin{bmatrix} 2(1+k_2), \quad k_2 \\ \lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_3 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \mathfrak{M} + (K'_1 + K_2k_2) \\ \mathfrak{M}'k_{n-1} + (K'_{n-2} + K_{n-1}k_{n-1}) \end{bmatrix}$$

$$+\lfloor 1,\ 2(1+k_{n-1})\rfloor \left[\mathbf{L}_{n-3}\left[\mathbf{L}_{n-5}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-5}\right]+\mathbf{P}_{n-3}\right]\Bigg]. \tag{66}$$

Comparing this result with Eq. 24, the term due to loading condition is added in the postmultiplied matrix in the above equation.

Solution 3 (Fig. 15).

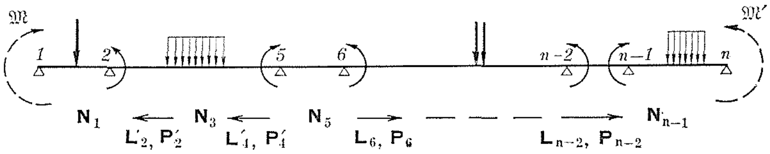


Fig. 15.

Taking any odd numbered eigenmatrix, for instance, $\mathbf{N}_5$ as standard, the final equation becomes

$$\begin{bmatrix} M_5 \\ M_6 \\ M_2 \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} -\mathbf{L}'_2\mathbf{L}'_4, & \begin{matrix}0, & 0 \\ 1, & 0\end{matrix} \\ -\mathbf{L}_{n-2}\cdots\mathbf{L}_6, & \begin{matrix}0, & 1 \\ 0, & 0\end{matrix} \end{bmatrix}^{-1}$$

$$\times\left[\begin{bmatrix} -\mathfrak{M} \\ 0 \\ 0 \\ -\mathfrak{M}' \end{bmatrix} + \begin{bmatrix} \mathbf{L}'_2\mathbf{P}'_4+\mathbf{P}'_2 \\ \mathbf{L}_{n-2}\left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_6\cdots)+\mathbf{P}_{n-4}\right]+\mathbf{P}_{n-2} \end{bmatrix}\right]. \tag{67}$$

Solution 4 (Fig. 16).

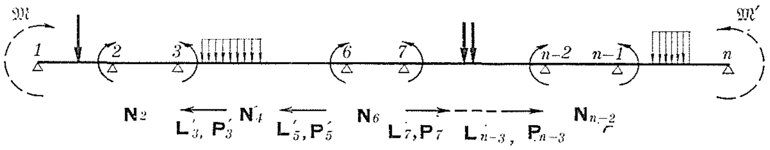


Fig. 16.

Taking $\mathbf{N}_6$ as standard, the final equation becomes

$$\mathbf{N}_6=\begin{bmatrix} M_6 \\ M_7 \end{bmatrix} = -\begin{bmatrix} \lfloor 2(1+k_2),\ k_2\rfloor\,\mathbf{L}'_3\mathbf{L}'_5 \\ \lfloor 1,\ 2(1+k_{n-1})\rfloor\,\mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_7 \end{bmatrix}^{-1}$$

$$\times\begin{bmatrix} \mathfrak{M} + (K'_1 + K_2k_2) \\ \mathfrak{M}'k_{n-1} + (K'_{n-2} + K_{n-1}k_{n-1}) \end{bmatrix} \begin{matrix} + \lfloor 2(1+k_2),\ k_2 \rfloor \begin{bmatrix} \mathbf{L}'_3\mathbf{P}'_5 + \mathbf{P}'_3 \end{bmatrix} \\ + \lfloor 1,\ 2(1+k_{n-1}) \rfloor \begin{bmatrix} \mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_7\cdots) + \mathbf{P}_{n-5}] + \mathbf{P}_{n-3} \end{bmatrix} \end{matrix} \Bigg]. \tag{68}$$

2. Continuous Beam with Even-Number Spans Subjected to Lateral Load.

As in the preceding solutions, the final equation for this case will also take the form postmultiplied by a certain matrix due to external load.

Assuming that, there acts an arbitrary series of external loads on the continuous beam as shown in Figs. 8, 9, 10, and 11, solutions for each case can be derived using the same procedures as described previously.

Solution 5.

Taking $\mathbf{N}_1$ as standard, and shifting to the rightward direction as shown in Fig. 8, the final equation becomes

$$M_2 = -\left[\lfloor 1,\ 2(1+k_n) \rfloor \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]^{-1}$$

$$\times\left[\lfloor 1, 2(1+k_n) \rfloor \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} + \mathfrak{M}'k_n + (K'_{n-1} + K_nk_n)\right.$$

$$\left. + \lfloor 1,\ 2(1+k_n) \rfloor \begin{bmatrix} \mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots) + \mathbf{P}_{n-4}] + \mathbf{P}_{n-2} \end{bmatrix}\right]. \tag{69}$$

Solution 6.

Taking $\mathbf{N}_2$ as standard in Fig. 9, the final equation becomes

$$\begin{bmatrix} M_2 \\ M_3 \\ M_n \end{bmatrix} = -\begin{bmatrix} 2(1+k_2), & k_2, & 0 \\ & & 1 \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3, & & \\ & & 0 \end{bmatrix}^{-1}$$

$$\times\begin{bmatrix} \mathfrak{M} + (K'_1 + K_2k_2) \\ \begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} - \begin{bmatrix} \mathbf{L}_{n-1}[\mathbf{L}_{n-3}(\cdots\mathbf{P}_3\cdots) + \mathbf{P}_{n-3}] + \mathbf{P}_{n-1} \end{bmatrix} \end{bmatrix}. \tag{70}$$

Solution 7.

Any odd numbered eigenmatrix is taken as standard in this case. Taking $\mathbf{N}_5$ as standard, the final equation is written to be

$$\begin{bmatrix} M_2 \\ M_5 \\ M_6 \end{bmatrix} = - \begin{bmatrix} 0, & -\mathbf{L}'_2\mathbf{L}'_4 \\ 1, & \\ 0, & \lfloor 1, 2(1+k_n) \rfloor \mathbf{L}_{n-2}\cdots\mathbf{L}_6 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \begin{bmatrix} \mathfrak{M} \\ 0 \end{bmatrix} - (\mathbf{L}'_2\mathbf{P}'_4 + \mathbf{P}'_2) \\ \mathfrak{M}'k_n + (K'_{n-1} + K_n k_n) \\ + \lfloor 1, \ 2(1+k_n) \rfloor \left[\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_6\cdots) + \mathbf{P}_{n-4}] + \mathbf{P}_{n-2} \right] \end{bmatrix}. \tag{71}$$

Solution 8.

Any even numbered eigenmatrix is taken as standard in this case. Taking $\mathbf{N}_4$ as standard, the final equation is written as follows, and may be compared with Eq. 49 which was derived for the unloaded case.

$$\begin{bmatrix} M_4 \\ M_5 \\ M_n \end{bmatrix} = - \begin{bmatrix} \lfloor 2(1+k_2), & k_2 \rfloor \mathbf{L}'_3, & 0 \\ & & 1 \\ - \mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_5, & & \\ & & 0 \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} & \mathfrak{M} + (K'_1 + K_2 k_2) + \lfloor 2(1+k_2), \ k_2 \rfloor \mathbf{P}'_3 \\ 0, & \\ \mathfrak{M}', & - \left[\mathbf{L}_{n-1}[\mathbf{L}_{n-3}(\cdots\mathbf{P}_5\cdots) + \mathbf{P}_{n-3}] + \mathbf{P}_{n-1} \right] \end{bmatrix}. \tag{72}$$

3. Clamped End of a Continuous Beam Subjected to Lateral Load.

Using the same procedure as described in the previous fundamental solution, the clamped end of a continuous beam can be analyzed for given loading conditions.

Clamped at the Left End.

In this case, the additional term due to loading condition

$$-6\left(\frac{\mathfrak{B}_0}{\infty} + \frac{\mathfrak{U}_1}{I_1}\right) \tag{73}$$

is to be added in the right-hand side of Eq. 50, and the following relation is readily obtained:

$$M_2 = -2M_1 - K_1. \tag{74}$$

In addition, writing down the three moment equation for the spans l_1 and l_2,

$$M_1 - (4M_1 + 2K_1)(1 + k_2) + M_3 k_2 = -(K'_1 + K_2 k_2), \tag{75}$$

the third support moment is obtained to be

$$M_3 = \frac{1}{k_2}\left[(3 + 4k_2) M_1 + 2(1 + k_2) K_1 - (K'_1 + K_2 k_2)\right]. \tag{76}$$

In a similar manner, all support moments of a continuous beam can be represented by the clamped end moment M_1, and therefore the problem can be solved if the right boundary condition is given.

