On the Elastic Waves with the Discrepancy between Tangential Displacements at the Junction of Two-Dimensional Layered Media

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Synopsis. In this article, we propose the existence of some dispersive *Rayleigh*-waves propagated within the superficial layer of two-dimensional layered media. They are not only *M*-waves and M_2-waves taken as the unique solutions whenever some physical constants are given, but also the other elastic waves propagated with some slide at the junction. If we regard *M*-waves and M_2-waves with no slide whatever at the junction as the upper limit, we may regard some elastic waves with such slide that makes the strain-energy of the whole elastic system minimum as the lower limit. In this article, we shall find the lower limit and illustrate the solutions of such elastic waves are not unique but exist in certain ranges.

1. Introduction

In the historical studies of elastic waves, we can find the studies that RAYLEIGH gave in 1887 the elastic waves propagated over the surface of semi-infinite medium without any layer with a velocity and the waves of this kind are called "*Rayleigh*-waves," and that LOVE published in 1911 the waves propagated within the superficial layer of semi-infinite layered medium whose particle each was horizontally moved in normal to the progressed direction, and he illustrated the dispersion, that is, the wave-velocity increased as the wave-length increased. The waves of this kind are called "*Love*-waves." In 1924, STONELEY researched any kind of waves which were called "*Stoneley*-waves" and propagated along the bottom of the ocean. In 1927, SEZAWA showed the plane waves of *Rayleigh*-wave type propagated within the layer of semi-infinite layered medium, and he called them the dispersive *M*-waves. In 1935, SEZAWA and KANAI published M_2-waves which were not the higher order of *M*-waves but the other waves, and then full paticulars were also given to these natures.

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Thus we have been able to give many elastic waves investigated in the past, but all these have been solved under the continuous conditions, that is, at the junction the superficial layer and the lower semi-infinite medium are not strictly separated and no slide is allowed to either of them. TANIMOTO has maintained the other idea with respect to the continuous conditions of such contact problems, namely, though any separation in the vertical direction at the junction is not caused tentatively, we can not affirm that any discrepancy in the tangential direction which takes place so that the strain-energy of the whole system may be minimum is not always caused. If we apply this idea to the problems of elastic waves, we shall not have M-waves and M_2-waves as the solutions, but we shall have the other elastic waves propagated causing some slide and allowing some discrepancy of tangential displacements. This idea has by far more universality as one of the continuous conditions.

Thus we shall appoint the new lower limit called M_u-waves and M_{u_2}-waves if we assume M-waves and M_2-waves as upper limit. As these waves are treated as the two-dimensional problems, we can not criticize three-dimensional seismic waves, but basing on the idea, if we apply it to the analysis of seismic waves, we hope that we may explain the observed values that have not yet been plotted on the analytical curves. We shall make any slide cause at the junction so that the whole strain-energy may be minimum and research analytically the limit waves, but we suppose that in actual seismic phenomena, the superficial layer has not slided perfectly till the strain-energy is minimum, but many phenomena caused the slide on the way and in the higher states of strain-energy have been observed.

In this article, we shall illustrate theoretically the two-dimensional limit waves as one step to reach the analysis of three-dimensional seismic waves and show some numerical calculations. And we shall propose that the solutions of elastic waves propagated within the superficial layer of two-dimensional layered medium have not the uniqueness but rather an extensive range. Let these matters cover with a fragment of some elastic waves existing under suitable conditions as LOVE expressed in "Some Problems of Geodynamics."

2. Equations of Motion

In the case of the plane waves propagated along the surface as given in Fig. 1, where the semi-infinite medium with the density ρ and the elastic constants λ and μ is covered by the superficial layer which has the uniform thickness H_0, the density ρ' and the elastic constants λ' and μ', the equations of motion of semi-infinite medium may be expressed in the forms

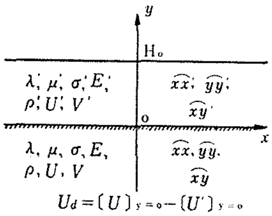

Fig. 1.

$$\left.\begin{aligned} \rho\frac{\partial^2 \varDelta}{\partial t^2} &= (\lambda + 2\mu)\left(\frac{\partial^2 \varDelta}{\partial x^2} + \frac{\partial^2 \varDelta}{\partial y^2}\right), \\ \rho\frac{\partial^2 \varOmega}{\partial t^2} &= \mu\left(\frac{\partial^2 \varOmega}{\partial x^2} + \frac{\partial^2 \varOmega}{\partial y^2}\right), \end{aligned}\right\} \tag{1}$$

where

$$\varDelta = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \quad 2\varOmega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y},$$

in which U and V denote the components of displacement in the directions of x- and y-axis, respectively. If we entirely neglect the effect of gravity and apply only the harmonic waves propagated within the media, we take as the solutions of equations (1)

$$\varDelta = Ae^{ry+i(pt-fx)}, \tag{2}$$

$$2\varOmega = Be^{sy+i(pt-fx)}, \tag{3}$$

in which

$$r^2 = f^2 - h^2, \qquad s^2 = f^2 - k^2, \qquad h^2 = \frac{\rho p^2}{\lambda + 2\mu}, \qquad k^2 = \frac{\rho p^2}{\mu}, \tag{4}$$

then A and B denote the determining coefficients, respectively, and $L = \frac{2\pi}{f}$ is the wave-length and $T = \frac{2\pi}{p}$ is the period. These equations (2) and (3) show that two kinds of waves are propagated. If U_1 and V_1 denote the components of displacement that satisfy the equation (2), we take

$$\left.\begin{aligned} U_1 &= \frac{if}{h^2}Ae^{ry+i(pt-fx)}, \\ V_1 &= -\frac{r}{h^2}Ae^{ry+i(pt-fx)}, \end{aligned}\right\} \tag{5}$$

if U_2 and V_2 denote the components of displacement that satisfy the equation (3), we take

$$\left.\begin{aligned} U_2 &= \frac{s}{k^2}Be^{sy+i(pt-fx)}, \\ V_2 &= \frac{if}{k^2}Be^{sy+i(pt-fx)}, \end{aligned}\right\} \tag{6}$$

the former are the equations of motion due to the longitudinal waves and the latter the equations of motion due to the transverse waves.

The equations of motion of the superficial layer, which has the uniform thickness and the physical properties to differ from the semi-infinite medium, may be expressed in the forms

$$\left.\begin{aligned} \rho'\frac{\partial^2 \Delta'}{\partial t^2} &= (\lambda' + 2\mu')\Big(\frac{\partial^2 \Delta'}{\partial x^2} + \frac{\partial^2 \Delta'}{\partial y^2}\Big), \\ \rho'\frac{\partial^2 \Omega'}{\partial t^2} &= \mu'\Big(\frac{\partial^2 \Omega'}{\partial x^2} + \frac{\partial^2 \Omega'}{\partial y^2}\Big), \end{aligned}\right\} \tag{7}$$

where

$$\Delta' = \frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y}, \qquad 2\Omega' = \frac{\partial V'}{\partial x} - \frac{\partial U'}{\partial y},$$

then U' and V' denote the components of displacement in the directions of x– and y–axis. Hence the solutions of equations (7), if we neglect the effect of gravity, are expressed by the equations

$$\left.\begin{aligned} \Delta' &= (C\cosh r'y + D\sinh r'y)e^{i(pt-fx)}, \\ 2\Omega' &= (E\cosh s'y + F\sinh s'y)e^{i(pt-fx)}, \end{aligned}\right\} \tag{8}$$

in which

$$r'^2 = f^2 - h'^2, \quad s'^2 = f^2 - k'^2, \quad h'^2 = \frac{\rho' p^2}{\lambda' + 2\mu'}, \quad k'^2 = \frac{\rho' p^2}{\mu'}; \tag{9}$$

thene C, D, E and F denote the determining coefficients, respectively. In the solutions of equations (8), the components of displacement due to the longitudinal waves are expressed in the forms

$$\left.\begin{aligned} U_1' &= \frac{if}{h'^2}(C\cosh r'y + D\sinh r'y)e^{i(pt-fx)}, \\ V_1' &= -\frac{r'}{h'^2}(C\sinh r'y + D\cosh r'y)e^{i(pt-fx)}, \end{aligned}\right\} \tag{10}$$

and the components of displacement due to the transverse waves are expressed in the forms

$$U_2' = \frac{s'}{k'^2}(E\sinh s'y + F\cosh s'y)e^{i(pt-fx)},$$

$$V_2' = \frac{if}{k'^2}(E\cosh s'y + F\sinh s'y)e^{i(pt-fx)}. \qquad (11)$$

As these components of displacement caused due to the longitudinal and transverse waves have been superposed actually, we should treat the total displacements, which are respectively given by the equations, for the semi-infinite medium:

$$U = U_1 + U_2, \qquad V = V_1 + V_2, \qquad (12)$$

and for the superficial layer:

$$U' = U_1' + U_2', \qquad V' = V_1' + V_2'. \qquad (13)$$

And the stress-components in the elastic systems may be expressed in the forms, for the semi-infinite medium:

$$\widehat{xx} = \lambda\Delta + 2\mu\frac{\partial U}{\partial x}, \qquad \widehat{yy} = \lambda\Delta + 2\mu\frac{\partial V}{\partial y}, \qquad \widehat{xy} = \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right), \qquad (14)$$

and for the superficial leyer:

$$\widehat{xx}' = \lambda'\Delta' + 2\mu'\frac{\partial U'}{\partial x}, \quad \widehat{yy}' = \lambda'\Delta' + 2\mu'\frac{\partial V'}{\partial y}, \quad \widehat{xy}' = \mu'\left(\frac{\partial U'}{\partial y} + \frac{\partial V'}{\partial x}\right). \quad (15)$$

Now, if the harmonic vibration of $Pe^{i(pt-fx)}$ is uniformlly applied over the surface of these elastic systems, we have at the surface ($y = H_0$) as the boundary conditions:

$$\widehat{yy}' = Pe^{i(pt-fx)}, \qquad \widehat{xy}' = 0, \qquad (16)$$

at the junction ($y = 0$) as the continuous conditions:

$$U = U' + U_d, \quad V = V', \quad \widehat{yy} = \widehat{yy}', \quad \widehat{xy} = \widehat{xy}'. \qquad (17)$$

We obtain six equations for determining coefficients A, B, C, D, E and F from the equations (5), (6), (10), (11), (12), (13), (14), (15), (16) and (17), and when we solve these, we shall get six determining coefficients which consist of both parts of the discrepancy of tangential displacements U_d and the harmonic motion $Pe^{i(pt-fx)}$.