Clamped at the Right End.

By the same procedure as described above, the following relations are obtained:

$$M_n = -2M_{n+1} - K'_n \tag{77}$$

$$M_{n-1} = (4 + 3k_n) M_{n+1} + 2(1 + k_n) K'_n - (K'_{n-1} + K_n k_n). \tag{78}$$

In these equations, M_{n+1} represents the bending moment at the right clamped end.

SETTLEMENT OF SUPPORT AND DIFFERENCE IN TEMPERATURE

Taking the influence due to settlement of support and difference in temperature into consideration, the shift formulas can be represented by

$$\mathbf{N}_{r+1} = \mathbf{L}_r \mathbf{N}_{r-1} + \mathbf{P}_r + \mathbf{S}_r + \mathbf{T}_r, \tag{79}$$

and

$$\mathbf{N}_{r-1} = \mathbf{L}'_r \mathbf{N}_{r+1} + \mathbf{P}'_r + \mathbf{S}'_r + \mathbf{T}'_r, \tag{80}$$

in which $\mathbf{L}_r$ and $\mathbf{L}'_r$ have been defined by Eqs. 12 and 13, and $\mathbf{P}_r$ and $\mathbf{P}'_r$ by Eqs. 60 respectively.

$\mathbf{S}_r$, $\mathbf{S}'_r$, $\mathbf{T}_r$, and $\mathbf{T}'_r$ are 2–by–1 matrices which are defined by the following formulas:

Shiftors for Settlement of Support.

$$\mathbf{S}_r = \frac{1}{k_r k_{r+1}} \begin{bmatrix} -\delta_r(\alpha_r w_{r-1} - \alpha_r w_r - w_r + w_{r+1}) k_{r+1} \\ -\delta_{r+1}(\alpha_{r+1} w_r - \alpha_{r+1} w_{r+1} - w_{r+1} + w_{r+2}) k_r \\ + 2\delta_r(\alpha_r w_{r-1} - \alpha_r w_r - w_r + w_{r+1})(1 + k_{r+1}) \end{bmatrix}, \quad (81)$$

$$\mathbf{S}'_r = \begin{bmatrix} -\delta_r(\alpha_r w_{r-1} - \alpha_r w_r - w_r + w_{r+1}) + 2\delta_{r+1}(\alpha_{r+1} w_r - \alpha_{r+1} w_{r+1} \\ - w_{r+1} + w_{r+2})(1 + k_r) \\ -\delta_{r+1}(\alpha_{r+1} w_r - \alpha_{r+1} w_{r+1} - w_{r+1} + w_{r+2}) \end{bmatrix}. \quad (82)$$

Shiftors for Temperature Change.

$$\mathbf{T}_r = \frac{\beta}{k_r k_{r+1}} \begin{bmatrix} (Z_{r-1}\Delta T_{r-1} + Z_r \Delta T_r k_r) k_{r+1} \\ -2(Z_{r-1}\Delta T_{r-1} + Z_r \Delta T_r k_r (1 + k_{r+1}) \\ + (Z_r \Delta T_r + Z_{r+1}\Delta T_{r+1} k_{r+1}) k_r \end{bmatrix}, \quad (83)$$

$$\mathbf{T}'_r = \beta \begin{bmatrix} (Z_{r-1}\Delta T_{r-1} + Z_r \Delta T_r k_r) - 2(Z_r \Delta T_r + Z_{r+1}\Delta T_{r+1} k_{r+1})(1 + k_r) \\ (Z_r \Delta T_r + Z_{r+1}\Delta T_{r+1} k_{r+1}) \end{bmatrix}. \quad (84)$$

Using the same procedure as described in the case of external lateral load, the influence due to settlement of support and difference in temperature can also be taken into the analysis of continuous beams.

PARTICULAR CASE

In the case of a continuous beam with equal span and equal cross-section, the shiftors take the following simple form:

$$\mathbf{L}_r = \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}, \quad (85)$$

$$\mathbf{L}'_r = \begin{bmatrix} 15, & 4 \\ -4, & -1 \end{bmatrix}, \quad (86)$$

$$\mathbf{P}_r = \begin{bmatrix} -(K'_{r-1} + K_r) \\ 4(K'_{r-1} + K_r) - (K'_r + K_{r+1}) \end{bmatrix}, \quad (87)$$

$$\mathbf{P}'_r = \begin{bmatrix} 4(K'_r + K_{r+1}) - (K'_{r-1} + K_r) \\ -(K'_r + K_{r+1}) \end{bmatrix}, \tag{88}$$

$$\mathbf{S}_r = \delta \begin{bmatrix} -w_{r-1} + 2w_r - w_{r+1} \\ 4w_{r-1} - 9w_r + 6w_{r+1} - w_{r+2} \end{bmatrix}, \tag{89}$$

$$\mathbf{S}'_r = \delta \begin{bmatrix} -w_{r-1} + 6w_r - 9w_{r+1} + 4w_{r+2} \\ -w_r + 2w_{r+1} - w_{r+2} \end{bmatrix}, \tag{90}$$

$$\mathbf{T}_r = 2\beta Z \Delta T \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \tag{91}$$

$$\mathbf{T}'_r = 2\beta Z \Delta T \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \tag{92}$$

GENERALIZED SOLUTION

As in the previous investigation of a continuous beam with simply supported ends, the common form of final equation consists of

(1) n-by-n inverse to be determined by the beam construction,

(2) n-by-1 column matrix representing the edge moment, and

(3) n-by-1 column matrix representing the loading condition (the number n representing 1, 2, 3, or 4).

Then the generalized solution can be written in the form

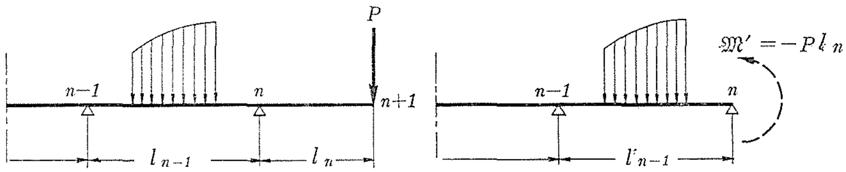
$$\mathbf{N} = \mathbf{R}[\mathbf{M} + \mathbf{Q}]. \tag{93}$$

Here $\mathbf{R}$ represents an n-by-n inverse matrix and is designated as the "Premultiplier." In the same manner, $\mathbf{M}$ is the "Edge Moment Matrix," and $\mathbf{Q}$ is the "Load Matrix."

When the both ends of a continuous beam are clamped, the final equation takes the form

$$\mathbf{N} = \mathbf{RQ}. \tag{94}$$

In the case of a continuous beam with overhanging end (Fig. 17a), the load on the overhanging part can be reduced to the edge moment effect as shown in Fig. 17b. Therefore, the final equation is the same form as Eq. 93.

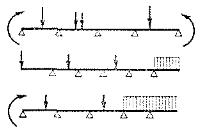
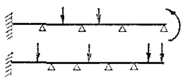


(a) (b)

Fig. 17.

The generalized formulas for all kinds of continuous beam are shown in Table I. The values of $\mathbf{R}$, $\mathbf{M}$, and $\mathbf{Q}$ are summarized in Appendix II.

Table I. Generalized Formulas for Continuous Beam

Type of Beam		Formula	Corresponding Table
Simple∼Simple Free ∼Free Simple∼Free		$\mathbf{N}=\mathbf{R}[\mathbf{M}+\mathbf{Q}]$	Table III, IV
Clamp ∼Simple Clamp ∼Free		$\mathbf{N}=\mathbf{R}[\mathbf{M}+\mathbf{Q}]$	Table V, VI
Clamp ∼Clamp		$\mathbf{N}=\mathbf{RQ}$	Table VII, VIII

EXAMPLES

Practical applications of the present method will be given in the following examples.

Example 1.

Determine the bending moments at the supports of a continuous beam with seven equal spans when the middle span alone is loaded by a uniformly distributed load q.[6)]

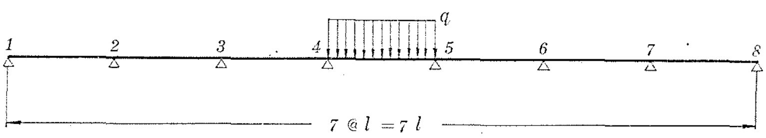


Fig. 18.

From Table II, the load term is obtained to be

$$K_4 = K'_4 = \frac{1}{4} q l^2. \tag{95}$$

Substituting the above value into Eqs. 87 and 88,

$$\left.\begin{aligned}
\mathbf{P}_3 &= \begin{bmatrix} 0 \\ -K_4 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}, & \mathbf{P}'_3 &= \begin{bmatrix} 4K_4 \\ -K_4 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \\
\mathbf{P}_4 &= \begin{bmatrix} -K_4 \\ 4K_4 - K'_4 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}, & \mathbf{P}'_4 &= \begin{bmatrix} 4K'_4 - K_4 \\ -K'_4 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \\
\mathbf{P}_5 &= \begin{bmatrix} -K'_4 \\ 4K'_4 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} -1 \\ 4 \end{bmatrix}, & \mathbf{P}'_5 &= \begin{bmatrix} -K'_4 \\ 0 \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} -1 \\ 0 \end{bmatrix},
\end{aligned}\right\} \tag{96}$$

and

$$\mathbf{P}_2 = \mathbf{P}'_2 = \mathbf{P}_6 = \mathbf{P}'_6 = 0. \tag{97}$$

Then the shiftors $\mathbf{L}_r$ and $\mathbf{L}'_r$ are given by Eqs. 85 and 86, and reduce to

$$\mathbf{L} = \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}, \qquad \mathbf{L}' = \begin{bmatrix} 15, & 4 \\ -4, & -1 \end{bmatrix}. \tag{98}$$

Using the Solution 2 in Table III, the eigenmatrix is obtained as follows:

$$\left.\begin{aligned}
\mathbf{R} &= -\begin{bmatrix} 2(1+k_2), & k_2 \\ \lfloor 1, 2(1+k_2) \rfloor \mathbf{L}_5 \mathbf{L}_3 \end{bmatrix}^{-1} = -\begin{bmatrix} 4 & 1 \\ \lfloor 1, 4 \rfloor \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}^2 \end{bmatrix}^{-1} \\
&= \frac{1}{2\,911} \begin{bmatrix} -780, & 1 \\ 209, & -4 \end{bmatrix}, \\
\mathbf{M} &= 0, \\
\mathbf{Q} &= \begin{bmatrix} (K'_1 + K_2 k_2) \\ (K'_6 + K_7 k_7) + \lfloor 1, 2(1+k_7) \rfloor [\mathbf{L} \mathbf{P}_3 + \mathbf{P}_5] \end{bmatrix} \\
&= \frac{1}{4} q l^2 \begin{bmatrix} 0 \\ \lfloor 1, 4 \rfloor \left[\begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right] \end{bmatrix} = \frac{1}{4} q l^2 \begin{bmatrix} 0 \\ -41 \end{bmatrix},
\end{aligned}\right\} \tag{99}$$

from which

$$\mathbf{N}_2=\begin{bmatrix}M_2\\M_3\end{bmatrix}=\mathbf{R}[\mathbf{M}+\mathbf{Q}]=\frac{ql^2}{284}\begin{bmatrix}-1\\4\end{bmatrix}. \tag{100}$$

By the right shift operation, the unknown support moments are determined as

$$\left.\begin{aligned}\mathbf{N}_4&=\begin{bmatrix}M_4\\M_5\end{bmatrix}=\mathbf{L}\mathbf{N}_2+\mathbf{P}_3=\frac{ql^2}{284}\left[\begin{bmatrix}-1, & -4\\4, & 15\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix}+\begin{bmatrix}0\\-71\end{bmatrix}\right]=\frac{ql^2}{284}\begin{bmatrix}-15\\-15\end{bmatrix},\\ \mathbf{N}_6&=\begin{bmatrix}M_6\\M_7\end{bmatrix}=\mathbf{L}\mathbf{N}_4+\mathbf{P}_5=\frac{ql^2}{284}\begin{bmatrix}4\\-1\end{bmatrix}.\end{aligned}\right\} \tag{101}$$

The check calculation for the above obtained values may be carried out by the left shift operation as follows :

$$\left.\begin{aligned}\mathbf{N}_7&=\begin{bmatrix}M_7\\M_8\end{bmatrix}=\frac{ql^2}{284}\begin{bmatrix}-1\\0\end{bmatrix}, && \text{O. K.}\\ \mathbf{N}_5&=\mathbf{L}'\mathbf{N}_7=\frac{ql^2}{284}\begin{bmatrix}-15\\4\end{bmatrix}, && \text{O. K.}\\ \mathbf{N}_3&=\mathbf{L}'\mathbf{N}_5+\mathbf{P}'_4=\frac{ql^2}{284}\begin{bmatrix}4\\-15\end{bmatrix}, && \text{O. K.}\\ \mathbf{N}_1&=\mathbf{L}'\mathbf{N}_3=\frac{ql^2}{284}\begin{bmatrix}0\\-1\end{bmatrix}. && \text{O. K.}\end{aligned}\right\} \tag{102}$$

Example 2.

The main pile of a cofferdam is strengthened by wales and struts as shown in Fig. 19a. Calculate the bending moments at the strengthened points of the main pile. The wales are arranged with ten equal spaces on the main pile.

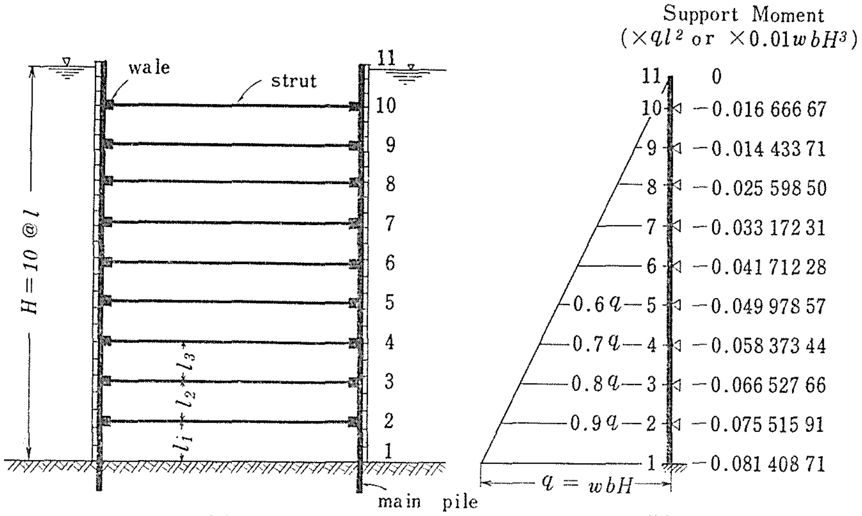


Fig. 19.

The main pile is subjected to the hydrostatic pressure and can be considered as a continuous beam with the lower end clamped and the upper end free as shown in Fig. 19b.

Using Table II, the load term is written down easily as follows :

$$\left.\begin{aligned}
K_1 &= \frac{l^2}{60}(8q_a+7q_b) = \frac{ql^2}{60}(8+6.3) = \frac{ql^2}{60}\times 14.3 = 14.3\lambda, \\
K'_1 &= \frac{l^2}{60}(7q_a+8q_b) = \frac{ql^2}{60}(7+7.2) = \frac{ql^2}{60}\times 14.2 = 14.2\lambda, \\
K_2 &= 12.8\lambda, \quad K'_2 = 12.7\lambda, \quad K_3 = 11.3\lambda, \quad K'_3 = 11.2\lambda, \\
K_4 &= 9.8\lambda, \quad K'_4 = 9.7\lambda, \quad K_5 = 8.3\lambda, \quad K'_5 = 8.2\lambda, \\
K_6 &= 6.8\lambda, \quad K'_6 = 6.7\lambda, \quad K_7 = 5.3\lambda, \quad K'_7 = 5.2\lambda, \\
K_8 &= 3.8\lambda, \quad K'_8 = 3.7\lambda, \quad K_9 = 2.3\lambda, \quad K'_9 = 2.2\lambda.
\end{aligned}\right\} \qquad (103)^{*)}$$

Substituting the above values into Eq. 87, it follows that

$$\left.\begin{aligned}
\mathbf{P}_2 &= \begin{bmatrix} -(K'_1+K_2) \\ 4(K'_1+K_2)-(K'_2+K_3) \end{bmatrix} = \lambda\begin{bmatrix} -27 \\ 84 \end{bmatrix}, \\
\mathbf{P}_4 &= \lambda\begin{bmatrix} -21 \\ 66 \end{bmatrix}, \quad \mathbf{P}_6 = \lambda\begin{bmatrix} -15 \\ 48 \end{bmatrix}, \quad \mathbf{P}_8 = \lambda\begin{bmatrix} -9 \\ 30 \end{bmatrix}.
\end{aligned}\right\} \qquad (104)$$

*) For simplicity, the notation $\lambda = ql^2/60$ is adopted for use in this example.