A	B	C	D
$i\frac{f}{h^2}$	$\frac{sf}{k^2}$	$-i\frac{f}{h'^2}$	0
$-\frac{rf}{h^2}$	$i\frac{f^2}{k^2}$	0	$\frac{r'f}{h'^2}$
$2i\frac{rf}{h^2}$	$\frac{f^2+s^2}{k^2}$	0	$-2i\frac{r'f}{h'^2}\frac{\mu'}{\mu}$
$\frac{\lambda}{\mu}-\frac{2r^2}{h^2}$	$2i\frac{sf}{k^2}$	$-\left(\frac{\lambda'}{\mu}-2\frac{r'^2}{h'^2}\frac{\mu'}{\mu}\right)$	0
0	0	$\left(\frac{\lambda'}{\mu}-2\frac{r'^2}{h'^2}\frac{\mu'}{\mu}\right)X_1$	$\left(\frac{\lambda'}{\mu}-2\frac{r'^2}{h'^2}\frac{\mu'}{\mu}\right)X_2$
0	0	$2i\frac{r'f}{h'^2}X_2$	$2i\frac{r'f}{h'^2}X_1$

where

$$X_1 = \cosh r'H_0, \quad Y_1 = \sinh r'H_0, \quad X_2 = \cosh s'H_0, \quad Y_2 = \sinh s'H_0.$$

Now, the velocities of longitudinal and transverse waves are expressed in the forms, for the semi-infinite medium :

$$C_p = \sqrt{\frac{\lambda+2\mu}{\rho}}, \qquad C_s = \sqrt{\frac{\mu}{\rho}}, \tag{19}$$

and for the superficial layer:

$$C_p' = \sqrt{\frac{\lambda'+2\mu'}{\rho'}}, \qquad C_s' = \sqrt{\frac{\mu'}{\rho'}}, \tag{20}$$

where if we assume

$$\lambda = \mu, \qquad \lambda' = \mu', \qquad m = \frac{\rho'}{\rho}, \qquad n = \frac{\mu'}{\mu},$$

we have

$$C_p = C_s'\sqrt{\frac{3m}{n}}, \qquad C_s = C_s'\sqrt{\frac{m}{n}}, \qquad C_p' = C_s'\sqrt{3} \tag{21}$$

and the equations (4) and (9) may be expressed in the forms

E	F	$= \text{const.}$
0	$-\dfrac{s'f}{k'^2}$	$fU_d e^{-i(pt-fx)}$
$-i\dfrac{f^2}{k'^2}$	0	0
$-\dfrac{(f^2+s'^2)}{k'^2}\dfrac{\mu'}{\mu}$	0	0
0	$-2i\dfrac{s'f}{k'^2}\dfrac{\mu'}{\mu}$	0
$2i\dfrac{s'f\mu'}{k'^2\mu}Y_2$	$2i\dfrac{s'f\mu'}{k'^2\mu}Y_1$	$\dfrac{P}{\mu}$
$\dfrac{(f^2+s'^2)}{k'^2}Y_1$	$\dfrac{(f^2+s'^2)}{k'^2}Y_2$	0

(18)

$$\left.\begin{aligned} &\frac{h^2}{f^2}=\frac{n}{3m}v_0^2, \quad \frac{k^2}{f^2}=\frac{n}{m}v_0^2, \quad \frac{r^2}{f^2}=1-\frac{n}{3m}v_0^2, \quad \frac{s^2}{f^2}=1-\frac{n}{m}v_0^2, \\ &\frac{h'^2}{f^2}=\frac{1}{3}v_0^2, \quad \frac{k'^2}{f^2}=v_0^2, \quad \frac{r'^2}{f^2}=1-\frac{1}{3}v_0^2, \quad \frac{s'^2}{f^2}=1-v_0^2, \end{aligned}\right\} \tag{22}$$

in which

$$v_0=\frac{p}{f}\Big/C_s',$$

then p/f is the velocity of propagation of waves in the direction of x-axis.

In the case of solving the equations (18), when we put $P=0$, we can take six determining coefficients with respect to U_d in the forms

$$\left.\begin{aligned} &A_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_a, \quad B_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_b, \quad C_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_c, \\ &D_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_d, \quad E_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_e, \quad F_u=\frac{fU_d e^{-i(pt-fx)}}{\mathfrak{D}}\phi_f, \end{aligned}\right\} \tag{23}$$

if we put $U_d=0$, we can take them with respect to P in the forms

$$\left.\begin{aligned} A_P &= \frac{P}{\mu\mathfrak{D}}\varphi_a, & B_P &= \frac{P}{\mu\mathfrak{D}}\varphi_b, & C_P &= \frac{P}{\mu\mathfrak{D}}\varphi_c, \\ D_P &= \frac{P}{\mu\mathfrak{D}}\varphi_d, & E_P &= \frac{P}{\mu\mathfrak{D}}\varphi_e, & F_P &= \frac{P}{\mu\mathfrak{D}}\varphi_f. \end{aligned}\right\} \quad (24)$$

Hence, the determining coefficients A, B, C, D, E and F are expressed by superposing respectively equations (23) and (24) in the forms

$$\left.\begin{aligned} A &= \frac{1}{\mathfrak{D}}\left\{\varphi_a\frac{P}{\mu} + \phi_a f U_d e^{-i(pt-fx)}\right\}, \\ B &= \frac{1}{\mathfrak{D}}\left\{\varphi_b\frac{P}{\mu} + \phi_b f U_d e^{-i(pt-fx)}\right\}, \\ C &= \frac{1}{\mathfrak{D}}\left\{\varphi_c\frac{P}{\mu} + \phi_c f U_d e^{-i(pt-fx)}\right\}, \\ D &= \frac{1}{\mathfrak{D}}\left\{\varphi_d\frac{P}{\mu} + \phi_d f U_d e^{-i(pt-fx)}\right\}, \\ E &= \frac{1}{\mathfrak{D}}\left\{\varphi_e\frac{P}{\mu} + \phi_e f U_d e^{-i(pt-fx)}\right\}, \\ F &= \frac{1}{\mathfrak{D}}\left\{\varphi_f\frac{P}{\mu} + \phi_f f U_d e^{-i(pt-fx)}\right\}, \end{aligned}\right\} \quad (25)$$

in which

$$\mathfrak{D} = 4n(2-v_0^2)\left\{\left(2-2n-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)\right.$$

$$\left. -2(1-n)(2-2n+nv_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}$$

$$+n\left[-(2-v_0^2)^2\left(2-2n-\frac{n}{m}v_0^2\right)^2-4\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)^2+4\{(1-n)^2(2-v_0^2)^2\right.$$

$$\left.+(2-2n+nv_0^2)^2\}\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right]\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}\cosh r'H_0\cosh s'H_0$$

$$+\frac{n^3}{m}v_0^4\left\{4(1-v_0^2)\sqrt{1-\frac{n}{3m}v_0^2}-(2-v_0^2)^2\sqrt{1-\frac{n}{m}v_0^2}\right\}\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\sinh s'H_0$$

$$+\frac{n^3}{m}v_0^4\left\{4\left(1-\frac{v_0^2}{3}\right)\sqrt{1-\frac{n}{m}v_0^2}-(2-v_0^2)^2\sqrt{1-\frac{n}{3m}v_0^2}\right\}\sqrt{1-v_0^2}\sinh r'H_0\cosh s'H_0$$

$$+n\Bigg[(2-v_0^2)^2\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)^2+4(1-v_0^2)\left(1-\frac{v_0^2}{3}\right)\left(2-2n-\frac{n}{m}v_0^2\right)$$

$$-\left\{16(1-n)(1-v_0^2)\left(1-\frac{v_0^2}{3}\right)+(2-v_0^2)^2(2-2n+nv_0^2)^2\right\}$$

$$\times\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\Bigg]\sinh r'H_0\sinh s'H_0, \qquad (26)$$

$$\phi_a=i\frac{n^3}{3m}v_0^2\Bigg[2(2-v_0^2)\left\{\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)+(2-v_0^2)\left(2-2n-\frac{n}{m}v_0^2\right)\right\}$$

$$\times(1-\cosh r'H_0\cosh s'H_0)$$

$$+2v_0^2(2-v_0^2)^2\sqrt{1-\frac{n}{m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\sinh s'H_0$$

$$-8v_0^2\left(1-\frac{v_0^2}{3}\right)\sqrt{1-\frac{n}{m}v_0^2}\sqrt{1-v_0^2}\sinh r'H_0\cosh s'H_0$$

$$+\left\{(2-v_0^2)^3\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)+8(1-v_0^2)\left(1-\frac{v_0^2}{3}\right)\right.$$

$$\left.\times\left(2-2n-\frac{n}{m}v_0^2\right)\right\}\sinh r'H_0\sinh s'H_0\Bigg], \qquad (27)$$

$$\phi_b=\frac{n^3}{m}v_0^2\Bigg[4(2-v_0^2)\{(1-n)(2-v_0^2)+(2-2n+nv_0^2)\}\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}$$

$$\times\sqrt{1-v_0^2}(1-\cosh r'H_0\cosh s'H_0)$$

$$+v_0^2(2-v_0^2)^2\left(2-\frac{n}{m}v_0^2\right)\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\sinh s'H_0$$

$$-4v_0^2\left(1-\frac{v_0^2}{3}\right)\left(2-\frac{n}{m}v_0^2\right)\sqrt{1-v_0^2}\sinh r'H_0\cosh s'H_0$$

$$+\left\{16(1-n)(1-v_0^2)\left(1-\frac{v_0^2}{3}\right)-(2-v_0^2)^3(2-2n+nv_0^2)\right\}$$

$$\left.\times \sinh r'H_0 \sinh s'H_0\right], \qquad (28)$$

$$\phi_c = i\frac{v_0^2}{3}\left[2n(2-v_0^2)\left\{\left(2-\frac{n}{m}v_0^2\right)\left(2-2n-\frac{n}{m}v_0^2\right)\right.\right.$$

$$\left.-4(1-n)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}$$

$$+4n\left\{-\left(2-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)\right.$$

$$\left.+2(2-2n+nv_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}\cosh r'H_0\cosh s'H_0$$

$$+8\frac{n^3}{m}v_0^2(1-v_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\sinh s'H_0$$

$$-2\frac{n^3}{m}v_0^2(2-v_0^2)^2\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-v_0^2}\,\sinh r'H_0\cosh s'H_0$$

$$+n(2-v_0^2)^2\left\{\left(2-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)-2(2-2n+nv_0^2)\right.$$

$$\left.\left.\times\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\sinh r'H_0\sinh s'H_0\right], \qquad (29)$$

$$\phi_d = i\frac{v_0^2}{3}\left[-2\frac{n^3}{m}v_0^2(2-v_0^2)^2\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-v_0^2}(1-\cosh r'H_0\cosh s'H_0)\right.$$

$$+n(2-v_0^2)^2\left\{-\left(2-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)+2(2-2n+nv_0^2)\right.$$

$$\left.\times\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\cosh r'H_0\sinh s'H_0$$

$$+4n\left\{\left(2-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)-2(2-n+nv_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}$$

$$\times\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}\sinh r'H_0\cosh s'H_0$$

$$-8\frac{n^3}{m}v_0^2(1-v_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}\sinh r'H_0\sinh s'H_0\Bigg], \tag{30}$$

$$\begin{aligned}
\phi_e = nv_0^2\Bigg[&4\frac{n^2}{m}v_0^2(2-v_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}(1-\cosh r'H_0\cosh s'H_0)\\
&+(2-v_0^2)^2\left\{-\left(2-\frac{n}{m}v_0^2\right)\left(2-2n-\frac{n}{m}v_0^2\right)+4(1-n)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\\
&\qquad\times\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\sinh s'H_0\\
&+4\left(1-\frac{v_0^2}{3}\right)\left\{\left(2-\frac{n}{m}v_0^2\right)\left(2-2n-\frac{n}{m}v_0^2\right)-4(1-n)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\\
&\qquad\times\sqrt{1-v_0^2}\sinh r'H_0\cosh s'H_0\\
&+\frac{n^2}{m}v_0^2(2-v_0^2)^3\sqrt{1-\frac{n}{3m}v_0^2}\sinh r'H_0\sinh s'H_0\Bigg],
\end{aligned} \tag{31}$$

$$\begin{aligned}
\phi_f = nv_0^2\Bigg[&2(2-v_0^2)\left\{-\left(2-\frac{n}{m}v_0^2\right)\left(2-2n+nv_0^2-\frac{n}{m}v_0^2\right)+2(2-2n+nv_0^2)\right.\\
&\qquad\left.\times\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\sqrt{1-\frac{v_0^2}{3}}\\
&+(2-v_0^2)^2\left\{\left(2-\frac{n}{m}v_0^2\right)\left(2-2n-\frac{n}{m}v_0^2\right)-4(1-n)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\\
&\qquad\times\sqrt{1-\frac{v_0^2}{3}}\cosh r'H_0\cosh s'H_0\\
&+4\frac{n^2}{m}v_0^2(2-v_0^2)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{v_0^2}{3}}\sqrt{1-v_0^2}\cosh r'H_0\sinh s'H_0\\
&-\frac{n}{m}v_0^2(2-v_0^2)^3\sqrt{1-\frac{n}{3m}v_0^2}\sinh r'H_0\cosh s'H_0\\
&+4\left(1-\frac{v_0^2}{3}\right)\left\{-\left(2-\frac{n}{m}v_0^2\right)\left(2-2n-\frac{n}{m}v_0^2\right)+4(1-n)\sqrt{1-\frac{n}{3m}v_0^2}\sqrt{1-\frac{n}{m}v_0^2}\right\}\\
&\qquad\times\sqrt{1-v_0^2}\sinh r'H_0\sinh s'H_0\Bigg],
\end{aligned} \tag{32}$$

$$\varphi_a = \frac{n^2}{3m} v_0^4 \Bigg\{ 2\left(2 - 2n + nv_0^2 - \frac{n}{m} v_0^2\right) \sqrt{1 - \frac{v_0^2}{3}} \sqrt{1 - v_0^2} \cosh r' H_0$$

$$- 4(1 - n)\left(1 - \frac{v_0^2}{3}\right) \sqrt{1 - \frac{n}{m} v_0^2} \sqrt{1 - v_0^2} \sinh r' H_0$$

$$- (2 - v_0^2)\left(2 - 2n - \frac{n}{m} v_0^2\right) \sqrt{1 - \frac{v_0^2}{3}} \sqrt{1 - v_0^2} \cosh s' H_0$$

$$+ (2 - v_0^2)(2 - 2n + nv_0^2) \sqrt{1 - \frac{n}{m} v_0^2} \sqrt{1 - \frac{v_0^2}{3}} \sinh s' H_0 \Bigg\}, \tag{33}$$

$$\varphi_b = i \frac{n^2}{m} v_0^4 \Bigg\{ - 2(2 - 2n + nv_0^2) \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - \frac{v_0^2}{3}} \sqrt{1 - v_0^2} \cosh r' H_0$$

$$+ 2\left(1 - \frac{v_0^2}{3}\right)\left(2 - 2n - \frac{n}{m} v_0^2\right) \sqrt{1 - v_0^2} \sinh r' H_0$$

$$+ 2(1 - n)(2 - v_0^2) \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - \frac{v_0^2}{3}} \sqrt{1 - v_0^2} \cosh s' H_0$$

$$- (2 - v_0^2)\left(2 - 2n + nv_0^2 - \frac{n}{m} v_0^2\right) \sqrt{1 - \frac{v_0^2}{3}} \sinh s' H_0 \Bigg\}, \tag{34}$$

$$\varphi_c = \frac{v_0^2}{3} \sqrt{1 - \frac{v_0^2}{3}} \Bigg[2 \Bigg\{ - \left(2 - 2n - \frac{n}{m} v_0^2\right)\left(2 - 2n + nv_0^2 - \frac{n}{m} v_0^2\right)$$

$$- 2(1 - n)(2 - 2n + nv_0^2) \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - \frac{n}{m} v_0^2} \Bigg\} \sqrt{1 - v_0^2} \cosh r' H_0$$

$$+ (2 - v_0^2) \Bigg\{ \left(2 - 2n - \frac{n}{m} v_0^2\right)^2 - 4(1 - n)^2 \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - \frac{n}{m} v_0^2} \Bigg\} \sqrt{1 - v_0^2} \cosh s' H_0$$

$$+ \frac{n^2}{m} v_0^4 (2 - v_0^2) \sqrt{1 - \frac{n}{m} v_0^2} \sinh s' H_0 \Bigg] \tag{35}$$

$$\varphi_d = \frac{v_0^2}{3} \Bigg[2 \Bigg\{ \left(2 - 2n - \frac{n}{m} v_0^2\right)\left(2 - 2n + nv_0^2 - \frac{n}{m} v_0^2\right) - 2(1 - n)(2 - 2n + nv_0^2)$$

$$\times \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - \frac{n}{m} v_0^2} \Bigg\} \sqrt{1 - \frac{v_0^2}{3}} \sqrt{1 - v_0^2} \sinh r' H_0$$

$$+ nv_0^4 (2 - v_0^2) \sqrt{1 - \frac{n}{3m} v_0^2} \sqrt{1 - v_0^2} \cosh s' H_0$$

$$+ (2 - v_0^2)\left\{-\left(2 - 2n + nv_0^2 - \frac{n}{m}v_0^2\right)^2\right.$$

$$\left.\left. + (2 - 2n + nv_0^2)^2\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{n}{m}v_0^2}\right\}\sinh s'H_0\right], \qquad (36)$$

$$\varphi_e = iv_0^2\left[-2\frac{n^2}{m}v_0^4\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{v_0^2}{3}}\sqrt{1 - v_0^2}\cosh r'H_0\right.$$

$$+ 2\left(1 - \frac{v_0^2}{3}\right)\left\{-\left(2 - 2n - \frac{n}{m}v_0^2\right)^2\right.$$

$$\left. + 4(1 - n)^2\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{n}{m}v_0^2}\right\}\sqrt{1 - v_0^2}\sinh r'H_0$$

$$+ (2 - v_0^2)\left\{\left(2 - 2n - \frac{n}{m}v_0^2\right)\left(2 - 2n + nv_0^2 - \frac{n}{m}v_0^2\right) - 2(1 - n)(2 - 2n + nv_0^2)\right.$$

$$\left.\left.\times\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{n}{m}v_0^2}\right\}\sqrt{1 - \frac{v_0^2}{3}}\cosh s'H_0\right], \qquad (37)$$

$$\varphi_f = iv_0^2\left[2\left\{\left(2 - 2n + nv_0^2 - \frac{n}{m}v_0^2\right)^2\right.\right.$$

$$\left. - (2 - 2n + nv_0^2)^2\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{n}{m}v_0^2}\right\}\sqrt{1 - \frac{v_0^2}{3}}\cosh r'H_0$$

$$+ 2\frac{n^2}{m}v_0^4\left(1 - \frac{v_0^2}{3}\right)\sqrt{1 - \frac{n}{m}v_0^2}\cosh s'H_0$$

$$+ (2 - v_0^2)\left\{-\left(2 - 2n - \frac{n}{m}v_0^2\right)\left(2 - 2n + nv_0^2 - \frac{n}{m}v_0^2\right) + 2(1 - n)(2 - 2n + nv_0^2)\right.$$

$$\left.\left.\times\sqrt{1 - \frac{n}{3m}v_0^2}\sqrt{1 - \frac{n}{m}v_0^2}\right\}\sqrt{1 - \frac{v_0^2}{3}}\sinh s'H_0\right]. \qquad (38)$$