Here the shiftors $\mathbf{L}_r$ and $\mathbf{L}'_r$ are given by Eqs. 85 and 86 and have constant value for the entire span.

The load on the overhanging part in Fig. 19b may be reduced to one of the edge moment effect on the support *10*, which is given by

$$\mathfrak{M}' = -\frac{1}{60} q l^2 = -\lambda. \tag{105}$$

Then, using the Solution 9 in Table V, it follows that ($n = 10$ in this case)

$$\left.\begin{aligned}
\mathbf{R} &= \left[\mathbf{L}_8\mathbf{L}_6\mathbf{L}_4\mathbf{L}_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]^{-1} = \left[\begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}^4 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]^{-1} \\
&= \frac{1}{70\,226}\begin{bmatrix} 0, & -1 \\ -70\,226, & -18\,817 \end{bmatrix}, \\
\mathbf{M} &= \begin{bmatrix} 0 \\ \mathfrak{M}' \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\
\mathbf{Q} &= \mathbf{L}^4 \begin{bmatrix} 0 \\ K_1 \end{bmatrix} - \mathbf{L}^3\mathbf{P}_2 - \mathbf{L}^2\mathbf{P}_4 - \mathbf{L}\mathbf{P}_6 - \mathbf{P}_8 \\
&= \lambda\left[\begin{bmatrix} -2\,911, & -10\,864 \\ 10\,864, & 40\,545 \end{bmatrix}\begin{bmatrix} 0 \\ 14.3 \end{bmatrix} - \begin{bmatrix} -209, & -780 \\ 780, & 2\,911 \end{bmatrix}\begin{bmatrix} -27 \\ 84 \end{bmatrix}\right. \\
&\qquad \left. - \begin{bmatrix} -15, & -56 \\ 56, & 209 \end{bmatrix}\begin{bmatrix} -21 \\ 66 \end{bmatrix} - \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}\begin{bmatrix} -15 \\ 48 \end{bmatrix} - \begin{bmatrix} -9 \\ 30 \end{bmatrix}\right] \\
&= \lambda \begin{bmatrix} -91\,911.2 \\ 343\,021.5 \end{bmatrix},
\end{aligned}\right\} \tag{106}$$

from which

$$\mathbf{N} = \begin{bmatrix} M_1 \\ M_9 \end{bmatrix} = \mathbf{R}[\mathbf{M} + \mathbf{Q}] = \frac{\lambda}{70\,226}\begin{bmatrix} -343\,020.5 \\ -60\,817.3 \end{bmatrix}. \tag{107}$$

Using Eq. 74,

$$\mathbf{N}_1 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} M_1 - \begin{bmatrix} 0 \\ K_1 \end{bmatrix} = \frac{\lambda}{70\,226}\begin{bmatrix} -343\,020.5 \\ -318\,190.8 \end{bmatrix}. \tag{108}$$

Shifting to the upward direction, each support moment is determined successively as follows :

$$\left.\begin{aligned}
\mathbf{N}_3 &= \begin{bmatrix} M_3 \\ M_4 \end{bmatrix} = \mathbf{L}\mathbf{N}_1 + \mathbf{P}_2 = \frac{\lambda}{70\,226}\begin{bmatrix} -280\,318.3 \\ -245\,960.0 \end{bmatrix}, \\
\mathbf{N}_5 &= \begin{bmatrix} M_5 \\ M_6 \end{bmatrix} = \mathbf{L}\mathbf{N}_3 + \mathbf{P}_4 = \frac{\lambda}{70\,226}\begin{bmatrix} -210\,587.7 \\ -175\,757.2 \end{bmatrix}, \\
\mathbf{N}_7 &= \begin{bmatrix} M_7 \\ M_8 \end{bmatrix} = \mathbf{L}\mathbf{N}_5 + \mathbf{P}_6 = \frac{\lambda}{70\,226}\begin{bmatrix} -139\,773.5 \\ -107\,860.8 \end{bmatrix}, \\
\mathbf{N}_9 &= \begin{bmatrix} M_9 \\ M_{10} \end{bmatrix} = \mathbf{L}\mathbf{N}_7 + \mathbf{P}_8 = \frac{\lambda}{70\,226}\begin{bmatrix} -60\,817.3 \\ -70\,226.0 \end{bmatrix}.
\end{aligned}\right\} \tag{109}$$

Comparing Eq. 109d with Eq. 105 or 107, we can see the correctness of computations. Thus the computation of this method can be carried out with an automatic procedure. The values of support moments are given in Fig. 19b.

Example 3.

Find the bending moments at supports of the continuous beam with variable cross-section shown in Fig. 20.

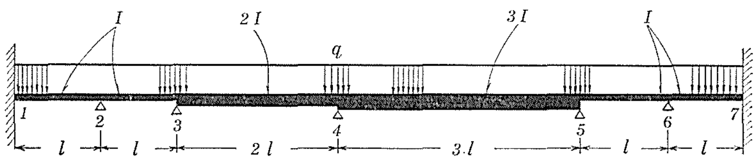


Fig. 20.

From the given configuration of the beam, it is evident that

$$k_r = 1 \qquad (r = 2,\ 3,\ 4,\ 5,\ 6), \tag{110}$$

and therefore

$$\mathbf{L}_r = \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}. \tag{111}$$

From Table II, the load term is obtained thus :

$$\left.\begin{aligned} &K_1 = K'_1 = \frac{1}{4} ql^2 = \mu, \\ &K_2 = K'_2 = K_5 = K'_5 = K_6 = K'_6 = \mu, \\ &K_3 = K'_3 = 4\mu, \\ &K_4 = K'_4 = 9\mu. \end{aligned}\right\} \qquad (112)^{*)}$$

Substituting the above values into Eq. 87, it follows that

$$\left.\begin{aligned} &\mathbf{P}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}\mu, \\ &\mathbf{P}_4 = \begin{bmatrix} -13 \\ 42 \end{bmatrix}\mu, \\ &\mathbf{P}_5 = \begin{bmatrix} -10 \\ 38 \end{bmatrix}\mu. \end{aligned}\right\} \qquad (113)$$

Using the Solution 21 in Table VIII,

$$\left.\begin{aligned} \mathbf{R} &= \left[\mathbf{L}_4\mathbf{L}_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4-3k_6 \\ 2 \end{bmatrix}\right]^{-1} \\ &= \left[\begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}^2\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \end{bmatrix}\right]^{-1} = -\frac{1}{2\,340}\begin{bmatrix} 2, & 7 \\ 362, & 97 \end{bmatrix}, \\ \mathbf{Q} &= \mathbf{L}_4\mathbf{L}_2\begin{bmatrix} 0 \\ K_1 \end{bmatrix} + \begin{bmatrix} 4K'_6-(K'_5+K_6) \\ -K'_6 \end{bmatrix} - \mathbf{L}_4\mathbf{P}_2 - \mathbf{P}_4 \\ &= \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\mu + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\mu - \begin{bmatrix} -1, & -4 \\ 4, & 15 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}\mu - \begin{bmatrix} -13 \\ 42 \end{bmatrix}\mu \\ &= \begin{bmatrix} -31 \\ 129 \end{bmatrix}\mu, \end{aligned}\right\} \qquad (114)$$

---

*) For simplicity, the notation $\mu = \frac{1}{4} ql^2$ is adopted for use.

from which

$$\mathbf{N}=\begin{bmatrix}M_1\\M_7\end{bmatrix}=\mathbf{RQ}=\frac{\mu}{2\,340}\begin{bmatrix}-\ \ 841\\-1\,291\end{bmatrix}. \tag{115}$$

The eigenmatrix for the first span becomes by Eq. 74 as follows :

$$\mathbf{N}_1=\begin{bmatrix}M_1\\M_2\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}M_1-\begin{bmatrix}0\\K_1\end{bmatrix}=\frac{\mu}{2\,340}\begin{bmatrix}-841\\-658\end{bmatrix}. \tag{116}$$

The unknown support moments are determined by the right shift operation as follows :

$$\left.\begin{aligned}\mathbf{N}_3&=\mathbf{LN}_1+\mathbf{P}_2=\frac{\mu}{2\,340}\begin{bmatrix}-1\,207\\-6\,214\end{bmatrix},\\ \mathbf{N}_5&=\mathbf{LN}_3+\mathbf{P}_4=\frac{\mu}{2\,340}\begin{bmatrix}-4\,357\\242\end{bmatrix}.\end{aligned}\right\} \tag{117}$$

The values above obtained may be checked by the following shift operation

$$\mathbf{N}_6=\begin{bmatrix}M_6\\M_7\end{bmatrix}=\mathbf{LN}_4+\mathbf{P}_5=\frac{\mu}{2\,340}\begin{bmatrix}-1, & -4\\4, & 15\end{bmatrix}\begin{bmatrix}-6\,214\\-4\,357\end{bmatrix}+\begin{bmatrix}-10\\38\end{bmatrix}\mu$$

$$=\frac{\mu}{2\,340}\begin{bmatrix}242\\-1\,291\end{bmatrix}.\quad \text{O.K.} \tag{118}$$

CONCLUSIONS

In conclusion, the following notes are given :

1. The exact solution for all kinds of continuous beam can be obtained by simple matrix algebra.
2. The bending moment at any support of a continuous beam can be determined directly.
3. Compared with the moment distribution method, this method has a higher efficiency and perfect exactness. The efficiency will grow greater and greater as the system become complicated.
4. The computation can be designed with the simultaneous checking method.
5. The calculated result is checked arbitrarily and effectively by the right or left shiftors.

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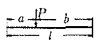
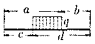
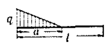
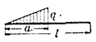
APPENDIX I. LOAD TERM

Table II. LOAD TERM FOR OPERATIONAL METHOD

Loading Condition	K	K'
	$\frac{ab}{l^2}(l+b)P$	$\frac{ab}{l^2}(l+a)P$
	$\frac{1}{4}ql^2$	$\frac{1}{4}ql^2$
	$\frac{qa^2}{4l^2}(2l-a)^2$	$\frac{qa^2}{4l^2}(2l^2-a^2)$
	$\frac{q}{4l^2}(d^2-b^2)(2l^2-b^2-d^2)$	$\frac{q}{4l^2}(a^2-c^2)(2l^2-a^2-c^2)$
	$\frac{7}{60}ql^2$	$\frac{8}{60}ql^2$
	$\frac{ql^2}{60}\left(1+\frac{b}{l}\right)\left(7-3\frac{b^2}{l^2}\right)$	$\frac{ql^2}{60}\left(1+\frac{a}{l}\right)\left(7-3\frac{a^2}{l^2}\right)$
	$\frac{qa^2}{60l^2}(3a^2-15al+20l^2)$	$\frac{qa^2}{60l^2}(10l^2-3a^2)$
	$\frac{qa^2}{60l^2}(12a^2-45al+40l^2)$	$\frac{4qa^2}{60l^2}(5l^2-3a^2)$
	$\frac{l^2}{60}(8q_a+7q_b)$	$\frac{l^2}{60}(7q_a+8q_b)$
	$\frac{q}{4l}(l^3-2a^2l+a^3)$	$\frac{q}{4l}(l^3-2a^2l+a^3)$

Note: The following relations are seen between the load terms in the "Operational Method" and the "Slope Deflection Method":[3)]

$$K = -2H_{AB}, \qquad K' = 2H_{BA}.$$

APPENDIX II. VARIOUS FORMS OF SOLUTION FOR ALL KINDS OF CONTINUOUS BEAM

The solutions for all possible cases of continuous beams are collected and classified in the following Tables III, through VIII.

Using together with Table I, the analysis of all kinds of continuous beams can be carried readily and systematically.

Table III. SIMPLE~SIMPLE

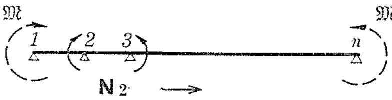
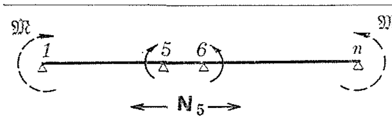
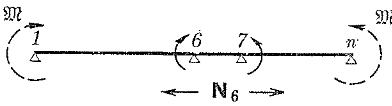
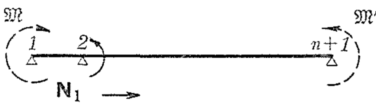
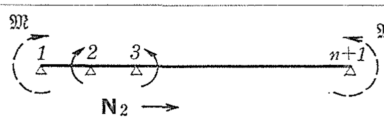
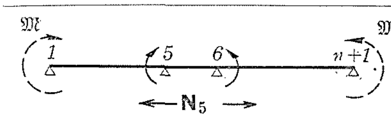
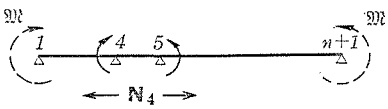
Shifting Procedure	N	R
 Solution 1.	$\begin{bmatrix} M_2 \\ M_{n-1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{bmatrix}^{-1}$
 Solution 2.	$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix}$	$-\begin{bmatrix} 2(1+k_2), & k_2 \\ \lfloor 1, & 2(1+k_{n-1})\rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_3 \end{bmatrix}^{-1}$
 Solution 3.	$\begin{bmatrix} M_5 \\ M_6 \\ M_2 \\ M_{n-1} \end{bmatrix}$	$\begin{bmatrix} -\mathbf{L}'_2\mathbf{L}'_4, & \begin{matrix} 0, & 0 \\ 1, & 0 \end{matrix} \\ -\mathbf{L}_{n-2}\cdots\mathbf{L}_6, & \begin{matrix} 0, & 1 \\ 0, & 0 \end{matrix} \end{bmatrix}^{-1}$
 Solution 4.	$\begin{bmatrix} M_6 \\ M_7 \end{bmatrix}$	$-\begin{bmatrix} \lfloor 2(1+k_2), \; k_2 \rfloor \mathbf{L}'_3\mathbf{L}'_5 \\ \lfloor 1, \; 2(1+k_{n-1})\rfloor \mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_7 \end{bmatrix}^{-1}$

Table IV. SIMPLE~SIMPLE

Shifting Procedure	N	R
 Solution 5.	$[M_2]$	$-\begin{bmatrix} \lfloor 1, \; 2(1+k_n)\rfloor \mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}^{-1}$
 Solution 6.	$\begin{bmatrix} M_2 \\ M_3 \\ M_n \end{bmatrix}$	$-\begin{bmatrix} 2(1+k_2), \; k_2, & 0 \\ & 1 \\ -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3, & 0 \end{bmatrix}^{-1}$
 Solution 7.	$\begin{bmatrix} M_2 \\ M_5 \\ M_6 \end{bmatrix}$	$-\begin{bmatrix} 0, & -\mathbf{L}'_2\mathbf{L}'_4 \\ 1, & \\ 0, & \lfloor 1, \; 2(1+kn)\rfloor \mathbf{L}_{n-2}\cdots\mathbf{L}_6 \end{bmatrix}^{-1}$
 Solution 8.	$\begin{bmatrix} M_4 \\ M_5 \\ M_n \end{bmatrix}$	$-\begin{bmatrix} \lfloor 2(1+k_2), k_2\rfloor \mathbf{L}'_3, & 0 \\ & 1 \\ -\mathbf{L}_{n-1}\cdots\mathbf{L}_5, & 0 \end{bmatrix}^{-1}$

(for Odd-Number Spans)