Hence the components of displacement may be written in the forms, for the semi-infinite medium:

$$U = \frac{1}{\mathfrak{D}}\left\{\left(\frac{if}{h^2}\varphi_a e^{ry} + \frac{s}{k^2}\varphi_b e^{sy}\right)\frac{P}{\mu}e^{i(pt-fx)} + \left(\frac{if}{h^2}\phi_a e^{ry} + \frac{s}{k^2}\phi_b e^{sy}\right)fU_d\right\}, \qquad (39)$$

$$V=\frac{1}{\mathfrak{D}}\Big\{\Big(-\frac{r}{h^2}\varphi_a e^{ry}+\frac{if}{k^2}\varphi_b e^{sy}\Big)\frac{P}{\mu}e^{i(pt-fx)}+\Big(-\frac{r}{h^2}\psi_a e^{ry}+\frac{if}{k^2}\psi_b e^{sy}\Big)fU_d\Big\}, \tag{40}$$

and for the superficial layer:

$$U'=\frac{1}{\mathfrak{D}}\Big[\Big\{\frac{if}{h'^2}(\varphi_c\cosh r'y+\varphi_d\sinh r'y)+\frac{s'}{k'^2}(\varphi_e\sinh s'y+\varphi_f\cosh s'y)\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$
$$+\Big\{\frac{if}{h'^2}(\psi_c\cosh r'y+\psi_d\sinh r'y)+\frac{s'}{k'^2}(\psi_e\sinh s'y+\psi_f\cosh s'y)\Big\}fU_d\Big], \tag{41}$$

$$V'=\frac{1}{\mathfrak{D}}\Big[\Big\{-\frac{r'}{h'^2}(\varphi_c\sinh r'y+\varphi_d\cosh r'y)+\frac{if}{k'^2}(\varphi_e\cosh s'y+\varphi_f\sinh s'y)\frac{P}{\mu}e^{i(pt-fx)}$$
$$+\Big\{-\frac{r'}{h'^2}(\psi_c\sinh r'y+\psi_d\cosh r'y)+\frac{if}{k'^2}(\psi_e\cosh s'y+\psi_f\sinh s'y)\Big\}fU_d\Big]. \tag{42}$$

The stress-components may be written in the forms, for the semi-infinite medium:

$$\widehat{xx}=\frac{\mu}{\mathfrak{D}}\Big[\Big\{\Big(1+\frac{2f^2}{h^2}\Big)\varphi_a e^{ry}-i\frac{2fs}{k^2}\varphi_b e^{sy}\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$
$$+\Big\{\Big(1+\frac{2f^2}{h^2}\Big)\psi_a e^{ry}-i\frac{2fs}{k^2}\psi_b e^{sy}\Big\}fU_d\Big], \tag{43}$$

$$\widehat{yy}=\frac{\mu}{\mathfrak{D}}\Big[\Big\{\Big(1-\frac{2r^2}{h^2}\Big)\varphi_a e^{ry}+i\frac{2fs}{k^2}\varphi_b e^{sy}\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$
$$+\Big\{\Big(1-\frac{2r^2}{h^2}\Big)\psi_a e^{ry}+i\frac{2fs}{k^2}\psi_b e^{sy}\Big\}fU_d\Big], \tag{44}$$

$$\widehat{xy}=\frac{\mu}{\mathfrak{D}}\Big\{\Big(i\frac{2fr}{h^2}\varphi_a e^{ry}+\frac{f^2+s^2}{k^2}\varphi_b e^{sy}\Big)\frac{P}{\mu}e^{i(pt-fx)}$$
$$+\Big(i\frac{2fr}{h^2}\psi_a e^{ry}+\frac{f^2+s^2}{k^2}\psi_b e^{sy}\Big)fU_d\Big], \tag{45}$$

and for the superficial layer:

$$\widehat{xx}'=\frac{\mu}{\mathfrak{D}}\Big[\ n\Big(1+\frac{2f^2}{h'^2}\Big)(\varphi_c\cosh r'y+\varphi_d\sinh r'y)$$

$$- i\frac{2nfs'}{k'^2}(\varphi_e \sinh s'y + \varphi_f \cosh s'y)\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$

$$+ \Big\{n\Big(1+\frac{2f^2}{h'^2}\Big)(\psi_c \cosh r'y + \psi_d \sinh r'y)$$

$$- i\frac{2nfs'}{k'^2}(\psi_e \sinh s'y + \psi_f \cosh s'y)\Big\}fU_d\Big], \tag{46}$$

$$\widehat{yy'} = \frac{\mu}{\mathfrak{D}}\Big[\Big\{n\Big(1-\frac{2r'^2}{h'^2}\Big)(\varphi_c \cosh r'y + \varphi_d \sinh r'y)$$

$$+ i\frac{2nfs'}{k'^2}(\varphi_e \sinh s'y + \varphi_f \cosh s'y)\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$

$$+ \Big\{n\Big(1-\frac{2r'^2}{h'^2}\Big)(\psi_c \cosh r'y + \psi_d \sinh r'y)$$

$$+ i\frac{2nfs'}{k'^2}(\psi_e \sinh s'y + \psi_f \cosh s'y)\Big\}fU_d\Big], \tag{47}$$

$$\widehat{xy'} = \frac{n\mu}{\mathfrak{D}}\Big[\Big\{i\frac{2fr'}{h'^2}(\varphi_c \sinh r'y + \varphi_d \cosh r'y)$$

$$+ \frac{f^2+s'^2}{k'^2}(\varphi_e \cosh s'y + \varphi_f \sinh s'y)\Big\}\frac{P}{\mu}e^{i(pt-fx)}$$

$$+ \Big\{i\frac{2fr'}{h'^2}(\psi_c \sinh r'y + \psi_d \cosh r'y)$$

$$+ \frac{f^2+s'^2}{k'^2}(\psi_e \cosh s'y + \psi_f \sinh s'y)\Big\}fU_d\Big], \tag{48}$$

in which U_d is still unknown.

3. Discrepancy between the Tangential Displacements

U_d is the discrepancy between the tangential displacements at the junction and the unknown value and it takes place as the strain-energy of the whole system becomes minimum. The total strain-energy per unit width in the direction of x-axis may be expressed by means of

$$W = \frac{1}{2E}\int_{-\infty}^{0}\left\{(\widehat{xx}^2 + \widehat{yy}^2) - 2\sigma\widehat{xx}\widehat{yy} + 2(1+\sigma)\widehat{xy}^2\right\}dy$$

$$+ \frac{1}{2E'}\int_{0}^{H_0}\left\{(\widehat{xx}'^2 + \widehat{yy}'^2) - 2\sigma'\widehat{xx}'\widehat{yy}' + 2(1+\sigma')\widehat{xy}'^2\right\}dy, \qquad (49)$$

in which E and E' are Young's moduli and σ and σ' are Poisson's ratios, respectively. When the integrations of each term are expressed by $I_{i(=1,2,\cdots,13)}$, the equation (49) may be written in the form

$$W = (\mu^2/2E\mathfrak{D}^2)[(P^2/\mu^2)e^{2i(pt-fx)}\{I_1\varphi_a^2 + I_2\varphi_a\varphi_b + I_3\varphi_b^2 + I_4(\varphi_c^2 + \varphi_d^2) + I_5(\varphi_e^2 + \varphi_f^2)$$

$$+ I_6\varphi_c\varphi_d + I_7\varphi_e\varphi_f + I_8(\varphi_c\varphi_e + \varphi_d\varphi_f) + I_9(\varphi_c\varphi_e - \varphi_d\varphi_f)$$

$$+ I_{10}(\varphi_d\varphi_e + \varphi_c\varphi_f) + I_{11}(\varphi_d\varphi_e - \varphi_c\varphi_f) + I_{12}(\varphi_c^2 - \varphi_d^2) + I_{13}(\varphi_e^2 - \varphi_f^2)\}$$

$$+ (P/\mu)fU_d e^{i(pt-fx)}\{2I_1\varphi_a\psi_a + I_2(\varphi_a\psi_b + \varphi_b\psi_a) + 2I_3\varphi_b\psi_b$$

$$+ 2I_4(\varphi_c\psi_c + \varphi_d\psi_d) + 2I_5(\varphi_e\psi_e + \varphi_f\psi_f) + I_6(\varphi_c\psi_d + \varphi_d\psi_c) + I_7(\varphi_e\psi_f + \varphi_f\psi_e)$$

$$+ I_8(\varphi_c\psi_e + \varphi_e\psi_c + \varphi_d\psi_f + \varphi_f\psi_d) + I_9(\varphi_c\psi_e + \varphi_e\psi_c - \varphi_d\psi_f - \varphi_f\psi_d)$$

$$+ I_{10}(\varphi_d\psi_e + \varphi_e\psi_d + \varphi_c\psi_f + \varphi_f\psi_c) + I_{11}(\varphi_d\psi_e + \varphi_e\psi_d - \varphi_c\psi_f - \varphi_f\psi_c)$$

$$+ 2I_{12}(\varphi_c\psi_c - \varphi_d\psi_d) + 2I_{13}(\varphi_e\psi_e - \varphi_f\psi_f)\}$$

$$+ f^2U_d^2\{I_1\psi_a^2 + I_2\psi_a\psi_b + I_3\psi_b^2 + I_4(\psi_c^2 + \psi_d^2) + I_5(\psi_e^2 + \psi_f^2) + I_6\psi_c\psi_d$$

$$+ I_7\psi_e\psi_f + I_8(\psi_c\psi_e + \psi_d\psi_f) + I_9(\psi_c\psi_e - \psi_d\psi_f) + I_{10}(\psi_d\psi_e + \psi_c\psi_f)$$

$$+ I_{11}(\psi_d\psi_e - \psi_c\psi_f) + I_{12}(\psi_c^2 - \psi_d^2) + I_{13}(\psi_e^2 - \psi_f^2)\}], \qquad (50)$$

in which

$$I_1 = \frac{1}{r}(5 - 3\sigma),$$

$$I_2 = i8(1+\sigma)\frac{(r-s)(f^2-rs)}{(r+s)h^2k^2}f,$$

$$I_3 = \frac{1}{s}(1+\sigma),$$

$$\left.\begin{aligned}
I_4 &= \frac{n}{r'}(5-3\sigma')\cosh r'H_0 \sinh r'H_0, \\
I_5 &= \frac{n}{s'}(1+\sigma')\cosh s'H_0 \sinh s'H_0, \\
I_6 &= \frac{n}{r'}(5-3\sigma')(\cosh 2r'H_0 - 1), \\
I_7 &= \frac{n}{s'}(1+\sigma')(\cosh 2s'H_0 - 1), \\
I_8 &= i2n(1+2\sigma')\frac{(r'-s')(f^2-r's')}{(r'+s')h'^2k'^2}f\{\cosh(r'+s')H_0 - 1\}, \\
I_9 &= i2n(1+2\sigma')\frac{(r'+s')(f^2+r's')}{(r'-s')h'^2k'^2}f\{\cosh(r'-s')H_0 - 1\}, \\
I_{10} &= i2n(1+2\sigma')\frac{(r'-s')(f^2-r's')}{(r'+s')h'^2k'^2}f\sinh(r'+s')H_0, \\
I_{11} &= i2n(1+2\sigma')\frac{(r'+s')(f^2+r's')}{(r'-s')h'^2k'^2}f\sinh(r'-s')H_0, \\
I_{12} &= n\left\{4(1-\sigma') + (1+\sigma')\left(1+\frac{8f^2r'^2}{h'^4}\right)\right\}H_0, \\
I_{13} &= n(1+\sigma')\left(1+\frac{8f^2s'^2}{h'^4}\right)H_0.
\end{aligned}\right\} \quad (51)$$

To simplify the equation (50), when we change to write the parts enclosed by brackets into $[I\varphi\varphi]$, $[I\varphi\psi]$ and $[I\psi\psi]$, respectively, the total strain-energy may be written in the form

$$W = \frac{\mu^2}{2E\mathfrak{D}^2}\left\{\frac{P^2}{\mu^2}e^{2i(pt-fx)}[I\varphi\varphi] + \frac{P}{\mu}fU_d e^{i(pt-fx)}[I\varphi\psi] + (fU_d)^2[I\psi\psi]\right\}, \quad (52)$$

which is expressed by a quadratic equation of U_d.
Hence

$$\frac{\partial W}{\partial U_d} = 0 \qquad \text{or} \qquad 2fU_d[I\psi\psi] + \frac{P}{\mu}e^{i(pt-fx)}[I\varphi\psi] = 0. \quad (53)$$

Now, if $[I\psi\psi] \neq 0$, we take

$$fU_d = -\frac{1}{2}\frac{[I\varphi\phi]}{[I\phi\phi]}\frac{P}{\mu}e^{i(pt-fx)}, \tag{54}$$

$$W = \frac{\mu^2}{2E\mathfrak{D}^2} \times$$

	$\frac{P^2}{\mu^2}e^{2i(pt-fx)}$
$\frac{4.25}{r}$	φa^2
$i10\frac{(r-s)(f^2-rs)}{(r+s)h^2k^2}f$	$\varphi a\varphi b$
$\frac{1.25}{s}$	φb^2
$4.25\frac{n}{r'}\cosh r'H_0 \sinh r'H_0$	$\varphi c^2 + \varphi d^2$
$1.25\frac{n}{s'}\cosh s'H_0 \sinh s'H_0$	$\varphi e^2 + \varphi f^2$
$4.25\frac{n}{r'}(\cosh 2r'H_0 - 1)$	$\varphi c\varphi d$
$1.25\frac{n}{s'}(\cosh 2s'H_0 - 1)$	$\varphi e\varphi f$
$i3n\frac{(r'-s')(f^2-r's')}{(r'+s')h'^2k'^2}f\{\cosh(r'+s')H_0 - 1\}$	$\varphi c\varphi e + \varphi d\varphi f$
$i3n\frac{(r'+s')(f^2+r's')}{(r'-s')h'^2k'^2}f\{\cosh(r'-s')H_0 - 1\}$	$\varphi c\varphi e - \varphi d\varphi f$
$i3n\frac{(r'-s')(f^2-r's')}{(r'+s')h'^2k'^2}f\sinh(r'+s')H_0$	$\varphi d\varphi e + \varphi c\varphi f$
$i3n\frac{(r'+s')(f^2+r's')}{(r'-s')h'^2k'^2}f\sinh(r'-s')H_0$	$\varphi d\varphi e - \varphi c\varphi f$
$n\left(4.25 + 10\frac{f^2r'^2}{h'^4}\right)H_0$	$\varphi c^2 - \varphi d^2$
$n\left(1.25 + 10\frac{f^2s'^2}{k'^4}\right)H_0$	$\varphi e^2 - \varphi f^2$

where the unknown value of U_d is found.

To simplify the numerical calculations of the total strain-energy, when we change to write the equation (50) into the form of table, we have as $\sigma = \sigma' = 0.25$

$\frac{P}{\mu} f U_d e^{i(pt-fx)}$	$(fU_d)^2$
$2\varphi a\phi a$	ϕa^2
$\varphi a\phi b + \varphi b\phi a$	$\phi a\phi b$
$2\varphi b\phi b$	ϕb^2
$2(\varphi c\phi c + \varphi d\phi d)$	$\phi c^2 + \phi d^2$
$2(\varphi e\phi e + \varphi f\phi f)$	$\phi e^2 + \phi f^2$
$\varphi c\phi d + \varphi d\phi c$	$\phi c\phi d$
$\varphi e\phi f + \varphi f\phi e$	$\phi e\phi f$
$(\varphi c\phi e + \varphi e\phi c) + (\varphi d\phi f + \varphi f\phi d)$	$\phi c\phi e + \phi d\phi f$
$(\varphi c\phi e + \varphi e\phi c) - (\varphi d\phi f + \varphi f\phi d)$	$\phi c\phi e - \phi d\phi f$
$(\varphi d\phi e + \varphi e\phi d) + (\varphi c\phi f + \varphi f\phi c)$	$\phi d\phi e + \phi c\phi f$
$(\varphi d\phi e + \varphi e\phi d) - (\varphi c\phi f + \varphi f\phi c)$	$\phi d\phi e - \phi c\phi f$
$2(\varphi c\phi c - \varphi d\phi d)$	$\phi c^2 - \phi d^2$
$2(\varphi e\phi e - \varphi f\phi f)$	$\phi e^2 - \phi f^2$

, (55)

in which each term in the first column has expressed I_i and if we sum up in the columns each product of the terms in the other columns with I_i on the same lines, we shall take $[I\varphi\varphi]$, $[I\varphi\phi]$ and $[I\phi\phi]$, respectively.

4. Dispersion Curves of M–, M_2–, M_u– and M_{u2}–Waves

We shall express the dispersion curves of M–, M_2–, M_u– and M_{u2}–waves in such cases that the densities of both media are equal and the ratios of the rigidity of the semi-infinite medium to one of the superficial layer are 2, 3, 4, 5 and ∞. M– and M_2–waves can be decided by putting $\mathfrak{D}=0$ and M_u– and M_{u2}–waves can be obtained by the principle of minimum strain-energy. The results are given in Tables 1∼17 and plotted in Fig. 2∼6.

Table 1. M–wave in the case of $n=1/2$.

v_0	0.9194	0.95	1.00	1.10	1.20	1.3002
$\frac{L}{H_0}$	0	1.498	2.077	3.236	6.286	∞

Table 2. M–wave in the case of $n=1/3$.

v_0	0.9194	0.95	1.00	1.10	1.20	1.30	1.40	1.50	1.5924
$\frac{L}{H_0}$	0	1.439	1.859	2.545	3.254	4.238	6.450	15.180	∞

Table 3. M–wave in the case of $n=1/4$.

v_0	0.9194	0.95	1.00	1.10	1.20	1.40	1.60	1.8388
$\frac{L}{H_0}$	0	1.362	1.789	2.334	2.921	4.214	7.830	∞

Table 4. M–wave in the case of $n=1/5$.

v_0	0.9194	0.95	1.00	1.20	1.40	1.60	1.70	2.0530
$\frac{L}{H_0}$	0	1.345	1.752	2.790	3.766	5.312	6.880	∞

Table 5. M_2-wave in the case of $n = 1/2$.

v_0	1.00	1.10	1.20	1.30	1.414	—
$\frac{L}{H_0}$	0	0.671	0.957	1.341	2.822	—

Table 6. M_2-wave in the case of $n = 1/3$.

v_0	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.732
$\frac{L}{H_0}$	0	0.646	0.887	1.106	1.376	1.930	2.976	3.924

Table 7. M_2-wave in the case of $n = 1/4$.

v_0	1.00	1.10	1.20	1.40	1.60	1.838	2.00
$\frac{L}{H_0}$	0	0.635	0.868	1.279	2.302	4.422	5.507

Table 8. M_2-wave in the case of $n = 1/5$.

v_0	1.00	1.10	1.20	1.40	1.60	1.70	2.053	2.236
$\frac{L}{H_0}$	0	0.621	0.859	1.246	2.020	3.056	5.517	6.638