M	Q
$\begin{bmatrix} -\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}\mathfrak{M}\\ 0\end{bmatrix} \\ +\begin{bmatrix}0\\ \mathfrak{M}'\end{bmatrix}\end{bmatrix}$	$-\left[\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots)+\mathbf{P}_{n-4}]+\mathbf{P}_{n-2}\right]$
$\begin{bmatrix}\mathfrak{M}\\ \mathfrak{M}'k_{n-1}\end{bmatrix}$	$\begin{bmatrix}(K'_1+K_2k_2)\\ (K'_{n-2}+K_{n-1}k_{n-1}) \\ +\lfloor 1,\ 2(1+k_{n-1})\rfloor\left[\mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-5}]+\mathbf{P}_{n-3}\right]\end{bmatrix}$
$\begin{bmatrix}-\mathfrak{M}\\ 0\\ 0\\ -\mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}\mathbf{L}'_2\mathbf{P}'_4+\mathbf{P}'_2\\ \mathbf{L}_{n-2}\left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_6\cdots)+\mathbf{P}_{n-4}\right]+\mathbf{P}_{n-2}\end{bmatrix}$
$\begin{bmatrix}\mathfrak{M}\\ \mathfrak{M}'k_{n-1}\end{bmatrix}$	$\begin{bmatrix}(K'_1+K_2k_2)\\ (K'_{n-2}+K_{n-1}k_{n-1}) \\ +2\lfloor(1+k_2),k_2\rfloor[\mathbf{L}'_3\mathbf{P}'_5+\mathbf{P}'_3] \\ +\lfloor 1,2(1+k_{n-1})\rfloor\left[\mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_7\cdots)+\mathbf{P}_{n-5}]+\mathbf{P}_{n-3}\right]\end{bmatrix}$

(for Even-Number Spans)

M	Q
$\begin{bmatrix}\lfloor 1,2(1+k_n)\rfloor\mathbf{L}_{n-2}\cdots\mathbf{L}_2 \\ \times\begin{bmatrix}\mathfrak{M}\\ 0\end{bmatrix}+\mathfrak{M}'k_n\end{bmatrix}$	$(K'_{n-1}+K_nk_n)$ $+\lfloor 1,\ 2(1+k_n)\rfloor\left[\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots)+\mathbf{P}_{n-4}]+\mathbf{P}_{n-2}\right]$
$\begin{bmatrix}\mathfrak{M}\\ 0\\ \mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}(K'_1+K_2k_2)\\ -\left[\mathbf{L}_{n-1}[\mathbf{L}_{n-3}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-3}]+\mathbf{P}_{n-1}\right]\end{bmatrix}$
$\begin{bmatrix}\mathfrak{M}\\ 0\\ \mathfrak{M}'k_n\end{bmatrix}$	$\begin{bmatrix}-(\mathbf{L}'_2\mathbf{P}'_4+\mathbf{P}'_2)\\ (K'_{n-1}+K_nk_n)+\lfloor 1,2(1+k_n)\rfloor\left[\mathbf{L}_{n-2}(\cdots\mathbf{P}_6\cdots)+\mathbf{P}_{n-2}\right]\end{bmatrix}$
$\begin{bmatrix}\mathfrak{M}\\ 0\\ \mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}(K'_1+K_2k_2)+\lfloor 2(1+k_2),k_2\rfloor\mathbf{P}'_3\\ -\left[\mathbf{L}_{n-1}[\mathbf{L}_{n-3}(\cdots\mathbf{P}_5\cdots)+\mathbf{P}_{n-3}]+\mathbf{P}_{n-1}\right]\end{bmatrix}$

Table V. CLAMP~SIMPLE

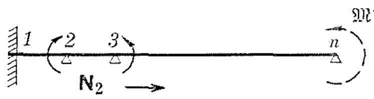
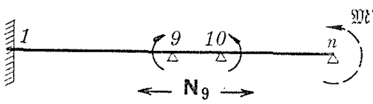
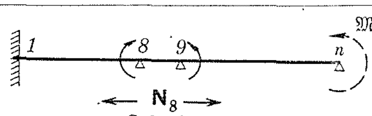
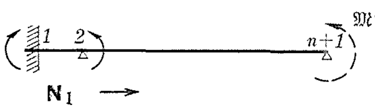
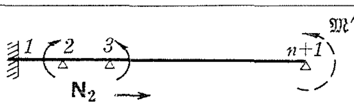
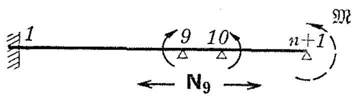
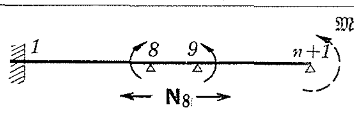
Shifting Procedure	N	R
 Solution 9.	$\begin{bmatrix} M_1 \\ M_{n-1} \end{bmatrix}$	$\left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right]^{-1}$
Solution 10a. 	$[M_1]$	$-\left[\lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\cdots\mathbf{L}_3 \begin{bmatrix} -2 \\ \dfrac{3+4k_2}{k_2} \end{bmatrix} \right]^{-1}$
Solution 10b.	$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$	$-\begin{bmatrix} 2, & 1, & 0 \\ -\dfrac{(3+4k_2)}{k_2}, & 0, & 1 \\ 0, & \multicolumn{2}{c}{\lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\cdots\mathbf{L}_3} \end{bmatrix}^{-1}$
 Solution 11.	$\begin{bmatrix} M_9 \\ M_{10} \\ M_1 \\ M_{n-1} \end{bmatrix}$	$-\begin{bmatrix} \mathbf{L}'_2\mathbf{L}'_4\mathbf{L}'_6\mathbf{L}'_8, & -1, & 0 \\ & 2, & 0 \\ & 0, & -1 \\ \mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_{10}, & 0, & 0 \end{bmatrix}^{-1}$
 Solution 12.	$\begin{bmatrix} M_1 \\ M_8 \\ M_9 \end{bmatrix}$	$-\begin{bmatrix} -2, & \\ \dfrac{3+4k_2}{k_2}, & -\mathbf{L}'_3\mathbf{L}'_5\mathbf{L}'_7 \\ 0, & \lfloor 1,\ 2(1+k_{n-1}) \rfloor \mathbf{L}_{n-3}\cdots\mathbf{L}_9 \end{bmatrix}^{-1}$

Table VI. CLAMP~SIMPLE

Shifting Procedure	N	R
Solution 13a. 	$[M_1]$	$-\left[\lfloor 1,\ 2(1+k_n) \rfloor \mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right]^{-1}$
Solution 13b.	$\begin{bmatrix} M_1 \\ M_{n-1} \\ M_n \end{bmatrix}$	$-\begin{bmatrix} -\mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, & 1, & 0 \\ & 0, & 1 \\ 0 \quad 1, & & 2(1+k_n) \end{bmatrix}^{-1}$
 Solution 14.	$\begin{bmatrix} M_2 \\ M_3 \\ M_1 \\ M_n \end{bmatrix}$	$-\begin{bmatrix} 1, & 0, & 2, & 0 \\ 0, & 1, & -\dfrac{(3+4k_2)}{k_2}, & 0 \\ -\mathbf{L}_{n-1}\cdots\mathbf{L}_3 & & 0, & 1 \\ & & 0, & 0 \end{bmatrix}^{-1}$
 Solution 15.	$\begin{bmatrix} M_1 \\ M_9 \\ M_{10} \end{bmatrix}$	$-\begin{bmatrix} -1, & \mathbf{L}'_2\mathbf{L}'_4\mathbf{L}'_6\mathbf{L}'_8 \\ 2, & \\ 0, & \lfloor 1, 2(1+k_n) \rfloor \mathbf{L}_{n-2}\cdots\mathbf{L}_{10} \end{bmatrix}^{-1}$
 Solution 16.	$\begin{bmatrix} M_1 \\ M_n \\ M_8 \\ M_9 \end{bmatrix}$	$-\begin{bmatrix} -2, & 0, & -\mathbf{L}'_3\mathbf{L}'_5\mathbf{L}'_7 \\ \dfrac{3+4k_2}{k_2} & 0, & \\ 0, & 1, & -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_9 \\ 0, & 0, & \end{bmatrix}^{-1}$

(for Odd-Number Spans)