Table 9. M_u-wave in the case of $n = 1/2$.

v_0	0.9194	0.95	1.00	1.10	1.20
$\frac{L}{H_0}$	0	1.81	2.51	3.83	8.00
$\frac{U_d}{C_0}$	—	$+i0.28$	$-i3.70$	$-i3.22$	$-i9.76$

Table 10. M_u-wave in the case of $n = 1/3$.

v_0	0.9194	1.00	1.10	1.20	1.30	1.40
$\frac{L}{H_0}$	0	2.14	3.00	3.74	5.52	8.62
$\frac{U_d}{C_0}$	—	$-i6.64$	$-i3.14$	$-i4.41$	$-i6.27$	$-i9.33$

Table 11. Mu-wave in the case of $n = 1/4$.

v_0	0.9194	1.00	1.10	1.20	1.40	1.60
$\frac{L}{H_0}$	0	2.00	2.58	3.65	5.52	12.05
$\frac{Ud}{C_0}$	—	$-i7.36$	$-i4.98$	$-i7.84$	$-i22.39$	$+i2.50$

Table 12. Mu-wave in the case of $n = 1/5$.

v_0	0.9194	1.00	1.10	1.20	1.40	1.60	1.70
$\frac{L}{H_0}$	0	1.90	2.72	3.21	5.00	8.69	20.00
$\frac{Ud}{C_0}$	—	$-i10.76$	$-i2.94$	$-i7.89$	$-i6.50$	$-i0.40$	$+i2.86$

Table 13. Mu_2-wave in the case of $n = 1/2$.

v_0	1.086	1.10	1.20	1.30
$\frac{L}{H_0}$	0	0.50	1.67	2.50
$\frac{Ud}{C_0}$	—	$+i0.42$	$+i0.36$	$+i0.09$

Table 14. Mu_2-wave in the case of $n = 1/3$.

v_0	1.060	1.20	1.30	1.40	1.45
$\frac{L}{H_0}$	0	1.67	2.00	2.57	3.00
$\frac{Ud}{C_0}$	—	$-i1.23$	$-i0.70$	$-i1.68$	$-i0.96$

Table 15. Mu_2-wave in the case of $n = 1/4$.

v_0	1.05	1.20	1.40	1.60	1.65
$\frac{L}{H_0}$	0	1.67	2.27	4.00	4.40
$\frac{Ud}{C_0}$	—	$-i2.67$	$-i2.06$	$-i2.41$	$-i1.65$

Table 16. M_{u_2}-wave in the case of $n = 1/5$.

v_0	1.045	1.20	1.40	1.60	1.70	1.75
$\frac{L}{H_0}$	0	1.64	2.59	3.13	3.26	4.00
$\frac{U_d}{C_0}$	—	$-i2.78$	$-i2.94$	$-i3.67$	$-i3.36$	$-i3.00$

Table 17. M- and M_2-wave in the case of $n = 1/\infty$.

v_0		0.9194	0.95	1.00	1.20	1.40	1.60	1.80	2.00	2.40	3.00	4.00	…
$\frac{L}{H_o}$	M	0	1.287	1.651	2.475	3.080	3.619	4.127	4.614	—	6.978	9.294	…
	M_2	—	—	0	0.833	1.176	1.654	3.108	4.614	7.021	10.071	14.584	…

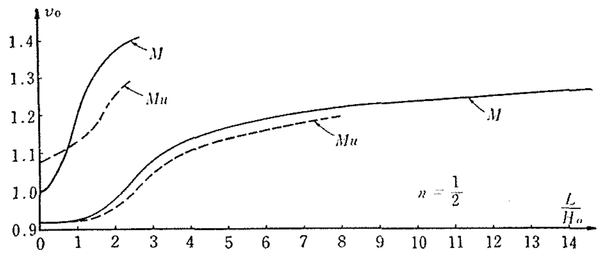

Fig. 2. The dispersion curves of M-, M_2-, M- and M_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/2$.

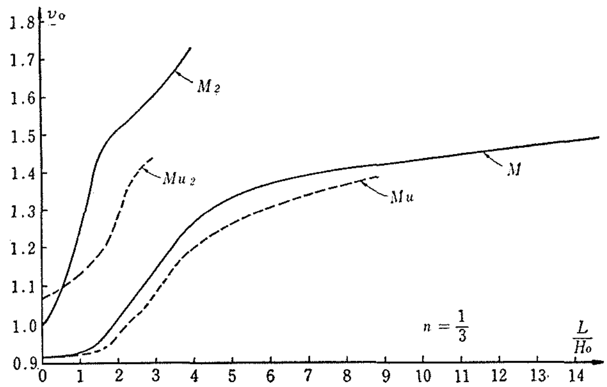

Fig. 3. The dispersion curves of M-, M_2-, M_u- and M_{u_2}-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/3$.

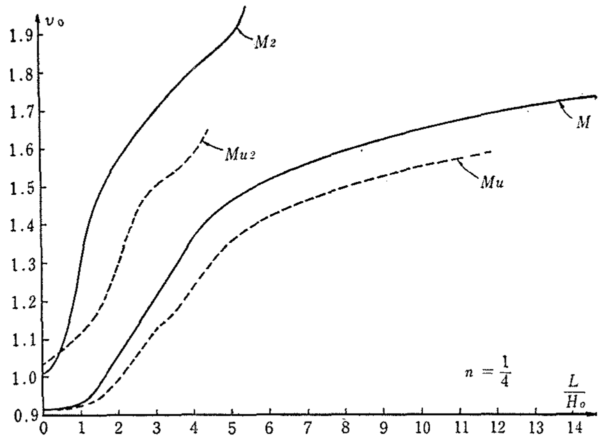

Fig. 4. The dispersion curves of M–, M_2–, M_u– and M_{u_2}–wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/4$.

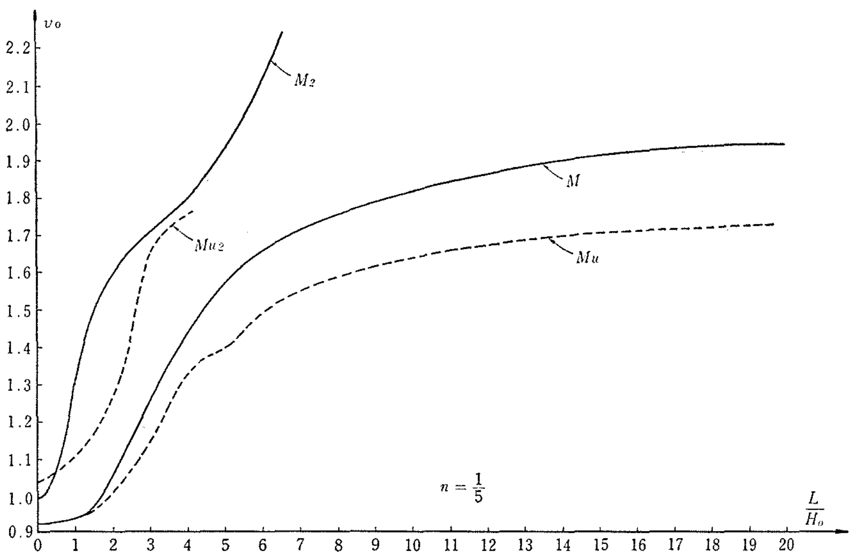

Fig. 5. The dispersion curves of M–, M_2–, M_u– and M_{u_2}–wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/5$.

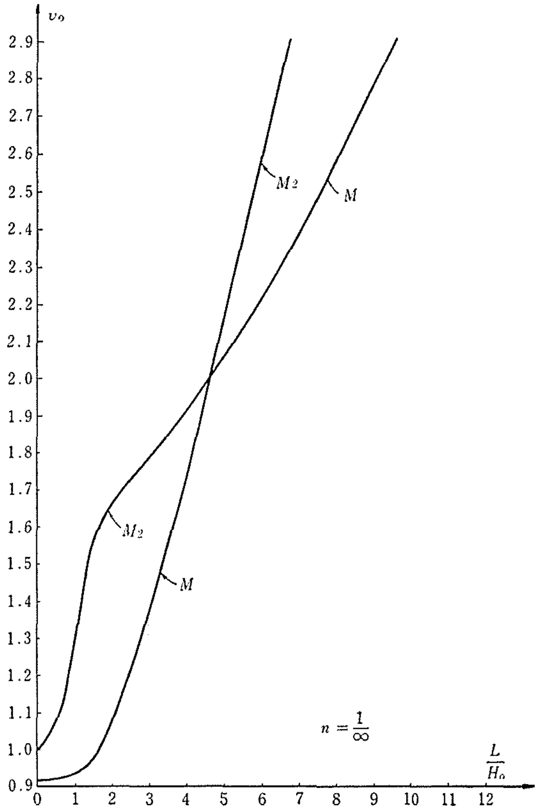

Fig. 6. The dispersion curves of M- and M_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/\infty$.

In the above figures, full lines have expressed the dispersion curves of M- and M_2-waves, and broken lines the dispersion curves of M_u- and M_{u2}-waves. It may be understood that M_u- and M_{u2}-waves exist within more narrow ranges than the ranges of existing M- and M_2-waves, and the smaller n becomes, the nearer M_u-waves get to M-waves and M_{u2}-waves to M_2-waves, respectively, and at the limit of n, that is $1/\infty$, M_u- and M_{u2}-waves should correspond with M- and M_2-waves, respectively.

We have thought that M_u- and M_{u2}-waves take place when the total strain-energy becomes minimum with the discrepancy between tangential displacements, but in the actual phenomena, the waves of this kind can not be always observed at the states of minimum strain-energy, but at the states of higher strain-energy which have taken place because of imperfect slide. Hence we suppose that the observed values exist within the ranges between M- and M_u-waves or M_2- and M_{u2}-waves, but never over the both limits.