$\mathbf{M}$	$\mathbf{Q}$
$\begin{bmatrix}0\\ \mathfrak{M}'\end{bmatrix}$	$\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}0\\ K_1\end{bmatrix}-\left[\mathbf{L}_{n-2}\left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots)+\mathbf{P}_{n-4}\right]+\mathbf{P}_{n-2}\right]$
$[\mathfrak{M}'k_{n-1}]$	$\lfloor 1,\ 2(1+k_{n-1})\rfloor\left[\mathbf{L}_{n-3}\left[\mathbf{L}_{n-5}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-5}\right]+\mathbf{P}_{n-3}+\mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_3\begin{bmatrix}-K_1\\ \frac{2}{k_2}(1+k_2)K_1-\frac{1}{k_2}(K'_1+K_2k_2)\end{bmatrix}\right]+K'_{n-2}+K_{n-1}k_{n-1}$
$\begin{bmatrix}0\\ 0\\ \mathfrak{M}'k_{n-1}\end{bmatrix}$	$\begin{bmatrix}K_1\\ \frac{1}{k_2}(K'_1+K_2k_2)-\frac{2}{k_2}(1+k_2)K_1\\ (K'_{n-2}+K_{n-1}k_{n-1})+\lfloor 1,\ 2(1+k_{n-1})\rfloor\left[\mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-5}]+\mathbf{P}_{n-3}\right]\end{bmatrix}$
$\begin{bmatrix}0\\ 0\\ 0\\ -\mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}0\\ K_1\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}\mathbf{L}'_2[\mathbf{L}'_4(\mathbf{L}'_6\mathbf{P}'_8+\mathbf{P}'_6)+\mathbf{P}'_4]+\mathbf{P}'_2\\ \mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_{10}\cdots)+\mathbf{P}_{n-4}]+\mathbf{P}_{n-2}\end{bmatrix}$
$\begin{bmatrix}0\\ 0\\ \mathfrak{M}'k_{n-1}\end{bmatrix}$	$\begin{bmatrix}\begin{bmatrix}-K_1\\ \frac{2}{k_2}(1+k_2)K_1-\frac{1}{k_2}(K'_1+K_2k_2)\end{bmatrix}-\mathbf{L}'_3(\mathbf{L}'_5\mathbf{P}'_7+\mathbf{P}'_5)-\mathbf{P}'_3\\ (K'_{n-2}+K_{n-1}k_{n-1})+\lfloor 1,\ 2(1+k_{n-1})\rfloor\left[\mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_9\cdots)+\mathbf{P}_{n-5}]+\mathbf{P}_{n-3}\right]\end{bmatrix}$

(for Even-Number Spans)

$\mathbf{M}$	$\mathbf{Q}$
$[\mathfrak{M}'k_n]$	$\left[K'_{n-1}+K_nk_n+\lfloor 1,\ 2(1+k_n)\rfloor\left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}0\\ -K_1\end{bmatrix}+\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots)+\mathbf{P}_{n-4}]+\mathbf{P}_{n-2}\right]\right]$
$\begin{bmatrix}0\\ 0\\ \mathfrak{M}'k_n\end{bmatrix}$	$\begin{bmatrix}-\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2\begin{bmatrix}0\\ -K_1\end{bmatrix}-\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots)+\mathbf{P}_{n-4}]-\mathbf{P}_{n-2}\\ (K'_{n-1}+K_nk_n)\end{bmatrix}$
$\begin{bmatrix}0\\ 0\\ 0\\ \mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}K_1\\ \frac{K'_1+K_2k_2}{k_2}-\frac{2(1+k_2)K_1}{k_2}\\ -\mathbf{L}_{n-1}\left[\mathbf{L}_{n-3}(\cdots\mathbf{P}_3\cdots)+\mathbf{P}_{n-3}\right]-\mathbf{P}_{n-1}\end{bmatrix}$
$\begin{bmatrix}0\\ 0\\ \mathfrak{M}'k_n\end{bmatrix}$	$\begin{bmatrix}\mathbf{L}'_2[\mathbf{L}'_4(\mathbf{L}'_6\mathbf{P}'_8+\mathbf{P}'_6)+\mathbf{P}'_4]+\mathbf{P}'_2+\begin{bmatrix}0\\ K_1\end{bmatrix}\\ (K'_{n-1}+K_nk_n)+\lfloor 1,\ 2(1+k_n)\rfloor\left[\mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_{10}\cdots)+\mathbf{P}_{n-4}]+\mathbf{P}_{n-2}\right]\end{bmatrix}$
$\begin{bmatrix}0\\ 0\\ 0\\ \mathfrak{M}'\end{bmatrix}$	$\begin{bmatrix}\begin{bmatrix}-K_1\\ \frac{2(1+k_2)K_1}{k_2}-\frac{K'_1-K_2k_2}{k_2}\end{bmatrix}-\mathbf{L}'_3(\mathbf{L}'_5\mathbf{P}'_7+\mathbf{P}'_5)-\mathbf{P}'_3\\ -\mathbf{L}_{n-1}[\mathbf{L}_{n-3}(\cdots\mathbf{P}_9\cdots)+\mathbf{P}_{n-3}]-\mathbf{P}_{n-1}\end{bmatrix}$

Table VII. CLAMP～CLAMP

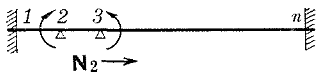
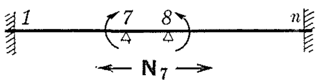
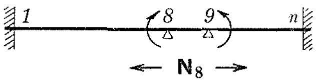
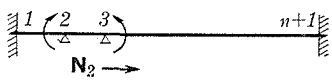
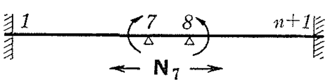
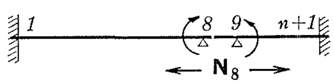
Shifting Procedure	N	R
 Solution 17.	$\begin{bmatrix} M_1 \\ M_n \end{bmatrix}$	$\left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right]^{-1}$
 Solution 18.	$\begin{bmatrix} M_1 \\ M_n \end{bmatrix}$	$\left[\mathbf{L}_{n-3}\mathbf{L}_{n-5}\cdots\mathbf{L}_3 \begin{bmatrix} -2 \\ \dfrac{3+4k_2}{k_2} \end{bmatrix}, \begin{bmatrix} -4-3k_{n-1} \\ 2 \end{bmatrix} \right]^{-1}$
 Solution 19.	$\begin{bmatrix} M_1 \\ M_n \\ M_7 \\ M_8 \end{bmatrix}$	$\begin{bmatrix} 1, & 0, & -\mathbf{L}'_2\mathbf{L}'_4\mathbf{L}'_6 \\ -2, & 0, & \\ 0, & -2, & -\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_8 \\ 0, & 1, & \end{bmatrix}^{-1}$
 Solution 20.	$\begin{bmatrix} M_1 \\ M_n \\ M_8 \\ M_9 \end{bmatrix}$	$\begin{bmatrix} -2, & 0, & -\mathbf{L}'_3\mathbf{L}'_5\mathbf{L}'_7 \\ \dfrac{3+4k_2}{k_2}, & 0, & \\ 0, & 4+3k_{n-1}, & -\mathbf{L}_{n-3}\cdots\mathbf{L}_9 \\ 0, & -2, & \end{bmatrix}^{-1}$

Table VIII. CLAMP～CLAMP

Shifting Procedure	N	R
 Solution 21.	$\begin{bmatrix} M_1 \\ M_{n+1} \end{bmatrix}$	$\left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4-3k_n \\ 2 \end{bmatrix} \right]^{-1}$
 Solution 22.	$\begin{bmatrix} M_1 \\ M_{n+1} \end{bmatrix}$	$\left[\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3 \begin{bmatrix} -2 \\ \dfrac{3+4k_2}{k_2} \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right]^{-1}$
 Solution 23.	$\begin{bmatrix} M_1 \\ M_{n+1} \\ M_7 \\ M_8 \end{bmatrix}$	$\begin{bmatrix} 1, & 2, & -\mathbf{L}'_2\mathbf{L}'_4\mathbf{L}'_6 \\ -2, & 0, & \\ 0, & (4+3k_n), & -\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_8 \\ 0, & -2, & \end{bmatrix}^{-1}$
 Solution 24.	$\begin{bmatrix} M_1 \\ M_{n+1} \\ M_8 \\ M_9 \end{bmatrix}$	$\begin{bmatrix} -2, & 0, & -\mathbf{L}'_3\mathbf{L}'_5\mathbf{L}'_7 \\ \dfrac{3+4k_2}{k_2}, & 0, & \\ 0, & -2, & -\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_9 \\ 0, & 1, & \end{bmatrix}^{-1}$