5. Group-Velocity of M-, M_2-, M_u- and M_{u2}-Waves

As mentioned above, we have expressed several dispersion curves of the waves, but as a matter of fact, the observed values are group-velocities resulted from many elastic waves. The equation of the group-velocity is expressed in the form

$$V_0 = v_0 - \frac{L}{H_0}\frac{\partial v_0}{\partial(L/H_0)}, \tag{56}$$

in which V_0 is expressed by a ratio of the group-velocity of the wave to the velocity of the transverse wave propagated within the superficial layer, and the group-velocity does not generally correspond with the phase-velocity. When we find the group-velocity with the graphical solution, we shall be able to express the group-velocity of M-and M_u-waves in Fig. 7~10, and one of M_2- and M_{u2}-waves in Fig. 11~14, in which the dispersion curves and group-velocities of M- and M_u-waves are expressed with full lines, and these of M_2- and M_{u2}-waves are expressed with broken lines, respectively.

We can find in Fig. 7~14 that the group-velocity of M_u- and M_{u2}-waves differ from one of M- and M_2-waves, and M_u-waves have a characteristic that two minimum values are expressed. The smaller n becomes, the smaller the minimum group-velocity becomes and when the minimum group-velocity is caused, the greater the phase-velocity and wave-length. As far these natures M_u-waves look like M-waves and M_{u2}-waves are similar to M_2-waves. We hope that there will be observed values within the ranges limited by two kinds of group-velocities.

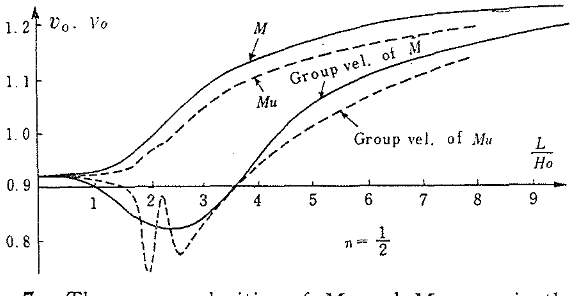

Fig. 7. The group-velocities of M- and M_u-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/2$.

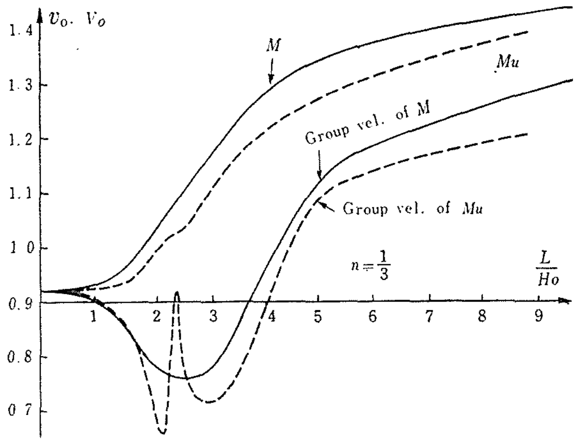

Fig. 8. The group-velocities of M- and Mu-wave in the case of $\lambda=\mu$, $\lambda'=\mu'$, $m=1$ and $n=1/3$.

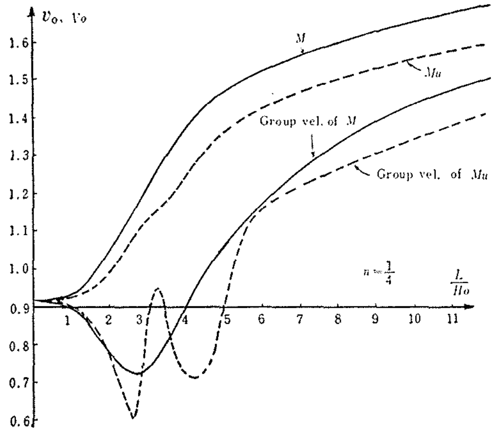

Fig. 9. The group-velocities of M- and Mu-wave in the case of $\lambda=\mu$, $\lambda'=\mu'$, $m=1$ and $n=1/4$.

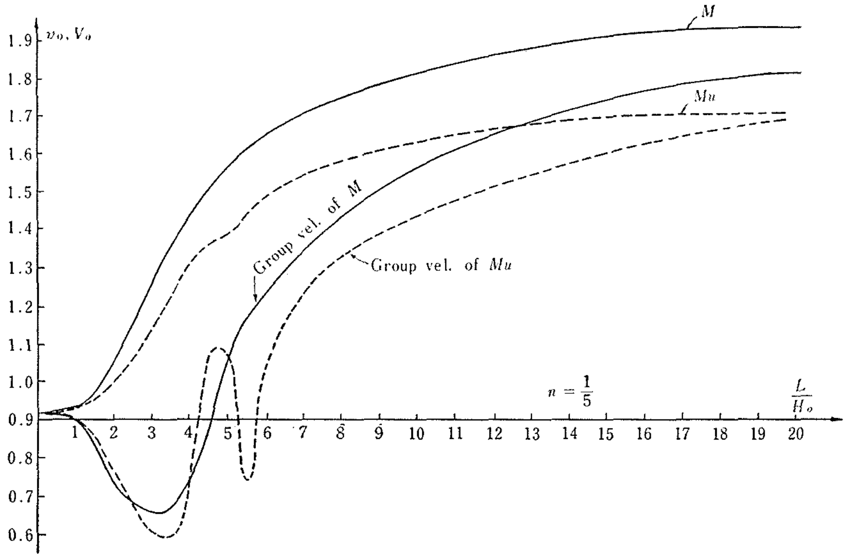

Fig. 10. The group-velocities of M- and Mu-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/5$.

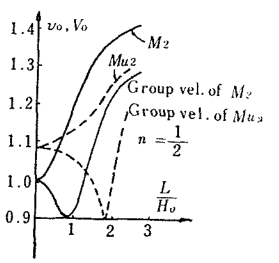

Fig. 11. The group-velocities of M_2- and Mu_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/2$.

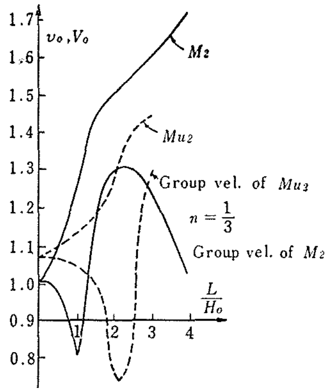

Fig. 12. The group-velocities of M_2- and Mu_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/3$.

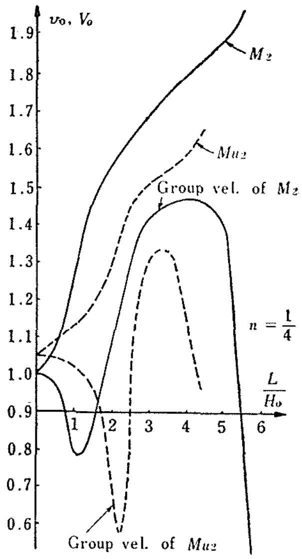

Fig. 13. The group-velocities of M_2- and Mu_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/4$.

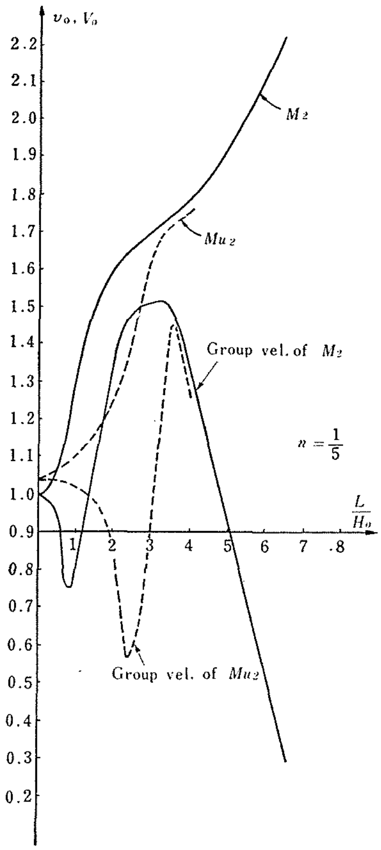

Fig. 14. The group-velocities of M_2- and Mu_2-wave in the case of $\lambda = \mu$, $\lambda' = \mu'$, $m = 1$ and $n = 1/5$.

6. Distribution of Displacements at Different Depths

We shall study the distribution of displacements at different depths. Horizontal and vertical components of displacement within the superficial layer and within the semi-infinite medium are reducted to

$$\left.\begin{aligned} U' &= \left\{\frac{if}{h'^2}(C\cosh r'y + D\sinh r'y) + \frac{s'}{k'^2}(E\sinh s'y + F\cosh s'y)\right\}e^{i(pt-fx)}, \\ V' &= \left\{-\frac{r'}{h'^2}(C\sinh r'y + D\cosh r'y) + \frac{if}{k'^2}(E\cosh s'y + F\sinh s'y)\right\}e^{i(pt-fx)}, \end{aligned}\right\} \quad (57)$$

$$\left.\begin{aligned} U &= \left(\frac{if}{h^2} Ae^{ry} + \frac{s}{k^2} Be^{sy}\right) e^{i(pt-fx)}, \\ V &= \left(-\frac{r}{h^2} Ae^{ry} + \frac{if}{k^2} Be^{sy}\right) e^{i(pt-fx)}. \end{aligned}\right\} \tag{58}$$

The results of calculations of displacements in the cases of corresponding to several points of the figures that express dispersion curves are shown in Fig. 15~24 for M_u-waves and in Fig. 25~35 for M_{u_2}-waves. In figures, let a and b denote amplitudes of horizontal and vertical displacement-components, respectively.

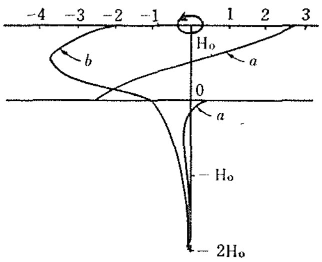

Fig. 15. The distributions of displacements in the case of corresponding to $n = 1/2$ and $v_0 = 1.10$ for M_u-wave.

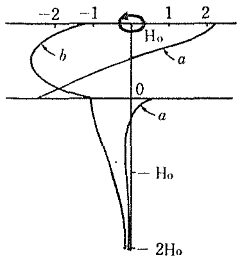

Fig. 16. The distributions of displacements in the case of corresponding to $n = 1/2$ and $v_0 = 1.20$ for M_u-wave.

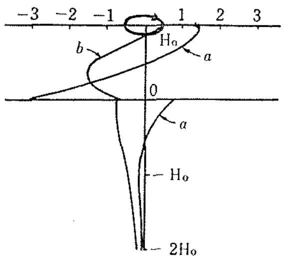

Fig. 17. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.20$ for M_u-wave.

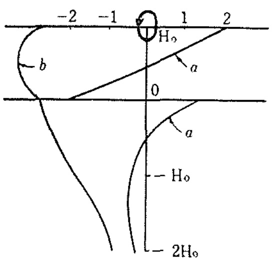

Fig. 18. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.30$ for M_u-wave.

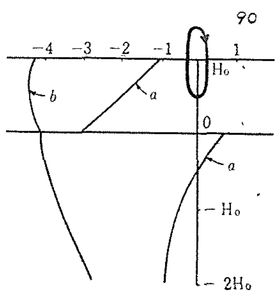

Fig. 19. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.40$ for M_u-wave.

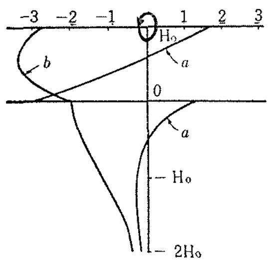

Fig. 20. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.20$ for M_u-wave.

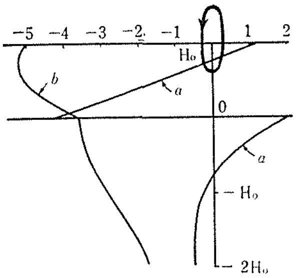

Fig. 21. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.40$ for M_u-wave.

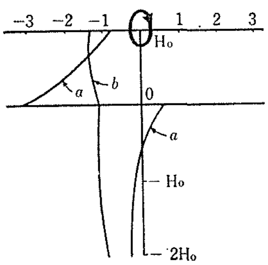

Fig. 22. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.60$ for M_u-wave.

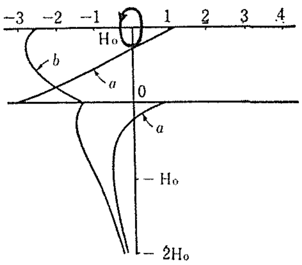

Fig. 23. The distributions of displacements in the case of corresponding to $n = 1/5$ and $v_0 = 1.20$ for M_u-wave.

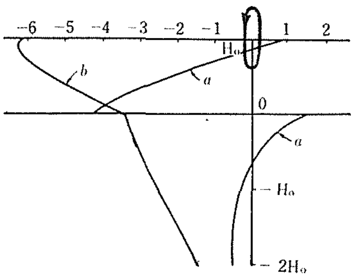

Fig. 24. The distributions of displacements in the case of corresponding to $n = 1/5$ and $v_0 = 1.60$ for M_u-wave.

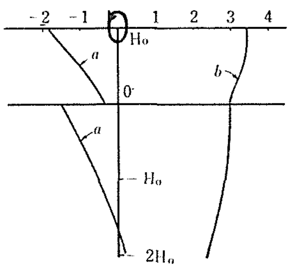

Fig. 25. The distributions of displacements in the case of corresponding to $n = 1/2$ and $v_0 = 1.20$ for M_{u_2}-wave.

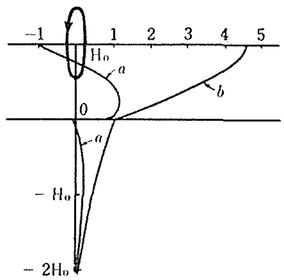

Fig. 26. The distributions of displacements in the case of corresponding to $n = 1/2$ and $v_0 = 1.30$ for M_{u_2}-wave.

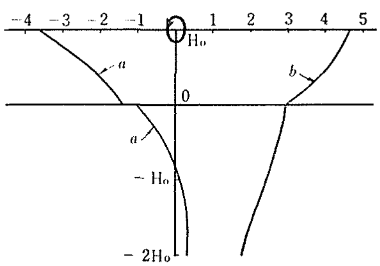

Fig. 27. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.20$ for M_{u_2}-wave.

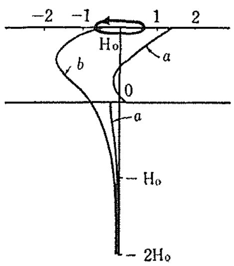

Fig. 28. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.30$ for M_{u_2}-wave.

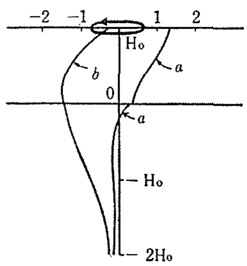

Fig. 29. The distributions of displacements in the case of corresponding to $n = 1/3$ and $v_0 = 1.40$ for M_{u_2}-wave.

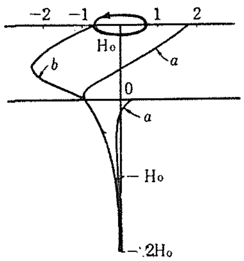

Fig. 30. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.20$ for Mu_2-wave.

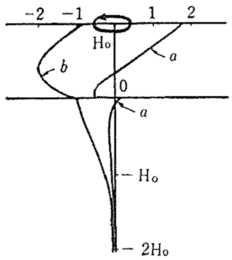

Fig. 31. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.40$ for Mu_2-wave.

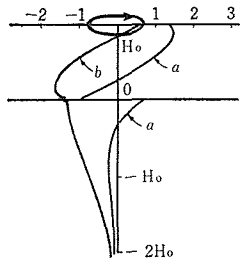

Fig. 32. The distributions of displacements in the case of corresponding to $n = 1/4$ and $v_0 = 1.60$ for Mu_2-wave.

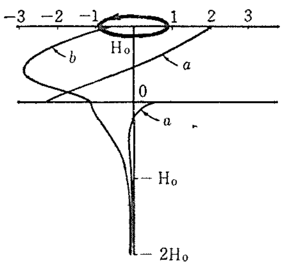

Fig. 33. The distributions of displacements in the case of corresponding to $n = 1/5$ and $v_0 = 1.20$ for Mu_2-wave.

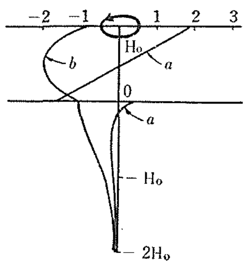

Fig. 34. The distributions of displacements in the case of corresponding to $n = 1/5$ and $v_0 = 1.40$ for Mu_2-wave.

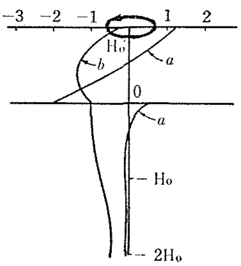

Fig. 35. The distributions of displacements in the case of corresponding to $n = 1/5$ and $v_0 = 1.60$ for Mu_2-wave.

It will be seen that there are two kinds of orbital motions of a particle at the surface of media and one of them is in the same sense as the motion of gravitational wave and the other is in the same sense as that of the usual *Rayleigh*-wave propagated over the surface of semi-infinite medium and they are elliptic motions. But though n is the same number, while the phase-velocity is relatively smaller, the elliptic orbital motion is counter-clockwise, and when the phase-velocity become relatively greater it becomes clockwise. On the M_u-waves, the vertical displacement is greater than the horizontal displacement at the surface, but on the M_{u_2}-waves, it is contrary. Hence we may find such phase-velocity that either horizontal motions of M_u-waves or vertical motions of M_{u_2}-waves are nothing.

As all vertical stress-components are compressive stresses, and tensile stresses are not at all caused at the junction, the motions do not accompany the separations as the initial assumption.

7. Conclusion

On the elastic waves propagated within the two-dimensional layered media, the results of numerical calculations in the case of allowing the discrepancy between tangential displacements at the junction have clearly expressed that M_u- and M_{u2}-waves are different from M- and M_2-waves which have been known already and that the natures of M_u-waves are different from one of M_{u2}-waves. The analysis is based on the condition that the superficial layer and the semi-infinite medium can be reciprocally slided at the junction to the last and the discrepancy between both media is decided so that the strain-energy of the whole system may become minimum. Hence, this continuous condition has more universality and more apropriateness and M_u- and M_{u2}-waves obtained under such a condition are one phase of limit waves. We have heard there is something hard to explain the larger discrepancy between the observed values and the analytical curves. We have been afraid lest the observed values with such discrepancy should be plotted for reasons that these have errors due to defects of the recording instruments or the discrepancy is so large on account of the inner structures of the earth having such elements as we can not ascertain.

Lately, we have expected that we may explain the very observed values being different from analytical curves by the application of this idea. In this article, we have treated the elastic waves as the two-dimensional problem and so we can not compare the three-dimensional seismic phenomena with these results, but we can give some suggestions of the tendency of waves. We may suppose the slide is caused at the junction between each layer in the earth, but we may doubt if the slide will take place till the strain-energy reaches the states of minimum. In the case of the imperfect slide, the observed values will not correspond with the analytical curves and be recorded within the narrow ranges limited by M-waves and M_u-waves or M_2-waves and M_{u2}-waves.

Acknowledgement

In conclusion, I should like to express my sincere thanks to Prof. Dr. B. Tanimoto, Department of Civil Engineering, Faculty of Engineering, Shinshu University, for his kind instructions of this idea and his constant encouragements, and to Prof. Dr. G. NISHIMURA, Tokyo University, and Prof. Dr. K. KANAI, Earthquake Research Institute, Tokyo University, for their kind supervision.

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