(for Odd-Number Spans)

$\mathbf{Q}$
$\left[\mathbf{L}_{n-2}\mathbf{L}_{n-4}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ K_1 \end{bmatrix} - \begin{bmatrix} K'_{n-1} \\ 0 \end{bmatrix} - \mathbf{L}_{n-2}\left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots) + \mathbf{P}_{n-4}\right] - \mathbf{P}_{n-2} \right]$
$\begin{bmatrix} 2(1+k_{n-1})K'_{n-1} - (K'_{n-2} + K_{n-1}k_{n-1}) \\ -K'_{n-1} \end{bmatrix} - \mathbf{L}_{n-3}\cdots\mathbf{L}_3 \begin{bmatrix} -K_1 \\ \frac{2(1+k_2)}{k_2}K_1 - \frac{K'_1 + K_2k_2}{k_2} \end{bmatrix} - [\mathbf{L}_{n-3}(\cdots\mathbf{P}_3\cdots) + \mathbf{P}_{n-3}]$
$\begin{bmatrix} 0 \\ K_1 \\ K'_{n-1} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{L}'_2(\mathbf{L}'_4\mathbf{P}'_6 + \mathbf{P}'_4) + \mathbf{P}'_2 \\ \mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_8\cdots) + \mathbf{P}_{n-4}] + \mathbf{P}_{n-2} \end{bmatrix}$
$\begin{bmatrix} K_1 \\ \frac{K'_1 + K_2k_2}{k_2} - \frac{2(1+k_2)}{k_2}K_1 \\ K'_{n-2} + K_{n-1}k_{n-1} - 2(1+k_{n-1})K'_{n-1} \\ K'_{n-1} \end{bmatrix} + \begin{bmatrix} \mathbf{L}'_3(\mathbf{L}'_5\mathbf{P}'_7 + \mathbf{P}'_5) + \mathbf{P}'_3 \\ \mathbf{L}_{n-3}[\mathbf{L}_{n-5}(\cdots\mathbf{P}_9\cdots) + \mathbf{P}_{n-5}] + \mathbf{P}_{n-3} \end{bmatrix}$

(for Even-Number Spans)

$\mathbf{Q}$
$\mathbf{L}_{n-2}\cdots\mathbf{L}_2 \begin{bmatrix} 0 \\ K_1 \end{bmatrix} + \begin{bmatrix} 2(1+k_n)K'_n - (K'_{n-1} + K_nk_n) \\ -K'_n \end{bmatrix} - \mathbf{L}_{n-2}[\mathbf{L}_{n-4}(\cdots\mathbf{P}_2\cdots) + \mathbf{P}_{n-4}] - \mathbf{P}_{n-2}$
$\mathbf{L}_{n-1}\mathbf{L}_{n-3}\cdots\mathbf{L}_3 \begin{bmatrix} K_1 \\ \frac{K'_1 + K_2k_2}{k_2} - \frac{2(1+k_2)}{k_2}K_1 \end{bmatrix} + \begin{bmatrix} -K'_n \\ 0 \end{bmatrix} - \mathbf{L}_{n-1}\left[\mathbf{L}_{n-3}(\cdots\mathbf{P}_3\cdots) + \mathbf{P}_{n-3}\right] - \mathbf{P}_{n-1}$
$\begin{bmatrix} 0 \\ K_1 \\ K'_{n-1} + K_nk_n - 2(1+k_n)K'_n \\ K'_n \end{bmatrix} + \begin{bmatrix} \mathbf{L}'_2(\mathbf{L}'_4\mathbf{P}'_6 + \mathbf{P}'_4) + \mathbf{P}'_2 \\ \mathbf{L}_{n-2}\left[\mathbf{L}_{n-4}(\cdots\mathbf{P}_8\cdots) + \mathbf{P}_{n-4}\right] + \mathbf{P}_{n-2} \end{bmatrix}$
$\begin{bmatrix} K_1 \\ \frac{K'_1 + K_2k_2}{k_2} - \frac{2(1+k_2)}{k_2}K_1 \\ K'_n \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{L}'_3(\mathbf{L}'_5\mathbf{P}'_7 + \mathbf{P}'_5) + \mathbf{P}'_3 \\ \mathbf{L}_{n-1}\left[\mathbf{L}_{n-3}(\cdots\mathbf{P}_9\cdots) + \mathbf{P}_{n-3}\right] + \mathbf{P}_{n-1} \end{bmatrix}$

APPENDIX III. NOTATION

The following symbols have been adopted for use in this paper :

r = support number countered from the extreme left support ;
n = any even number representing the number of constituent span of continuous beam ;
l_r = span length of r-th span countered from the extreme left span ;
M_r = bending moment at r-th support of continuous beam ;
I_r = moment of inertia of cross-section of beam in r-th span ;
I_c = standard moment of inertia ;
Z_r = section modulus of cross-section of beam in r-th span ;
E = modulus of elasticity ;
A_r = area of bending moment diagram of r-th span calculated as a simple beam for its loading ;
$\mathfrak{U}_r$, $\mathfrak{B}_r$ = reaction at the left or right support of r-th span calculated as a simple beam subjected to the moment diagram load A_r ;
w_r = settlement of r-th support ;
ε = coefficient of thermal expansion ;
h_r = depth of cross-section of beam in r-th span ;
ΔT_r = difference in temperature between the upper and lower sides of beam in r-th span ;
l'_r = $\frac{I_c}{I_r} l_r$, see Eq. 2a ;
α_r = $\frac{l_r}{l_{r-1}}$, see Eq. 2b ;
k_r = $\frac{l'_r}{l'_{r-1}}$, see Eq. 2c ;
K_r, K'_r = load term, see Eqs. 4a, 4b, and Table II ;
δ_r = $\frac{6EI_r}{l^2_r} k_r$, see Eq. 4c ;
β = $\frac{3}{2} E\varepsilon$, see Eq. 4d ;
$\mathbf{N}_r$ = "eigenmatrix," see Eq. 10 ;
$\mathbf{C}_r$ = connection matrix representing the connection conditions between two eigenmatrices $\mathbf{N}_{r-1}$ and $\mathbf{N}_{r+1}$, see Eq. 9 ;
$\mathbf{L}_r$, $\mathbf{L}'_r$ = 2-by-2 matrices called the "shift operator" between two physical quantities $\mathbf{N}_{r-1}$ and $\mathbf{N}_{r+1}$, see Eqs. 12, and 13 ;
$\mathbf{E}$ = 2-by-2 unit matrix ;
$\mathfrak{M}$, $\mathfrak{M}'$ = given external edge moments at the left and right extreme

supports of a continuous beam;

$\mathbf{K}_r$ = 2-by-1 matrix representing external loading conditions, see Eq. 58;

$\mathbf{P}_r$, $\mathbf{P}'_r$ = shift operators consisting of 2-by-1 matrix, representing external loading condition, see Eqs. 60;

$\mathbf{S}_r$, $\mathbf{S}'_r$ = shift operators consisting of 2-by-1 matrix, representing influence of settlement of support, see Eqs. 81, and 82;

$\mathbf{T}_r$, $\mathbf{T}'_r$ = shift operators consisting of 2-by-1 matrix, representing influence of temperature change, see Eqs. 83, and 84;

$\mathbf{R}$ = "premultiplier," see Eq. 93, Table I, and Appendix II;

$\mathbf{M}$ = "edge moment matrix," see Eq. 93, Table I, and Appendix II;

$\mathbf{Q}$ = "load matrix," see Eq. 93, Table I, and Appendix I;

$\lambda = \frac{ql^2}{60}$, see Example 2;

$\mu = \frac{ql^2}{4}$, see Example 3;

$\lfloor \quad \rfloor$ = row vector; and

$\{ \quad \}$ = column vector.

ERRATA

PAGE	LINE	FOR	READ
8	8	**Spans**	**Spans.**
13	5	beam subjected	beam is subjected
13	7	a	the
13	Eq. 54a	$M_{r+1}+\cdots$	$M_{r-1}+\cdots$
19	Eq. 77	K'_n	$K'_n,$
20	Eq. 83	$Z_r\Delta T_r k_r(1+k_{r+1})$	$Z_r\Delta \mathrm{T}_r k_r)(1+k_{r+1})$
21	Eq. 92	$\rfloor,$	$\rfloor.$
29	5	operation	operation: