Analysis of Rigid Frames in Space by Applying Slope-Deflection Formulas

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Synopsis. Rigid frames in space are successfully and exactly analysed by applying the slope-deflection equations to the flexural members and the torsion equations to the twisted members respectively. The results are compared to those obtained from the conventional two-dimensional analysis, the precision of which is numerically shown.

Introduction

In designing a rigid frame in space, it is usual to divide it up into several constituent plane frames and analyse each separately. Although this conventional procedure greatly facilitates the calculations, it does not determine the torsion effects which exist and sometimes amout to considerable magnitude. It is hoped to give an exact and easily applicable method which treats the frame as a whole.

Several three-dimensional analyses ever proposed[9),10),12)] are almost based upon the principles of virtual work or the like which are taking the stress functions as redundants. Accordingly the calculations become so tedious that the practical applications are limited to the comparatively simple problems, e.g., the symmetrical frames under symmetrical loadings or, if not symmetrical, joints are assumed not to translate.

The author presents here a three-dimensional analysis using the slope-deflection equations together with the torsion equations. It is well known that the slope-deflection method is so superior to the classical methods in treating the plane frames.[11)] This circumstance agrees equally with the space frames,[13)] and many advantages will be claimed in the illustrations which follow.

The space frames mainly considered here are of single-storied and of multi-bayed longitudinally and laterally, whose members are all straight and prismatic meeting each other at right angles.

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Chapter I. Fundamental Formulas

1. Notations

In dealing with a space frame, the slope-deflection method now familiar to us will also be the most convenient, since it takes the end-deformations of members for unknowns instead of stresses or reactions as in the other classical methods. On this account, the number of unknowns is greatly diminished, simple and clear sign conventions are established, the stress diagrams are made easy to draw, and moreover the deformed structure is readily visualized. Now, we proceed with illustrations of notations adopted.

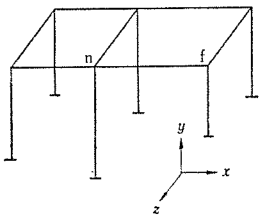


Fig. 1

The members of a space frame in Fig. 1 meet each other at right angles. In this connection, the orthogonal coordinate axes x, y and z are set up as shown. Note that any member lies parallel to one of these three axes. The member will be designated by its ends, being called the near end n and the far end f.

Consider, for example, a member nf which is parallel to x axis, Fig. 1. Moments acting at its end are denoted by M with three subscripts such as M_{xnf}, M_{ynf} and M_{znf}. The first subscript designates the axis about which M acts; the remaining two, the member and the end in question as usual. Thus, we readily tell that M_{xnf} is the moment at the near end about (an axis parallel to) x axis, i. e., the twisting moment; M_{ynf} and M_{znf} are the moments at the near end about y and z axes respectively, i. e., the bending moments.

The angular deflections produced by M's are denoted by θ with subscripts similarly as above. Thus, θ_{xnf} is the angular deflection at n about x axis, i. e., the angle of twist; θ_{ynf} and θ_{znf} are the angular deflections at n about y and z axes respectively, i. e., the end-slopes. Since the joint rotations (§9) are always taken up instead of end-slopes in our problems, the last subscript of θ designating the far end will often be omitted for convenience. For the member lying parallel to the axes other than x, the designations of M's and of θ's will be made by reading their first subscripts.

M's are positive which agree with the "right-hand screw rule" when the arrow-heads point negative directions of reference axes; θ's are positive which correspond to positive M's.

The relative displacements of joints tend the member to revolve from its original unstrained position. These angles of revolution are denoted by R, similarly accompanied by three subscripts; thus, by R_{ynf} and R_{znf} are meant that the revolutions are around y and z axes respectively. A clockwise revolution is taken as positive when the "center line of screw driver" through the member-end points to the negative direction of the reference axis.

2. Slope-Deflection Equations

If the member nf lies parallel to x axis, then it undergoes bending deformations in the planes xz and xy, and for the member nf the well known slope-deflection equations hold. See Fig. 2. In xz plane, we have

$$M_{ynf} = 2E\frac{I_{ynf}}{l_{nf}}(2\theta_{yn} + \theta_{yf} - 3R_{ynf}) + C_{ynf}, \tag{1}$$

and in xy plane

$$M_{znf} = 2E\frac{I_{znf}}{l_{nf}}(2\theta_{zn} + \theta_{zf} - 3R_{znf}) + C_{znf}, \tag{2}$$

where E = the modulus of elasticity.

I_y, I_z = the moments of inertia of the section referred to y and z axes respectively.

C_y, C_z = the fixed-end bending moments about y and z axes respectively. Their values will be found in many texts or pocket-books.

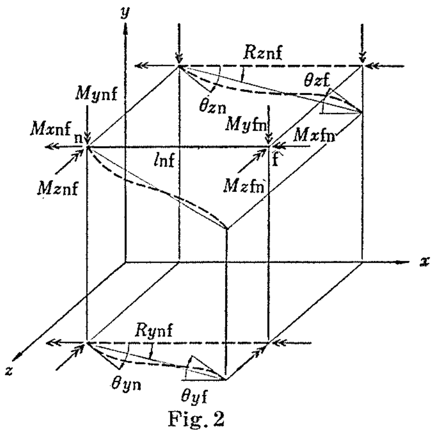


Fig. 2

Introducing the stiffness factors for flexure, these equations are modified:

$$M_{ynf} = 2EK_{ynf}(2\theta_{yn} + \theta_{yf} - 3R_{ynf}) + C_{ynf}, \tag{3}$$

$$M_{znf} = 2EK_{znf}(2\theta_{zn} + \theta_{zf} - 3R_{znf}) + C_{znf}, \tag{4}$$

where $K_y = I_y/l_{nf}$ = the stiffness factor for flexure referred to y axis,

$K_z = I_z/l_{nf}$ = do. referred to z axis.

3. *Torsion Equations* [1), 2), 3), 4), 5)]

We have, for a prismatic bar of length l, the relationship between the angle of twist θ of an end relative to the other and the applied torque M_T:

$$\theta = \frac{M_T l}{GJ}, \tag{5}$$

where G = the modulus of rigidity,

J = the torsional constant of the section.

Applying this to the member nf which lies, for example, parallel to x axis, Fig. 2, we obtain

$$M_{x\mathrm{nf}} = \frac{GJ_{x\mathrm{nf}}}{l_{\mathrm{nf}}}(\theta_{x\mathrm{n}} - \theta_{x\mathrm{f}}). \tag{6}$$

If the member is loaded with torques as in Fig. 3, their effects must be added to the right-hand side of eq. (6), which will be called the fixed-end torque and denoted by $C_{x\mathrm{nf}}$. Thus

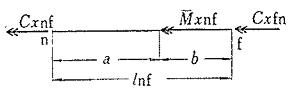


Fig. 3

$$M_{x\mathrm{nf}} = \frac{GJ_{x\mathrm{nf}}}{l_{\mathrm{nf}}}(\theta_{x\mathrm{n}} - \theta_{x\mathrm{f}}) + C_{x\mathrm{nf}}, \tag{7}$$

where

$$C_{x\mathrm{nf}} = -\sum \frac{b}{l_{\mathrm{nf}}} \bar{M}_{x\mathrm{nf}}. \tag{8}$$

Eq. (7) is written in the form

$$M_{x\mathrm{nf}} = GK_{x\mathrm{nf}}(\theta_{x\mathrm{n}} - \theta_{x\mathrm{f}}) + C_{x\mathrm{nf}}, \tag{9}$$

where $K_{x\mathrm{nf}} = J_{x\mathrm{nf}}/l_{\mathrm{nf}}$ = the stiffness factor for torsion.

4. *Evaluation of J*

The torsional constant J must be evaluated from the St. Venant's theory. We see that, if the section is circular, J is equal to I_p, the polar moment of inertia. But for non-circular sections, being the theory too long and complicated to evaluate it exactly, approximations are mostly being made. Only the formulas for the sections which seem to be important for our problems will be shown below:

1) Rectangular section of width a and depth b :

The St. Venant's exact expression is

$$J=\frac{a^3b}{3}\left(1-\frac{192}{\pi^5}\frac{a}{b}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}\right), \tag{10}$$

which is not so tedious to compute, because the series is very rapidly convergent. For practical purposes, we may take its first term only, thus

$$J=\frac{a^3b}{3}\left(1-\frac{192}{\pi^5}\frac{a}{b}\tanh\frac{\pi b}{2a}\right). \tag{11}$$

2) Narrow rectangular section:

If b/a is large, letting $\tanh(n\pi b/2a)=1$ and $\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^5}=1.0046$, we obtain from exp. (10)

$$J=\frac{a^3b}{3}\left(1-0.630\frac{a}{b}\right). \tag{12}$$

This gives good approximations for $b/a>1.6$.

If $b/a=1\sim1.6$ the Föppl's formula below is rather accurate.

$$J=\frac{a^3b^3}{3.6(a^2+b^2)}. \tag{13}$$

In the case where the width is very thin, the second term inside the parentheses of exp. (12) becomes neglegible, and we have

$$J=\frac{a^3b}{3}. \tag{14}$$

Note that this is the moment of inertia with respect to the longer side.

3) Square section:

For $a=b$, exp. (10) yields

$$J=0.1406a^4. \tag{15}$$

See that eq. (13) gives the value close enough to this.

4) Rolled profile sections:

For the sections such as angles, channels and I's, divide them up into rectangular strips and apply exp. (14) to each strip. The summations of the results give the satisfactory values:

$$J = \frac{1}{3}\sum at^3, \tag{16}$$

where $a =$ the width of the strip,

$t =$ the thickness of the strip.

5. *Relative Stiffnesses*

The torsional rigidity GJ of a member can be expressed by the flexural rigidity $E\bar{I}$ of any member chosen as reference:

$$GJ = \frac{mE}{2(m+1)}J = \left\{\frac{m}{2(m+1)}\frac{J}{\bar{I}}\right\}E\bar{I},$$

where $m =$ Poisson's number,

$\bar{I} =$ the moment of inertia of the section of the reference member.

The transformation gives

$$\frac{GJ}{l} = \left\{\frac{m}{2(m+1)}\frac{J/l}{\bar{I}/\bar{l}}\right\}\frac{E\bar{I}}{\bar{l}},$$

or

$$GK_t = \left\{\frac{m}{2(m+1)}\frac{K_t}{\bar{K}_b}\right\}E\bar{K}_b,$$

where $l,\ \bar{l} =$ the lengths of the member considered and of reference member respectively.

$K_t = J/l =$ the stiffness factor for the torsion of the member in question.

$\bar{K}_b = \bar{I}/\bar{l} =$ the stiffness factor for the flexure of the reference member.

This will be written in the simple form:

$$GK_t = 4\beta k_t E\bar{K}_b, \tag{17}$$

in which

$$\beta = \frac{G}{4E} = \frac{m}{8(m+1)}, \tag{18}$$

$$k_t = \frac{K_t}{\bar{K}_b}. \tag{19}$$

k_t denotes the stiffness factor for the torsion measured by the reference stiffness factor for flexure, which is called the relative stiffness or stiffness ratio for torsion.

Similarly, measuring the stiffness factor for flexure K_b of the member considered by $\overline{K}_b$, the stiffness ratio for flexure is defined:

$$k_b = \frac{K_b}{\overline{K}_b}, \tag{20}$$

whence we have, corresponding to eq. (17)

$$EK_b = k_b E\overline{K}_b. \tag{21}$$

6. *Equations for Practical Use*

To obtain facilities for practical calculations, the foregoing equations will be simplified by using the stiffness ratios and the symbols below:

$$\left.\begin{aligned} \varphi &= 2E\overline{K}_b\theta, \\ \psi &= -\ 6E\overline{K}_b R. \end{aligned}\right\} \tag{22}$$

Thus we have from the slope-deflection eqs. (3) and (4)

$$M_{ynf} = k_{ynf}(2\varphi_{yn} + \varphi_{yf} + \psi_{ynf}) + C_{ynf}, \tag{23}$$

$$M_{znf} = k_{znf}(2\varphi_{zn} + \varphi_{zf} + \psi_{znf}) + C_{znf}, \tag{24}$$

and from the torsion eq. (9)

$$M_{xnf} = 2\beta k_{xnf}(\varphi_{xn} - \varphi_{xf}) + C_{xnf}. \tag{25}$$

7. *Members Free to Rotate at Far End*

Consider the member is supported at the far end so as to rotate freely against bending but fixed against torsion. For such member the slope-deflection equations are also provided. For example, if the member in Fig. 1 is bending-free at the far end f about y axis only, we have, in xz plane, the end-moment expression at the near end n eliminating φ_{yf} from eq. (23) that is using the relation $M_{yfn}=0$, thus

$$M_{ynf} = \frac{1}{2}k_{ynf}(3\varphi_{yn} + \psi_{ynf}) + H_{ynf}, \tag{26}$$

where

$$H_{y\mathrm{nf}} = C_{y\mathrm{nf}} - \frac{1}{2}C_{y\mathrm{fn}}. \tag{27}$$

Further, if the far end of the member in question is torsion-free but bending-fixed, we readily find that

$$M_{x\mathrm{nf}} = C_{x\mathrm{nf}}. \tag{28}$$

This becomes zero when the member has no torque applied, therefore we need not consider such members in such a case.

8. End-Shears

If a member of the space frame acted upon by the loads, the reactions normal to its axis are induced in addition to the end-moments. These reactions, often called the end-shears, will be denoted by X with three subscripts as before. See Fig. 4.

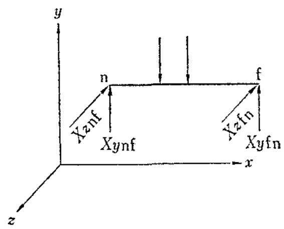


Fig. 4

Consider, for example, the member nf lying parallel to x axis, and write the equilibrium conditions in xy plane. Then we readily have the expressions for X's:

$$\left.\begin{aligned} X_{y\mathrm{nf}} &= -\frac{1}{l_{\mathrm{nf}}}(M_{z\mathrm{nf}} + M_{z\mathrm{fn}}) + \overline{X}_{y\mathrm{nf}}, \\ X_{y\mathrm{fn}} &= -\frac{1}{l_{\mathrm{nf}}}(M_{z\mathrm{nf}} + M_{z\mathrm{fn}}) + \overline{X}_{y\mathrm{fn}}, \end{aligned}\right\} \tag{29}$$

where $\overline{X}$'s denote the normal reactions produced by the loads when the member is assumed to be a simple beam.

X's and $\overline{X}$'s are taken as positive when they have the moments about the far end which agree with the right-hand screw rule. For the other reaction components such as $X_{z\mathrm{nf}}$ and $X_{z\mathrm{fn}}$, the similar expressions will be obtained.

Chapter II. Elastic Equations

9. Joint Equilibrium Equations

The condition of continuity at the joints states that the intersection angles between the member-axes remain unchanged even if the frame undergoes deformation, i. e., all the member-ends rotate by the same angle at the joint under consideration. This rotation angle, common to all member-ends, is termed the joint rotation angle or briefly the joint rotation. To satisfy the condition of continuity, it is only necessary to put the respective component joint rotations in the places of the component end-slopes in the end-moment equations.

Having this done, we write the equilibrium equations expressing that the component end-moments total to zero at each joint. For example, at joint a, we have

$$\left.\begin{aligned} &\text{about } x \text{ axis,} \quad \sum(-M_{x\mathrm{ai}}) + \overline{M}_{x\mathrm{a}} = 0,\\ &\text{about } y \text{ axis,} \quad \sum(-M_{y\mathrm{ai}}) + \overline{M}_{y\mathrm{a}} = 0,\\ &\text{about } z \text{ axis,} \quad \sum(-M_{z\mathrm{ai}}) + \overline{M}_{z\mathrm{a}} = 0, \end{aligned}\right\} \tag{30}$$

where i = the subscript designating the joint adjacent to a,
$\overline{M}$ = the component external moment existing at a.

Thus, we have equilibrium equations like these, each three at each joint, which coincide in total with the number of unknown φ's.

10. Horizontal-Shear Equations

A typical space frame is considered to be a system of plane frames standing parallel to xy or yz plane connected each other by a group of beams running parallel to z or y axis. In Fig. 5 the constituent plane frames are shown by heavy lines, the connecting beams by fine lines; in (a) the plane frames are arranged in row, and in (b) in column.

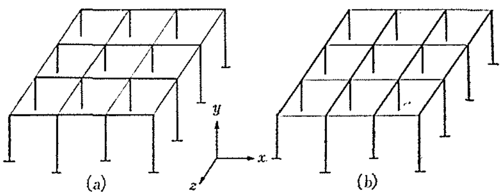


Fig. 5

Now consider a plane frame of any row as a free-body isolated from the whole assembly, and write the equilibrium of forces acting along the section through its column-tops. Considering the forces transmitted from the connecting beams, Fig. 6, we have

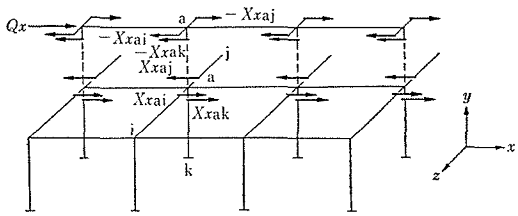


Fig. 6

$$\sum(-X_{x\mathrm{ak}}) + \sum(-X_{x\mathrm{ai}} - X_{x\mathrm{aj}}) + Q_x = 0, \tag{31}$$

where i, j = subscripts designating the adjacent joints, k = do. the column base,

Q_x = the x-component of the total horizontal load acting above the section considered.

This is written in the simple form:

$$\sum(-X_x) + Q_x = 0. \qquad (32)$$

Similarly, observing a plane-frame of any column, we obtain

$$\sum(-X_z) + Q_z = 0. \qquad (33)$$

These equations express the equilibrium condition of horizontal forces, and we will call them the horizontal-shear equations; see that the number of them equals the sum of numbers of rows and of columns.

11. Vertical-Shear Equations

The connecting beams spanning any two adjacent plane frames must be in equilibrium under the vertical forces. Considering, for example, any beam nf in row-arrangement shown in Fig. 7, we immediately have

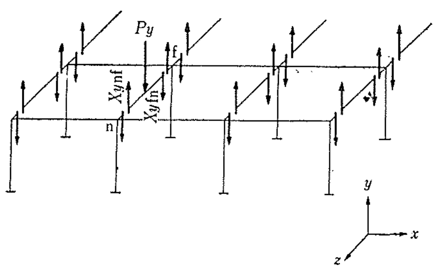


Fig. 7

$$X_{ynf} - X_{yfn} - P_y = 0, \qquad (34)$$

where P = the resultant vertical load acting on the beam.

Applying this relation to the beams in order and adding, we obtain

$$\sum(X_{ynf} - X_{yfn}) - \sum P_y = 0. \qquad (35)$$

The similar equations are also deduced in the case of column-arrangement. These will be referred to as the vertical-shear equations, the number of which agrees with the sum of numbers of bays of the plane frames in column-arrangement and of those in row-arrangement.

12. Compatibility Equations

When the members of the frame change their lengths due to the axial forces, temperature changes etc., the displacements of joints are resulted, which produce the revolutions of members. The matter is the same when the supports happen to displace. Since the members can revolve in two directions, the number of unknown R's (or ψ's) is equal to twice the number of the members. These R's must be compatible with the geometry of the frame to be mentioned below.

Take, for example, any space bounded by members either open or closed. Such a space is shown in Fig. 8 (a), where the skeleton after deformation is drawn by broken lines assuming the left support to be fixed in position.

Let the initial and final dimensions of the skeletons be as follows:

	Initial	Final
Length of member	s	$s + \Delta s$
Inclination of member	α	$\alpha_z - R_z$ (*)
Span	l	$l + \Delta l$
Relative height of the ends	h	$h + \Delta h$

(*) Cf Fig. 8 (d).

Then we have, by geometry, the relations after deformation:

$$\sum(s + \Delta s)\cos(\alpha_z - R_z) = l + \Delta l,$$

$$\sum(s + \Delta s)\sin(\alpha_z - R_z) = h + \Delta h.$$

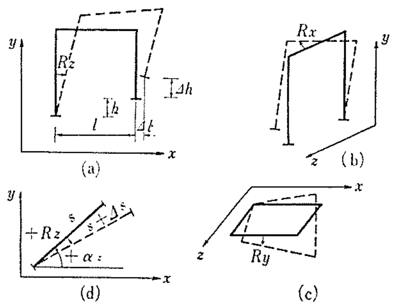


Fig. 8

Expand these equations considering the geometry before deformation, $\sum s\cos\alpha_z = 0$ and $\sum s\sin\alpha_z = h$; let $\sin R_z = R_z$ and $\cos R_z = 1$, and omit the small terms of higher order in the resulting expressions since Δs and R_z are very small. Then we finally have

$$\left.\begin{aligned} \sum\Delta s\cos\alpha_z + \sum R_z s\sin\alpha_z &= \Delta l, \\ \sum\Delta s\sin\alpha_z - \sum R_z s\cos\alpha_z &= \Delta h. \end{aligned}\right\} \quad (36)$$

These are the compatibility equations required. If we consider such spaces as in Fig. 8 (b) and (c), change the subscripts of α and R to x and y respectively.

On some particular cases it will be mentioned that:

1) When the supports do not displace, let $\Delta l = \Delta h = 0$.

2) For a closed space, also put $\Delta l = \Delta h = 0$.

3) For the case where the lengths of members remain unchanged and the

supports are fixed in positions, eqs. (36) reduce to the simple form:

$$\left.\begin{aligned}\sum Rs\sin\alpha &= 0,\\ \sum Rs\cos\alpha &= 0.\end{aligned}\right\} \tag{37}$$

The subscripts are here omitted.

13. *Establishment of Solution*

In order to carry out the analysis of the rigid frames in space, the equilibrium equations—joint equilibrium equations, horizontal- and vertical-shear equations—should be utilized together with the compatibility equations. To make sure the possibility of solution, we will here investigate the numbers of unknowns sought and of condition equations available. Take, for simplicity, a space frame of single story, and let m and n be the numbers of its constituent plane frames in row- and column-arrangements respectively.

Then, the joints, the numbers and the spaces are respectively enumerated as shown in Table 1.

Table 1

Item		No.	
Joint		mn	
Member	Column	mn	$3mn-(m+n)$
	Beam, parallel to x axis	$m(n-1)$	
	Do., 〃 z 〃	$n(m-1)$	
Space	Parallel to yz plane	$m(n-1)$	$3mn-2(m+n)+1$
	〃 xz 〃	$(m-1)(n-1)$	
	〃 xy 〃	$n(m-1)$	

With this, we can tell the numbers of condition equations as follows:

Joint equilibrium equations $\cdots 3mn$ (three times of no. of joints).

Horizontal-shear equations $\cdots m+n$ (sum of nos. of rows and columns).

Vertical-shear equations $\cdots (m-1)+(n-1)=m+n-2$ (sum of nos. of bays of constituent plane frames in both arrangements).

Compatibility equations $\cdots 6mn-4(m+n)+2$(twice the nos. of spaces)

Totals $\cdots 9mn-2(m+n)$.

On the other hand, the number of unknown φ's is $3mn$, and of ψ's $6mn-2(m+n)$, which evidently agrees, in all, with that of condition equations shown above. Therefore, we can conclude that the solution is always possible.

The similar discussion will hold for the frames of multi-story.

If we assume that the supports are immovable and the member-lenghths are immutable, which is the case usually considered, the compatibility equations become needless, because we can tell, by inspection, as follows:

1) The beams can revolve about y axis only. All the beams spanning adjacent two plane frames have the same ψ. Hence, the number of ψ's of them becomes $(m-1)+(n-1)=m+n-2$.

2) All the columns in any constituent plane frame have the same R, if their heights are all equal. Hence, the number of ψ's of columns is $m+n$.

3) Thus, the number of ψ's totals to $2(m+n-1)$, which just comes to the number of shear equations.

4) The matter is the same, if there is irregularity in the length of columns and beams.

Furthermore, if the joints do not displace due to the symmetry or to the lateral supports, ψ's become zeros, hence the solution is carried out by the use of the joint equilibrium equations only.

14. Comparison with Method of Redundancies

In order to obtain a statically determinate rigid connection in space by assembling k bars, we have to apply six constraints at each knot. Since there are $k-1$ knots, the required constraints must be $6(k-1)$ in all. The frame thus fabricated is then attached to the foundation to complete a statically determinate structure. For this purpose, we are again in need of six external constraints. Thus, we see that a statically determinate rigid frame should have the constraints of $6(k-1)+6=6k$ in total.

If the frame under consideration is statically indeterminate, it has more constraints than $6k$, and its degree of redundancy N is found to be

$$N=r+j-6k,$$

where $r=$ total external constraints,

$j=$ total internal constraints. At a rigid joint, j denotes the number of meeting members less than one multiplied by six.

The methods of redundancies, the classical methods, take the end-forces for unknowns, which are the constraints above mentioned. Therefore, N denotes the number of redundant forces to be found.

Let, for example, a single storied frame of m rows and n columns be considered. We obtain, referring to Table 1, $r = 6mn$, $j = 24mn - 12(m+n)$ and $k = 3mn - (m+n)$. Therefore

$$N = 12mn - 6(m+n).$$

Even in the case where the supports and the joints do not displace, we have to solve as many equations as this. If such a frame is analysed by the author's method, the number of unknowns to be found is $3mn$, for which only the joint equilibrium equations are enough. For a simple frame of $m = n = 2$, we see $N = 24$ and $3mn = 12$; the difference is indeed 12. This shows that how great facilities are obtained by the author's method.

Chapter III. Illustrative Examples

15. Torsion Effects in a Rigid Frame in Space

A symmetrical space frame in Fig. 9 subjects to symmetrical loadings as shown. The three-dimensional analysis by the author's method proceeds as follows:

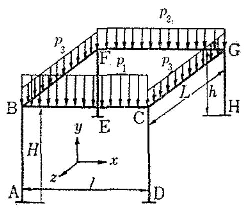


Fig. 9

1) Expressions of End-Moments

By inspection, we readily have the relations

$$\varphi_{xB} = \varphi_{xC},\quad \varphi_{xF} = \varphi_{xG},\quad \varphi_{yB} = \varphi_{yC} = \varphi_{yF} = \varphi_{yG} = 0,\quad \varphi_{zB} = -\varphi_{zC},\quad \varphi_{zF} = -\varphi_{zG}.$$

With these, the following end-moment expressions are written down.

$$\left.\begin{aligned}
&\text{For member AB:}\\
&\quad\text{About } x \text{ axis,} && M_{xAB} = k_{xAB}(\varphi_{xB} + \psi_{xAB}),\\
&\quad\text{Do.}\quad , && M_{xBA} = k_{xAB}(2\varphi_{xB} + \psi_{xAB}),\\
&\quad\text{About } z \text{ axis,} && M_{zAB} = k_{zAB}\varphi_{zB},\\
&\quad\text{Do.}\quad , && M_{zBA} = 2k_{zAB}\varphi_{zB}.\\
&\text{For member BC:}\\
&\quad\text{About } z \text{ axis,} && M_{zBC} = k_{zBC}\varphi_{zB} + C_{zBC}.\\
&\text{For member BF:}\\
&\quad\text{About } x \text{ axis,} && M_{xBF} = k_{xBF}(2\varphi_{xB} + \varphi_{xF}) + C_{xBF},\\
&\quad\text{Do.}\quad , && M_{xFB} = k_{xBF}(\varphi_{xB} + 2\varphi_{xF}) + C_{xFB},\\
&\quad\text{About } z \text{ axis,} && M_{zBF} = 2\beta k_{zBF}(\varphi_{zB} - \varphi_{zF}),\\
&\quad\text{Do.}\quad , && M_{zFB} = 2\beta k_{zBF}(\varphi_{zF} - \varphi_{zB}).\\
&\text{For member EF:}
\end{aligned}\right\}\quad\text{(a)}$$

About x axis, $M_{x\mathrm{EF}} = k_{x\mathrm{EF}}(\varphi_{x\mathrm{F}} + \psi_{x\mathrm{EF}}) = k_{x\mathrm{EF}}(\varphi_{x\mathrm{F}} + (H/h)\psi_{x\mathrm{AB}})$,

Do. , $M_{x\mathrm{FE}} = k_{x\mathrm{EF}}(2\varphi_{x\mathrm{F}} + \psi_{x\mathrm{EF}}) = k_{x\mathrm{EF}}(2\varphi_{x\mathrm{F}} + (H/h)\psi_{x\mathrm{AB}})$,

About z axis, $M_{z\mathrm{EF}} = k_{z\mathrm{EF}}\varphi_{z\mathrm{F}}$,

Do. , $M_{z\mathrm{FE}} = 2k_{z\mathrm{EF}}\varphi_{z\mathrm{F}}$.

For member FG:

About z axis, $M_{z\mathrm{FG}} = k_{z\mathrm{FG}}\varphi_{z\mathrm{F}} + C_{z\mathrm{FG}}$.

2) Elastic Equations

Condition equations to be satisfied by these end-moments are:

i) Joint Equilibrium Equations

At joint B:

About x axis, $M_{x\mathrm{BA}} + M_{x\mathrm{BF}} = 0$,

About z axis, $M_{z\mathrm{BA}} + M_{z\mathrm{BC}} + M_{z\mathrm{BF}} = 0$.

At joint F:

About x axis, $M_{x\mathrm{FB}} + M_{x\mathrm{FE}} = 0$,

About z axis, $M_{z\mathrm{FB}} + M_{z\mathrm{FE}} + M_{z\mathrm{FG}} = 0$.

ii) Horizontal Shear Equations

$$X_{z\mathrm{BA}} + X_{z\mathrm{FE}} = 0. \qquad \text{(b)}$$

Substituting the above end-moments (a) into these equilibrium equations (b), we get the following simultaneous equations.

$$2(k_{x\mathrm{AB}} + k_{x\mathrm{BF}})\varphi_{x\mathrm{B}} + k_{x\mathrm{BF}}\varphi_{x\mathrm{F}} + k_{x\mathrm{AB}}\psi_{x\mathrm{AB}} + C_{x\mathrm{BF}} = 0,$$

$$k_{x\mathrm{BF}}\varphi_{x\mathrm{B}} + 2(k_{x\mathrm{BF}} + k_{x\mathrm{EF}})\varphi_{x\mathrm{F}} + (H/h)k_{x\mathrm{EF}}\psi_{x\mathrm{AB}} + C_{x\mathrm{FB}} = 0,$$

$$(2k_{z\mathrm{AB}} + k_{z\mathrm{BC}} + 2\beta k_{z\mathrm{BF}})\varphi_{z\mathrm{B}} - 2\beta k_{z\mathrm{BF}}\varphi_{z\mathrm{F}} + C_{z\mathrm{BC}} = 0,$$

$$(2k_{z\mathrm{EF}} + k_{z\mathrm{FG}} + 2\beta k_{z\mathrm{BF}})\varphi_{z\mathrm{F}} - 2\beta k_{z\mathrm{BF}}\varphi_{z\mathrm{B}} + C_{z\mathrm{FG}} = 0,$$

$$3hk_{x\mathrm{AB}}\varphi_{x\mathrm{B}} + 3H\,k_{x\mathrm{EF}}\varphi_{x\mathrm{F}} + (2hk_{x\mathrm{AB}} + 2(H^2/h)k_{x\mathrm{EF}})\psi_{x\mathrm{AB}} = 0.$$

Solving these, we have:

Joint rotations at B,

$$\varphi_{x\mathrm{B}} = (-1/\nu)\{[(4k_{x\mathrm{BF}} + k_{x\mathrm{EF}})(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3(H^2/h)k^2_{x\mathrm{EF}}]C_{x\mathrm{BF}}$$
$$- [2k_{x\mathrm{BF}}(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3Hk_{x\mathrm{AB}}k_{x\mathrm{EF}}]C_{x\mathrm{FB}}\},$$

$$\varphi_{z\mathrm{B}} = (-1/\mu)\{(2k_{z\mathrm{EF}} + k_{z\mathrm{FG}} + 2\beta k_{z\mathrm{BF}})C_{z\mathrm{BC}} + 2\beta k_{z\mathrm{BF}}C_{z\mathrm{FG}}\}.$$

Joint rotations at F,

$$\varphi_{x\mathrm{F}} = (1/\nu)\{[2k_{x\mathrm{BF}}(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3Hk_{x\mathrm{AB}}k_{x\mathrm{EF}}]C_{x\mathrm{BF}}$$
$$- [4(k_{x\mathrm{AB}} + k_{x\mathrm{BF}})(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3hk^2_{x\mathrm{AB}}]C_{x\mathrm{FB}}\},$$

$$\varphi_{z\mathrm{F}} = (-1/\mu)\{(2k_{z\mathrm{AB}} + k_{z\mathrm{BC}} + 2\beta k_{z\mathrm{BF}})C_{z\mathrm{FG}} + 2\beta k_{z\mathrm{BF}} C_{z\mathrm{BC}}\}.$$

Revolution of member AB, (c)

$$\psi_{x\mathrm{AB}} = (-1/\nu)\{[3Hk_{x\mathrm{BF}}k_{x\mathrm{EF}} - 6hk_{x\mathrm{AB}}(k_{x\mathrm{BF}} + k_{x\mathrm{EF}})]C_{x\mathrm{BF}}$$
$$- [6Hk_{x\mathrm{EF}}(k_{x\mathrm{AB}} + k_{x\mathrm{BF}}) + 3hk_{x\mathrm{AB}}k_{x\mathrm{BF}}]C_{x\mathrm{BF}}\},$$

where,

$$\mu = (2k_{z\mathrm{AB}} + k_{z\mathrm{BC}} + 2\beta k_{z\mathrm{BF}})(2k_{z\mathrm{EF}} + k_{z\mathrm{FG}} + 2\beta k_{z\mathrm{BF}}) - 4\beta^2 k^2{}_{z\mathrm{BF}},$$

$$\nu = 2(k_{x\mathrm{AB}} + k_{x\mathrm{BF}})\{4(k_{x\mathrm{BF}} + k_{x\mathrm{EF}})(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3(H^2/h)k^2{}_{x\mathrm{EF}}\}$$
$$- k_{x\mathrm{BF}}\{2k_{x\mathrm{BF}}(hk_{x\mathrm{AB}} + (H^2/h)k_{x\mathrm{EF}}) - 3Hk_{x\mathrm{AB}}k_{x\mathrm{EF}}\}$$
$$+ 3hk_{x\mathrm{AB}}\{(H/h)k_{x\mathrm{BF}}k_{x\mathrm{EF}} - 2k_{x\mathrm{AB}}(k_{x\mathrm{BF}} + k_{x\mathrm{EF}})\}.$$

Finally, introducing these $\varphi-$ and $\psi-$ values into the expressions of end moments (a), we arrive at the solutions.

Consider now the case in which $k_{x\mathrm{BF}} = k_{z\mathrm{BF}} = 0$ and $C_{x\mathrm{BF}} = C_{x\mathrm{FB}} = 0$, then we have from (c)

$$\varphi_{x\mathrm{B}} = \varphi_{x\mathrm{F}} = \psi_{x\mathrm{AB}} = \psi_{x\mathrm{EF}} = 0,$$

$$\varphi_{z\mathrm{B}} = -C_{z\mathrm{BC}}/(2k_{z\mathrm{AB}} + k_{z\mathrm{BC}}),\quad \varphi_{z\mathrm{F}} = -C_{z\mathrm{FG}}/(2k_{z\mathrm{EF}} + k_{z\mathrm{FG}}).$$

These coincide with the solutions by the conventional two-dimensional method, that is, the solutions for the constituent plane frames ABCD or EFGH.

In the followings, we will consider in detail about $M_{z\mathrm{BA}}$, $M_{z\mathrm{BC}}$ and $M_{z\mathrm{BF}}$.

3) Relations between $k_{z\mathrm{EF}}$ and M's

Assuming that $p_1 = p_2$, $k_{z\mathrm{AB}} = k_{z\mathrm{BC}} = k_{z\mathrm{BF}} = k_{z\mathrm{FG}} = 1$ and that $\beta = 0.11$, putting $m = 6$ in eq. (18), we obtain $\mu = 6.44\, k_{z\mathrm{EF}} + 3.88$, $C_{z\mathrm{BC}} = C_{z\mathrm{FG}}$ and

$$\varphi_{z\mathrm{B}} = -(2k_{z\mathrm{EF}} + 1.44)C_{z\mathrm{BC}}/(6.44k_{z\mathrm{EF}} + 3.88),$$

$$\varphi_{z\mathrm{F}} = -3.44C_{z\mathrm{BC}}/(6.44k_{z\mathrm{EF}} + 3.88).$$

Hence we get, from eqs. (a),

$$M_{z\mathrm{BA}} = -(2k_{z\mathrm{EF}} + 1.44)C_{z\mathrm{BC}}/(3.22k_{z\mathrm{EF}} + 1.94),$$

$$M_{z\mathrm{BC}} = (2.22k_{z\mathrm{EF}} + 1.22)C_{z\mathrm{BC}}/(3.22k_{z\mathrm{EF}} + 1.94),$$

$$M_{z\mathrm{BF}} = -0.22(k_{z\mathrm{EF}} - 1)C_{z\mathrm{BC}}/(3.22k_{z\mathrm{EF}} + 1.94).$$

Thus we have the end-moments expressed by $k_{z\mathrm{EF}}$, the relations between them are shown in Table 2.

Table 2 $k_{zEF} \sim M$

M \ k_{zEF}	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
M_{zBA}	−0.74	−0.71	−0.69	−0.68	−0.67	−0.67	−0.66	−0.66	−0.65	−0.65	−0.65
M_{zBC}	0.63	0.64	0.65	0.66	0.66	0.67	0.67	0.67	0.67	0.67	0.67
M_{zBF}	0.11	0.07	0.04	0.02	0.01	0	−0.01	−0.01	−0.02	−0.02	−0.02

Multiplier: $C_{zBC} = -p_1 l^2/12$

See that as k_{zEF} increases, M_{zBC} increases and M_{zBA} and M_{zBF} decrease, but for $k_{zEF} > 1$ they remain almost constant. Again, when k_{zEF} becomes unity, i.e., the frame becomes also symmetrical about xz plane, the twisting moments vanish and the frame ABCD may be isolated and treated two-dimensionally.

4) Relations between C_{zFG}/C_{zBC} and M's

For convenience sake, letting all stiffness ratios be unities and β be 0.11, we have

$$\varphi_{zB} = -(0.31C_{zBC} + 0.02C_{zFG}),$$

$$\varphi_{zF} = -(0.31C_{zFG} + 0.02C_{zBC}),$$

and

$$M_{zBA} = -C_{zBC}(0.62 + 0.04C_{zFG}/C_{zBC}),$$

$$M_{zBC} = C_{zBC}(0.69 - 0.02C_{zFG}/C_{zBC}),$$

$$M_{zBF} = -0.06C_{zBC}(1 - C_{zFG}/C_{zBC}).$$

The relations between C_{zFG}/C_{zBC} and M's are shown in Table 3.

Table 3 $C_{zFG}/C_{zBC} \sim M$

M \ C_{zFG}/C_{zBC}	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
M_{zBA}	−0.62	−0.63	−0.64	−0.64	−0.65	−0.67	−0.67	−0.68	−0.68	−0.69	−0.70
M_{zBC}	0.69	0.69	0.68	0.68	0.67	0.67	0.67	0.66	0.66	0.65	0.65
M_{zBF}	−0.07	−0.06	−0.04	−0.02	−0.01	0	0.01	0.02	0.04	0.05	0.06

Miltiplier: $C_{zBC} = -p_1 l^2/12$

From this table it is perceived that M_{zBA} and M_{zBC} are not so different from the two-dimensional solution, $0.67\,C_{zBC}$, and that the magnitudes of twisting moments in the range considered are very small compared with those of bending moments. Thus, it may be concluded that the variation of C_{zFG}/C_{zBC} does not effect so much upon the end-moments analysed either three-dimensionally or two-dimensionally.

5) Relations between k_{zBF} and M's

To show the effects of the torsional rigidity of a member, consider the frame in Fig. 10 which is the special case in Fig. 9. For this frame we can put $k_{zEF} = k_{zFG} = \infty$ and $C_{zFG} = 0$ in the foregoing solution.

To investigate the relations between k_{zBF} and the end-moments, let, for example, $k_{zAB} = k_{zBC} = 1$. Then, we have

$$\varphi_{zB} = -C_{zBC}/(3 + 2\beta k_{zBF})$$

and

$$M_{zBA} = -2C_{zBC}/(3 + 0.22k_{zBF}),$$

$$M_{zBC} = 2(1 + 0.11k_{zBF})C_{zBC}/(3 + 0.22k_{zBF}),$$

$$M_{zBF} = -0.22k_{zBF}C_{zBC}/(3 + 0.22k_{zBF}).$$

Table 4 shows the relations thus obtained which are plotted in Fig. 11.

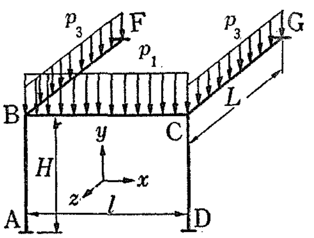


Fig. 10

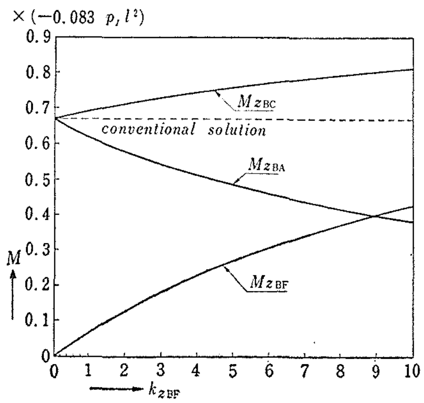


Fig. 11

Table 4 $k_{zBF} \sim M$

M \ k_{zBF}	0	0.2	0.4	0.6	0.8	1.0	2.0	4.0	6.0	8.0	10.0	∞
M_{zBA}	-0.67	-0.66	-0.65	-0.64	-0.63	-0.62	-0.58	-0.52	-0.46	-0.42	-0.38	0
M_{zBC}	0.67	0.67	0.68	0.68	0.69	0.69	0.71	0.74	0.77	0.79	0.81	1
M_{zBF}	0	-0.014	-0.028	-0.042	-0.055	-0.068	-0.128	-0.227	-0.306	-0.370	-0.423	—

Multiplier: $C_{zBC} = -p_1 l^2/12$

See that for $k_{zBF}=\infty$, M_{zBC} coincides with the end-moment of a both-ends-fixed-beam and M_{zBA} vanishes. Futhermore, with the decrease of k_{zBF}, M_{zBA} and M_{zBC} approach respectively to the values of the plane frame ABCD.

Table 5 shows the percentage errors in the end-moments resulting from the two-dimensional analysis; the values from three-dimensional analysis being taken as references. From this, it will be realized that the members BF and CG, which are attached to the plane frame ABCD, have considerable effects and they should never be omitted in the practical design. Provided that the members BF and CG are very slender, the conventional two-dimensional analysis will be satisfactorily applied. In practice, if we allow 10% errors, we must allot the values less than 2 to *k*.

Table 5 k_{zBF}～percentage errors

k_{zBF}	0	0.2	0.4	0.6	0.8	1.0	2.0	4.0	6.0	8.0	10.0	∞	
$(M_{zBA}-Mo)/M_{zBA}$	0	+1.5	+3.0	+4.5	+6.0	+7.5	+13.4	+22	+31	+37	+43	—	(%)
$(M_{zBC}-Mo)/M_{zBC}$	0	0	−1.5	−1.5	−3.0	−3.0	−9.0	−10	−15	−18	−21	—	(%)

From this numerical example, it is perceived that the rigid frame in space without side-sway may be analysed by the conventional two-dimensional method provided that the torsional stiffness of any member is not so different from the bending stiffnesses of other members; if it is not so, it is necessary to solve the frame three-dimensionally.

16. Unsymmetrical Rigid Frame in Space (1)

A space frame, Fig. 12, loaded laterally in *z* direction, has the dimensions and *k*-values as shown in Table 6. The procedures of analysis and some related discussions will be given below.

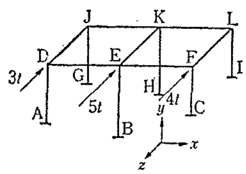


Fig. 12

Table 6

Member	AD	BE	CF	DE	EF	DJ	EK	FL	JK	KL	GJ	HK	IL	
Length	4.2	5.4	3.0	4.0	4.0	5.0	5.0	5.0	4.0	4.0	3.6	4.2	3.0	(m)
k_x	1.0	1.2	0.8	1.4	1.4	1.5	1.5	1.5	1.4	1.4	1.0	1.2	0.8	
k_y	1.0	1.0	1.0	1.2	1.2	1.0	1.0	1.0	1.2	1.2	1.0	1.2	0.8	
k_z	1.0	1.2	0.8	1.0	1.0	1.4	1.4	1.4	0.8	0.8	1.0	1.0	1.0	

1) Expressions of End-Moments

In the followings, we allot 0.107 to β assuming $m = 6$ in eq. (18).

For member AD :

$$M_{x\mathrm{AD}} = \varphi_{x\mathrm{D}} + \psi_{x\mathrm{AD}}, \qquad M_{x\mathrm{DA}} = 2\varphi_{x\mathrm{D}} + \psi_{x\mathrm{AD}}.$$

$$M_{y\mathrm{AD}} = -0.21\varphi_{y\mathrm{D}}, \qquad M_{y\mathrm{DA}} = 0.21\varphi_{y\mathrm{D}}.$$

$$M_{z\mathrm{AD}} = \varphi_{z\mathrm{D}} + \psi_{z\mathrm{AD}}, \qquad M_{z\mathrm{DA}} = 2\varphi_{z\mathrm{D}} + \psi_{z\mathrm{AD}}.$$

For member BE :

$$M_{x\mathrm{BE}} = 1.2\varphi_{x\mathrm{E}} + 1.2\psi_{x\mathrm{BE}}, \qquad M_{x\mathrm{EB}} = 2.4\varphi_{x\mathrm{E}} + 1.2\psi_{x\mathrm{BE}}.$$

$$M_{y\mathrm{BE}} = -0.21\varphi_{y\mathrm{E}}, \qquad M_{y\mathrm{EB}} = 0.21\varphi_{y\mathrm{E}}.$$

$$M_{z\mathrm{BE}} = 1.2\varphi_{z\mathrm{E}} + 0.93\psi_{z\mathrm{AD}}, \qquad M_{z\mathrm{EB}} = 2.4\varphi_{z\mathrm{E}} + 0.93\psi_{z\mathrm{AD}}.$$

For member CF :

$$M_{x\mathrm{CF}} = 0.8\varphi_{x\mathrm{F}} + 0.8\psi_{x\mathrm{CF}}, \qquad M_{x\mathrm{FC}} = 1.6\varphi_{x\mathrm{F}} + 0.8\psi_{x\mathrm{CF}}.$$

$$M_{y\mathrm{CF}} = -0.21\varphi_{y\mathrm{F}}, \qquad M_{y\mathrm{FC}} = 0.21\varphi_{y\mathrm{F}}.$$

$$M_{z\mathrm{CF}} = 0.8\varphi_{z\mathrm{F}} + 1.12\psi_{z\mathrm{AD}}, \qquad M_{z\mathrm{FC}} = 1.6\varphi_{z\mathrm{F}} + 1.12\psi_{z\mathrm{AD}}.$$

For member DE :

$$M_{x\mathrm{DE}} = 0.30(\varphi_{x\mathrm{D}} - \varphi_{x\mathrm{E}}), \qquad M_{x\mathrm{ED}} = 0.30(\varphi_{x\mathrm{E}} - \varphi_{x\mathrm{D}}).$$

$$M_{y\mathrm{DE}} = 1.2(2\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{E}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}}),$$

$$M_{y\mathrm{ED}} = 1.2(2\varphi_{y\mathrm{E}} + \varphi_{y\mathrm{D}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}}).$$

$$M_{z\mathrm{DE}} = 2\varphi_{z\mathrm{D}} + \varphi_{z\mathrm{E}}, \qquad M_{z\mathrm{ED}} = 2\varphi_{z\mathrm{E}} + \varphi_{z\mathrm{D}}.$$

For member EF :

$$M_{x\mathrm{EF}} = 0.3(\varphi_{x\mathrm{E}} - \varphi_{x\mathrm{F}}), \qquad M_{x\mathrm{FE}} = 0.3(\varphi_{x\mathrm{F}} - \varphi_{x\mathrm{E}}).$$

$$M_{y\mathrm{EF}} = 1.2(2\varphi_{y\mathrm{E}} + \varphi_{y\mathrm{F}} + 1.35\psi_{x\mathrm{BE}} - 0.75\psi_{x\mathrm{CF}}),$$

$$M_{y\mathrm{FE}} = 1.2(2\varphi_{y\mathrm{F}} + \varphi_{y\mathrm{E}} + 1.35\psi_{x\mathrm{BE}} - 0.75\psi_{x\mathrm{CF}}).$$

$$M_{z\mathrm{EF}} = 2\varphi_{z\mathrm{E}} + \varphi_{z\mathrm{F}}, \qquad M_{z\mathrm{FE}} = 2\varphi_{z\mathrm{F}} + \varphi_{z\mathrm{E}}.$$

For member DJ :

$$M_{x\mathrm{DJ}} = 1.5(2\varphi_{x\mathrm{D}} + \varphi_{x\mathrm{J}}), \qquad M_{x\mathrm{JD}} = 1.5(2\varphi_{x\mathrm{J}} + \varphi_{x\mathrm{D}}).$$

$$M_{y\mathrm{DJ}} = (2\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{J}} + 0.724\psi_{z\mathrm{GJ}} - 0.84\psi_{z\mathrm{AD}}),$$

$$M_{y\mathrm{JD}} = (2\varphi_{y\mathrm{J}} + \varphi_{y\mathrm{D}} + 0.724\psi_{z\mathrm{GJ}} - 0.84\psi_{z\mathrm{AD}}).$$

$$M_{z\mathrm{DJ}} = 0.3(\varphi_{z\mathrm{D}} - \varphi_{z\mathrm{J}}), \qquad M_{z\mathrm{JD}} = 0.3(\varphi_{z\mathrm{J}} - \varphi_{z\mathrm{D}}).$$

For member EK :

$M_{x\mathrm{EK}} = 1.5(2\varphi_{x\mathrm{E}} + \varphi_{x\mathrm{K}}),$ $\qquad M_{x\mathrm{KE}} = 1.5(2\varphi_{x\mathrm{K}} + \varphi_{x\mathrm{E}}).$

$M_{y\mathrm{EK}} = (2\varphi_{y\mathrm{E}} + \varphi_{y\mathrm{K}} + 0.72\phi_{z\mathrm{GJ}} - 0.84\phi_{z\mathrm{AD}}),$

$M_{y\mathrm{KE}} = (2\varphi_{y\mathrm{K}} + \varphi_{y\mathrm{E}} + 0.72\phi_{z\mathrm{GJ}} - 0.84\phi_{z\mathrm{AD}}).$

$M_{z\mathrm{EK}} = 0.3(\varphi_{z\mathrm{E}} - \varphi_{z\mathrm{K}}),$ $\qquad M_{z\mathrm{KE}} = 0.3(\varphi_{z\mathrm{K}} - \varphi_{z\mathrm{E}}).$

For member FL :

$M_{x\mathrm{FL}} = 1.5(2\varphi_{x\mathrm{F}} + \varphi_{x\mathrm{L}}),$ $\qquad M_{x\mathrm{LF}} = 1.5(2\varphi_{x\mathrm{L}} + \varphi_{x\mathrm{F}}).$

$M_{y\mathrm{FL}} = (2\varphi_{y\mathrm{F}} + \varphi_{y\mathrm{L}} + 0.72\phi_{z\mathrm{GJ}} - 0.84\phi_{z\mathrm{AD}}),$

$M_{y\mathrm{LF}} = (2\varphi_{y\mathrm{L}} + \varphi_{y\mathrm{F}} + 0.72\phi_{z\mathrm{GJ}} - 0.84\phi_{z\mathrm{AD}}).$

$M_{z\mathrm{FL}} = 0.3(\varphi_{z\mathrm{F}} - \varphi_{z\mathrm{L}}),$ $\qquad M_{z\mathrm{LF}} = 0.3(\varphi_{z\mathrm{L}} - \varphi_{z\mathrm{F}}).$

For member JK :

$M_{x\mathrm{JK}} = 0.3(\varphi_{x\mathrm{J}} - \varphi_{x\mathrm{K}}),$ $\qquad M_{x\mathrm{KJ}} = 0.3(\varphi_{x\mathrm{K}} - \varphi_{x\mathrm{J}}).$

$M_{y\mathrm{JK}} = 1.2(2\varphi_{y\mathrm{J}} + \varphi_{y\mathrm{K}} + 1.05\phi_{x\mathrm{AD}} - 1.35\phi_{x\mathrm{BE}}),$

$M_{y\mathrm{KJ}} = 1.2(2\varphi_{y\mathrm{K}} + \varphi_{y\mathrm{J}} + 1.05\phi_{x\mathrm{AD}} - 1.35\phi_{x\mathrm{BE}}).$

$M_{z\mathrm{JK}} = 0.8(2\varphi_{z\mathrm{J}} + \varphi_{z\mathrm{K}}),$ $\qquad M_{z\mathrm{KJ}} = 0.8(2\varphi_{z\mathrm{K}} + \varphi_{z\mathrm{J}}).$

For member KL :

$M_{x\mathrm{KL}} = 0.3(\varphi_{x\mathrm{K}} - \varphi_{x\mathrm{L}}),$ $\qquad M_{x\mathrm{LK}} = 0.3(\varphi_{x\mathrm{L}} - \varphi_{x\mathrm{K}}).$

$M_{y\mathrm{KL}} = 1.2(2\varphi_{y\mathrm{K}} + \varphi_{y\mathrm{L}} + 1.35\phi_{x\mathrm{BE}} - 0.75\phi_{x\mathrm{CF}}),$

$M_{y\mathrm{LK}} = 1.2(2\varphi_{y\mathrm{L}} + \varphi_{y\mathrm{K}} + 1.35\phi_{x\mathrm{BE}} - 0.75\phi_{x\mathrm{CF}}).$

$M_{z\mathrm{KL}} = 0.8(2\varphi_{z\mathrm{K}} + \varphi_{z\mathrm{L}}),$ $\qquad M_{z\mathrm{LK}} = 0.8(2\varphi_{z\mathrm{L}} + \varphi_{z\mathrm{K}}).$

For member GJ :

$M_{x\mathrm{GJ}} = \varphi_{x\mathrm{J}} + 1.17\phi_{x\mathrm{AD}},$ $\qquad M_{x\mathrm{JG}} = 2\varphi_{x\mathrm{J}} + 1.17\phi_{x\mathrm{AD}}.$

$M_{y\mathrm{GJ}} = -0.21\varphi_{y\mathrm{J}},$ $\qquad M_{y\mathrm{JG}} = 0.21\varphi_{y\mathrm{J}}.$

$M_{z\mathrm{GJ}} = \varphi_{z\mathrm{J}} + \phi_{z\mathrm{GJ}},$ $\qquad M_{z\mathrm{JG}} = 2\varphi_{z\mathrm{J}} + \phi_{z\mathrm{GJ}}.$

For member HK :

$M_{x\mathrm{HK}} = 1.2\varphi_{x\mathrm{K}} + 1.54\phi_{x\mathrm{BE}},$ $\qquad M_{x\mathrm{KH}} = 2.4\varphi_{x\mathrm{K}} + 1.54\phi_{x\mathrm{BE}}.$

$M_{y\mathrm{HK}} = -0.26\varphi_{y\mathrm{K}}$, $M_{y\mathrm{KH}} = 0.26\varphi_{y\mathrm{K}}$.

$M_{z\mathrm{HK}} = \varphi_{z\mathrm{K}} + 0.86\psi_{z\mathrm{GJ}}$, $M_{z\mathrm{KH}} = 2\varphi_{z\mathrm{K}} + 0.86\psi_{z\mathrm{GJ}}$.

For member IL :

$M_{x\mathrm{IL}} = 0.8\varphi_{x\mathrm{L}} + 0.8\psi_{x\mathrm{CF}}$, $M_{x\mathrm{LI}} = 1.6\varphi_{x\mathrm{F}} + 0.8\psi_{x\mathrm{CF}}$.

$M_{y\mathrm{IL}} = -0.17\varphi_{y\mathrm{L}}$, $M_{y\mathrm{LI}} = 0.17\varphi_{y\mathrm{L}}$.

$M_{z\mathrm{IL}} = \varphi_{z\mathrm{L}} + 1.2\psi_{z\mathrm{GJ}}$, $M_{z\mathrm{LI}} = 2\varphi_{z\mathrm{L}} + 1.2\psi_{z\mathrm{GJ}}$.

2) Expressions of End-Shears

$X_{z\mathrm{DA}} = -(0.71\varphi_{x\mathrm{D}} + 0.48\psi_{x\mathrm{AD}})$,

$X_{z\mathrm{DE}} = -(0.9\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{E}} + 0.63\psi_{x\mathrm{AD}} - 0.81\psi_{x\mathrm{BE}})$,

$X_{z\mathrm{JG}} = -(0.83\varphi_{x\mathrm{J}} + 0.65\psi_{x\mathrm{AD}})$,

$X_{z\mathrm{DE}} = X_{z\mathrm{ED}}$, $X_{z\mathrm{EB}} = -(0.67\varphi_{x\mathrm{E}} + 0.44\psi_{x\mathrm{BE}})$,

$X_{z\mathrm{KJ}} = -(0.9\varphi_{y\mathrm{J}} + 0.9\varphi_{y\mathrm{K}} + 0.63\psi_{x\mathrm{AD}} - 0.81\psi_{x\mathrm{BE}})$,

$X_{z\mathrm{JK}} = X_{z\mathrm{KJ}}$, $X_{z\mathrm{KH}} = -(0.86\varphi_{x\mathrm{K}} + 0.73\psi_{x\mathrm{BE}})$,

$X_{z\mathrm{FE}} = -(0.9\varphi_{y\mathrm{E}} + 0.9\varphi_{y\mathrm{F}} + 0.81\psi_{x\mathrm{BE}} - 0.45\psi_{x\mathrm{CF}})$,

$X_{z\mathrm{EF}} = X_{z\mathrm{FE}}$, $X_{z\mathrm{FC}} = -(0.8\varphi_{x\mathrm{F}} + 0.53\varphi_{x\mathrm{CF}})$,

$X_{z\mathrm{LK}} = -(0.9_{y\mathrm{K}} + 0.9\varphi_{y\mathrm{L}} + 0.81\psi_{x\mathrm{BE}} - 0.45\psi_{x\mathrm{CF}})$,

$X_{z\mathrm{KL}} = X_{z\mathrm{LK}}$, $X_{z\mathrm{LI}} = -(0.8\varphi_{x\mathrm{L}} + 0.53\psi_{x\mathrm{CF}})$,

$X_{x\mathrm{DA}} = -(0.71\varphi_{z\mathrm{D}} + 0.48\psi_{z\mathrm{AD}})$,

$X_{x\mathrm{EB}} = -(0.67\varphi_{z\mathrm{E}} + 0.34\psi_{z\mathrm{AD}})$, $X_{x\mathrm{FC}} = -(0.8\varphi_{z\mathrm{F}} + 0.75\psi_{z\mathrm{AD}})$,

$X_{x\mathrm{DJ}} = -(0.6\varphi_{y\mathrm{D}} + 0.6\varphi_{y\mathrm{J}} + 0.29\varphi_{z\mathrm{GJ}} - 0.34\psi_{z\mathrm{AD}})$,

$X_{x\mathrm{JD}} = X_{x\mathrm{DJ}}$,

$X_{x\mathrm{EK}} = -(0.6\varphi_{y\mathrm{E}} + 0.6\varphi_{y\mathrm{K}} + 0.29\varphi_{z\mathrm{GJ}} - 0.34\psi_{z\mathrm{AD}})$,

$X_{x\mathrm{KE}} = X_{x\mathrm{EK}}$,

$X_{x\mathrm{FL}} = -(0.6\varphi_{y\mathrm{F}} + 0.6\varphi_{y\mathrm{L}} + 0.29\psi_{z\mathrm{GJ}} - 0.34\psi_{z\mathrm{AD}})$,

$X_{x\mathrm{LF}} = X_{x\mathrm{FL}}$, $X_{x\mathrm{JG}} = -(0.83\varphi_{z\mathrm{J}} + 0.56\psi_{z\mathrm{GJ}})$,

$X_{x\mathrm{KH}} = -(0.71\varphi_{z\mathrm{K}} + 0.41\psi_{z\mathrm{GJ}})$, $X_{x\mathrm{LI}} = -(\varphi_{z\mathrm{L}} + 0.80\psi_{z\mathrm{GJ}})$.

3) Elastic Equations

i) Joint Equilibrium Equations

At joint D :

$M_{x\mathrm{DA}} + M_{x\mathrm{DE}} + M_{x\mathrm{DJ}} = 0.$ $\quad$ $M_{y\mathrm{DA}} + M_{y\mathrm{DE}} + M_{y\mathrm{DJ}} = 0.$

$M_{z\mathrm{DA}} + M_{z\mathrm{DE}} + M_{z\mathrm{DJ}} = 0.$

At joint E :

$M_{x\mathrm{ED}} + M_{x\mathrm{EB}} + M_{x\mathrm{EF}} + M_{x\mathrm{EK}} = 0.$ $\quad$ $M_{y\mathrm{ED}} + M_{y\mathrm{EB}} + M_{y\mathrm{EF}} + M_{y\mathrm{EK}} = 0.$

$M_{z\mathrm{ED}} + M_{z\mathrm{EB}} + M_{z\mathrm{EF}} + M_{z\mathrm{EK}} = 0.$

At joint F :

$M_{x\mathrm{FE}} + M_{x\mathrm{FC}} + M_{x\mathrm{FL}} = 0.$ $\quad$ $M_{y\mathrm{FE}} + M_{y\mathrm{FC}} + M_{y\mathrm{FL}} = 0.$

$M_{z\mathrm{FE}} + M_{z\mathrm{FC}} + M_{z\mathrm{FL}} = 0.$

At joint J :

$M_{x\mathrm{JG}} + M_{x\mathrm{JK}} + M_{x\mathrm{JD}} = 0.$ $\quad$ $M_{y\mathrm{JG}} + M_{y\mathrm{JK}} + M_{y\mathrm{JD}} = 0.$

$M_{z\mathrm{JG}} + M_{z\mathrm{JK}} + M_{z\mathrm{JD}} = 0.$

At joint K :

$M_{x\mathrm{KJ}} + M_{x\mathrm{KH}} + M_{x\mathrm{KL}} + M_{x\mathrm{KE}} = 0.$ $\quad$ $M_{y\mathrm{KJ}} + M_{y\mathrm{KH}} + M_{y\mathrm{KL}} + M_{y\mathrm{KE}} = 0.$

$M_{z\mathrm{KJ}} + M_{z\mathrm{KH}} + M_{z\mathrm{KL}} + M_{z\mathrm{KE}} = 0.$

At joint L :

$M_{x\mathrm{LK}} + M_{x\mathrm{LI}} + M_{x\mathrm{LF}} = 0.$ $\quad$ $M_{y\mathrm{LK}} + M_{y\mathrm{LI}} + M_{y\mathrm{LF}} = 0.$

$M_{z\mathrm{LK}} + M_{z\mathrm{LI}} + M_{z\mathrm{LF}} = 0.$

These conditions give eqs. (1)~(18) in Table 7.

ii) Shear Equations

For the 1st column-frame : $X_{z\mathrm{DA}} + X_{z\mathrm{DE}} + X_{z\mathrm{JG}} + X_{z\mathrm{JK}} - 3 = 0,$

〃 2nd 〃 : $X_{z\mathrm{ED}} - X_{z\mathrm{EB}} - X_{z\mathrm{EF}} + X_{z\mathrm{KJ}} - X_{z\mathrm{KH}} - X_{z\mathrm{KL}} + 5 = 0,$

〃 3rd 〃 : $X_{z\mathrm{FE}} - X_{z\mathrm{FC}} + X_{z\mathrm{LK}} - X_{z\mathrm{LI}} + 4 = 0,$

For the 1st row-frame : $X_{x\mathrm{DA}} + X_{x\mathrm{EB}} + X_{x\mathrm{FC}} - X_{x\mathrm{DJ}} - X_{x\mathrm{EK}} - X_{x\mathrm{FL}} = 0,$

〃 2nd 〃 : $X_{x\mathrm{JG}} + X_{x\mathrm{KH}} + X_{x\mathrm{LI}} + X_{x\mathrm{JD}} + X_{x\mathrm{KE}} + X_{x\mathrm{LF}} = 0.$

From these conditions we get eqs. (19)~(23) in Table 7.

Table 7 Elastic equations

Eq.	Left-hand side											
	φ_{xD}	φ_{xE}	φ_{xF}	φ_{xJ}	φ_{xK}	φ_{xL}	φ_{yD}	φ_{yE}	φ_{yF}	φ_{yJ}	φ_{yK}	φ_{yL}
(1)	5.30	−0.30		1.50								
(2)	−0.30	6.00	−0.30		1.50							
(3)		−0.03	4.90			1.50						
(4)	1.50			5.30	−0.30							
(5)		1.50		−0.30	6.00	−0.30						
(6)			1.50		−0.30	4.90						
(7)							4.61	1.20		1.00		
(8)							1.20	7.01	1.20		1.00	
(9)								1.20	4.61			1.00
(10)							1.00			4.61	1.20	
(11)								1.00		1.20	7.06	1.20
(12)									1.00		1.20	4.57
(13)												
(14)												
(15)												
(16)												
(17)												
(18)												
(19)	0.71			0.83			0.90	0.90		0.90	0.90	
(20)		0.67			0.86		−0.90		0.90	−0.90		0.90
(21)			0.80			0.80		−0.90	−0.90		−0.90	−0.90
(22)							−0.60	−0.60	−0.60	−0.60	−0.60	−0.60
(23)							0.60	0.60	0.60	0.60	0.60	0.60

Thus, we finally have twenty-three simultaneous equations in total, which give the solutions as follows:

Joint rotations at D, $\varphi_{xD} = +0.615$, $\varphi_{yD} = -0.175$, $\varphi_{zD} = -0.008,5$,

Do. at E, $\varphi_{xE} = +0.627$, $\varphi_{yE} = +0.040$, $\varphi_{zE} = -0.002,9$,

Do. at F, $\varphi_{xF} = +0.733$, $\varphi_{yF} = +0.239$, $\varphi_{zF} = -0.010,6$,

Do. at J, $\varphi_{xJ} = +0.833$, $\varphi_{yJ} = -0.175$, $\varphi_{zJ} = +0.008,9$,

Do. at K, $\varphi_{xK} = +0.944$, $\varphi_{yK} = +0.039$, $\varphi_{zK} = +0.003,2$,

Do. at L, $\varphi_{xL} = +0.761$, $\varphi_{yL} = +0.243$, $\varphi_{zL} = +0.010,8$,

Table 7—continued—

φ_{zD}	φ_{zE}	φ_{zF}	φ_{zJ}	φ_{zK}	φ_{zL}	ψ_{xAD}	ψ_{xBE}	ψ_{xCF}	ψ_{zAD}	ψ_{zGJ}	Right-hand side
						1.00					
							1.20				
								0.80			
						1.17					
							1.54				
								0.80			
						1.26	−1.62		−0.84	0.72	
						1.26		−0.90	−0.84	0.72	
							1.62	−0.90	−0.84	0.72	
						1.26	−1.62		−0.84	0.72	
						1.26		−0.90	−0.84	0.72	
							1.62	−0.90	−0.84	0.72	
4.30	1.00		−0.30						1.00		
1.00	6.70	1.00		−0.30					0.93		
	1.00	3.90			−0.30				1.12		
−0.30			3.90	0.80						1.00	
	−0.30		0.80	5.50	0.80					0.86	
		−0.30		0.80	3.90					1.20	
						2.39	−1.62				−3.00
						−1.26	4.41	−0.90			5.00
							−1.62	1.96			4.00
0.71	0.67	0.80							2.58	−0.86	
			0.83	0.71	1.00				−1.01	2.63	

and revolutions

$$\psi_{xAD} = -4.32, \qquad \psi_{xBE} = -3.98, \qquad \psi_{xCF} = -5.68,$$

$$\psi_{zAD} = +0.042,2, \qquad \psi_{zGJ} = -0.039,7.$$

The end-moments are found substituting these φ's and ψ's into the expressions in 1). The results are shown in Table 8 and in Fig. 13~16.

The horizontal and vertical reactions at supports are determined from the end-moments by statics.

Table 8 Values of end-moments (t-m)

Member	AD		BE		CF		DE		EF
End	A	D	B	E	C	F	D	E	E
M_x	−3.705	−3.090	−4.024	−3.271	−3.958	−3.371	−0.004	+0.004	−0.032
M_y	+0.037	−0.037	−0.008	+0.008	−0.050	+0.050	+0.632	+0.890	−0.953
M_z	+0.034	+0.025	+0.036	+0.032	+0.039	+0.030	−0.020	−0.014	−0.016

Member	EF	DJ		EK		FL		JK	
End	F	D	J	E	K	F	L	J	K
M_x	+0.032	+3.095	+3.422	+3.297	+3.773	+3.341	+3.383	−0.033	+0.033
M_y	−0.714	−0.589	−0.588	+0.055	+0.054	+0.657	+0.660	+0.632	+0.888
M_z	−0.024	−0.005	+0.005	−0.002	+0.002	−0.006	+0.006	+0.017	+0.012

Member	KL		GJ		HK		IL	
End	K	L	G	J	H	K	I	L
M_x	+0.055	−0.055	−4.221	−3.388	−4.996	−3.864	−3.935	−3.326
M_y	−0.951	−0.707	+0.037	−0.037	−0.010	+0.010	−0.041	+0.041
M_z	+0.014	+0.020	−0.031	−0.022	−0.031	−0.028	−0.037	−0.026

4) End-Moments Analysed Two-Dimensionally

The usual two-dimensional analysis treats the constituent plane frames ADJG, BEKH and CFLI separately. The solutions are easily carried out, and we have the results shown by broken lines in Fig. 13, which compares with the exact values, shown by full lines. Note that the two-dimensional analysis does not produce the values of M_y's and M_z's.

In Table 9 the end-moments are compared, and the errors produced by the conventional analysis are shown. It should be mentioned that the errors are remarkable and that they appear either on the safe side or on the dangerous side. The moments in Figs. 14, 15 and 16 are never been found by the conventional analysis, which leads us to the conclusion that the economical design can only be attained through the rigorous three-dimensional analysis. Especially, the rigid frames in space accompanying side-sways should be analysed three-dimensionally, otherwise, the design will come far from economy and considerable dangers will arise at times.

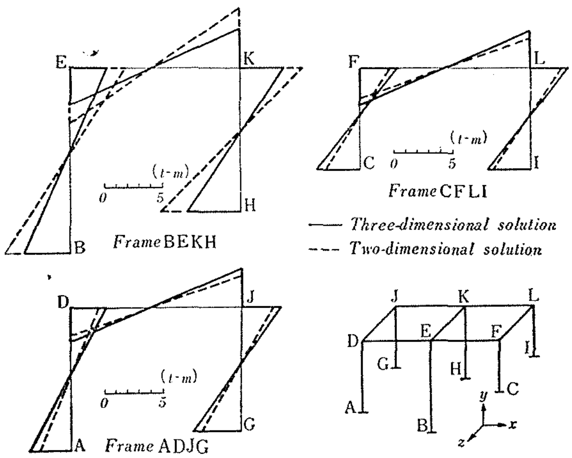


Fig. 13 M_x diagram

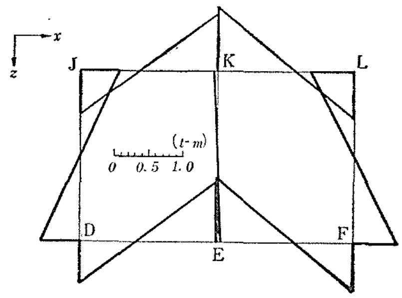


Fig. 14 M_y diagram

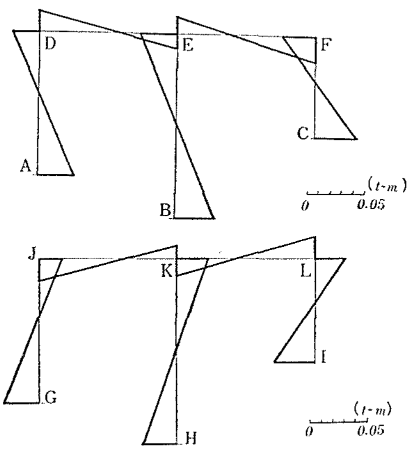


Fig. 15 M_z diagram

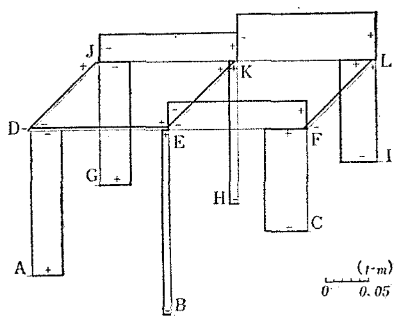


Fig. 16 Torsional moment diagram

Table 9 Percentage errors

(a) Frame ADJG

	M_{xAD}	M_{xDA}	M_{xDJ}	M_{xJD}	M_{xJG}	M_{xGJ}
Three-dim. (t-m)	−3.71	−3.10	+3.10	+3.42	−3.39	−4.22
Two-dim. (t-m)	−2.98	−2.48	+2.48	+2.73	−2.73	−3.39
Error (%)	−19.7	−20.0	−20.0	−20.0	−19.5	−19.7

(b) Frame BEKH

	M_{xBE}	M_{xEB}	M_{xEK}	M_{xKE}	M_{xKH}	M_{xHK}
Three-dim. (t-m)	−4.02	−3.27	+3.30	+3.77	−3.86	−5.00
Two-dim. (t-m)	−5.84	−4.77	+4.77	+5.54	−5.53	−7.21
Error (%)	+45.2	+45.9	+44.6	+47.0	+43.2	+44.2

(c) Frame CFLI

	M_{xCF}	M_{xFC}	M_{xFL}	M_{xLF}	M_{xLI}	M_{xIL}
Three-dim (t-m)	−3.96	−3.37	+3.34	+3.38	−3.33	−3.94
Two-dim. (t-m)	−3.24	−2.76	+2.75	+2.75	−2.75	−3.24
Error (%)	−18.2	−18.1	−17.7	−18.7	−17.4	−17.8

Note: Three-dim. : Three-dimensional solution
Two-dim. : Two-dimensional solution

17. Unsymmetrical Rigid Frame in Space (2)

Fig. 17 shows an unsymmetrical rigid frame in space. The member-lengths and stiffnesses are tabulated in Table 10. The calculations will be carried out assuming, for simplicity, $\beta = 0.1$.

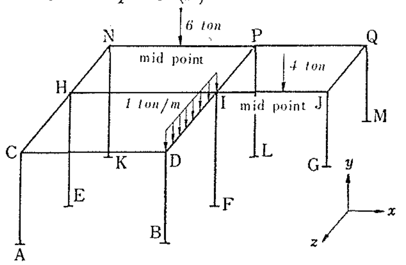


Fig. 17

Table 10

Member	A C	B D	C D	C H	C I	E H	F I	G J	H I	I J	H N	I P	J Q	K N	L P	M Q	N P	P Q
Length	5.0	5.0	8.0	5.0	5.0	6.0	6.0	4.0	8.0	6.0	4.0	4.0	4.0	6.0	6.0	4.0	8.0	6.0 (m)
k_x	0.8	0.8	1.0	1.5	1.0	1.2	1.0	1.1	1.2	1.5	0.7	1.0	1.6	1.2	1.2	1.3	1.2	1.0
k_y	1.2	1.1	0.9	1.2	0.7	1.4	1.0	1.3	1.0	1.3	0.9	0.8	1.4	0.8	1.0	0.8	0.6	0.6
k_z	1.0	1.0	0.7	0.8	0.9	1.2	1.0	0.9	0.8	1.4	1.3	0.9	0.9	1.2	1.2	1.4	1.2	1.0

Though the frame seems to have twenty-six unknown ψ's, they can be reduced to six by inspection or by referring to the compatibility condition in § *12*. Taking ψ_{xAC}, ψ_{xBD}, ψ_{xGJ}, ψ_{zAC}, ψ_{zEH} and ψ_{zKN} for the independent unknowns, the others are denoted as follows:

$\psi_{xEH} = (5/6) \times \psi_{xAC} = 0.833, 3\psi_{xAC},$

$\psi_{xKN} = (5/6) \times \psi_{xAC} = 0.833, 3\psi_{xAC},$

$\psi_{xFI} = (5/6) \times \psi_{xBD} = 0.833, 3\psi_{xBD},$

$\psi_{xLP} = (5/6) \times \psi_{xBD} = 0.833, 3\psi_{xBD},$

$\psi_{xMQ} = (4/4) \times \psi_{xGJ} = \psi_{xGJ},$

$\psi_{zBD} = (5/5) \times \psi_{zAC} = \psi_{zAC},$

$\psi_{zFI} = (6/6) \times \psi_{zEH} = \psi_{zEH},$

$\psi_{zGJ} = (6/4) \times \psi_{zEH} = 1.5\psi_{zEH},$

$\psi_{zLP} = (6/6) \times \psi_{zKN} = \psi_{zKN},$

$\psi_{zMQ} = (6/4) \times \psi_{zKN} = 1.5\psi_{zKN},$

$\psi_{yCD} = (5/8)\,\psi_{xAC} - (5/8)\psi_{xBD} = 0.625\psi_{xAC} - 0.625\psi_{xBD},$

$\psi_{yHI} = \psi_{yCD} = 0.625\psi_{xAC} - 0.625\psi_{xBD},$

$\psi_{yNP} = \psi_{yCD} = 0.625\psi_{xAC} - 0.625\psi_{xBD},$

$\psi_{yIJ} = (6/6)\psi_{xEI} - (4/6)\psi_{xGJ} = \psi_{xFI} - 0.666, 7\psi_{xGJ} = 0.833, 3\psi_{xBD} - 0.666, 7\psi_{xGJ},$

$\psi_{yPQ} = \psi_{yIJ} = 0.833, 3\psi_{xBD} - 0.666, 7\psi_{xGJ},$

$\psi_{yCH} = (6/5)\psi_{zEH} - (5/5)\psi_{zAC} = 1.2\psi_{zEH} - \psi_{zAC},$

$\psi_{yDI} = \psi_{yCH} = 1.2\psi_{zEH} - \psi_{zAC},$

$\psi_{yHN} = (6/4)\psi_{zKN} - (6/4)\psi_{zEH} = 1.5\psi_{zKN} - 1.5\psi_{zEH},$

$\psi_{yIP} = \psi_{yHN} = 1.5\psi_{zKN} - 1.5\psi_{zEH},$

$\psi_{yJQ} = \psi_{yHN} = 1.5\psi_{zKN} - 1.5\psi_{zEH}.$

Using these, we have the necessary expressions which follow:

1) Expressions of End-Moments

For member AC :

$M_{xAC} = 0.8(\varphi_{xC} + \psi_{xAC}),$ $\qquad M_{xCA} = 0.8(2\varphi_{xC} + \psi_{xAC}),$

$M_{yAC} = -0.24\varphi_{yC},$ $\qquad M_{yCA} = 0.24\varphi_{yC},$

$M_{zAC} = \varphi_{zC} + \psi_{zAC},$ $\qquad M_{zCA} = 2\varphi_{zC} + \psi_{zAC}.$

For member BD :

$M_{xBD} = 0.8(\varphi_{xD} + \psi_{xBD}),$ $\qquad M_{xDB} = 0.8(2\varphi_{xD} + \psi_{xBD}),$

$M_{y\mathrm{BD}} = -0.22\varphi_{y\mathrm{D}}$, $\quad M_{y\mathrm{DB}} = 0.22\varphi_{y\mathrm{D}}$,

$M_{z\mathrm{BD}} = \varphi_{z\mathrm{D}} + \psi_{z\mathrm{AC}}$, $\quad M_{z\mathrm{DB}} = 2\varphi_{z\mathrm{D}} + \psi_{z\mathrm{AC}}$.

For member CD :

$M_{x\mathrm{CD}} = 0.2(\varphi_{x\mathrm{C}} - \varphi_{x\mathrm{D}})$, $\quad M_{x\mathrm{DC}} = 0.2(\varphi_{x\mathrm{D}} - \varphi_{x\mathrm{C}})$,

$M_{y\mathrm{CD}} = 0.9(2\varphi_{y\mathrm{C}} + \varphi_{y\mathrm{D}} + 0.625\psi_{x\mathrm{AC}} - 0.625\psi_{x\mathrm{BD}})$,

$M_{y\mathrm{DC}} = 0.9(\varphi_{y\mathrm{C}} + 2\varphi_{y\mathrm{D}} + 0.625\psi_{x\mathrm{AC}} - 0.625\psi_{x\mathrm{BD}})$,

$M_{z\mathrm{CD}} = 0.7(2\varphi_{z\mathrm{C}} + \varphi_{z\mathrm{D}})$, $\quad M_{z\mathrm{DC}} = 0.7(\varphi_{z\mathrm{C}} + 2\varphi_{z\mathrm{D}})$.

For member CH :

$M_{x\mathrm{CH}} = 1.5(2\varphi_{x\mathrm{C}} + \varphi_{x\mathrm{H}})$, $\quad M_{x\mathrm{HC}} = 1.5(\varphi_{x\mathrm{C}} + 2\varphi_{x\mathrm{H}})$,

$M_{y\mathrm{CH}} = 1.2(2\varphi_{y\mathrm{C}} + \varphi_{y\mathrm{H}} + 1.2\psi_{z\mathrm{EH}} - \psi_{z\mathrm{AC}})$,

$M_{y\mathrm{HC}} = 1.2(\varphi_{y\mathrm{C}} + 2\varphi_{y\mathrm{H}} + 1.2\psi_{z\mathrm{EH}} - \psi_{z\mathrm{AC}})$,

$M_{z\mathrm{CH}} = 0.16(\varphi_{z\mathrm{C}} - \varphi_{z\mathrm{H}})$, $\quad M_{z\mathrm{HC}} = 0.16(\varphi_{z\mathrm{H}} - \varphi_{z\mathrm{C}})$.

For member DI :

$M_{x\mathrm{DI}} = 2\varphi_{x\mathrm{D}} + \varphi_{x\mathrm{I}} - 2.083,3$, $\quad M_{x\mathrm{ID}} = \varphi_{x\mathrm{D}} + 2\varphi_{x\mathrm{I}} + 2.083,3$,

$M_{y\mathrm{DI}} = 0.7(2\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{I}} + 1.2\psi_{z\mathrm{EH}} - \psi_{z\mathrm{AC}})$,

$M_{y\mathrm{ID}} = 0.7(\varphi_{y\mathrm{D}} + 2\varphi_{y\mathrm{I}} + 1.2\psi_{z\mathrm{EH}} - \psi_{z\mathrm{AC}})$,

$M_{z\mathrm{DI}} = 0.18(\varphi_{z\mathrm{D}} - \varphi_{z\mathrm{I}})$, $\quad M_{z\mathrm{ID}} = 0.18(\varphi_{z\mathrm{I}} - \varphi_{z\mathrm{D}})$.

For member EH :

$M_{x\mathrm{EH}} = 1.2(\varphi_{x\mathrm{H}} + 0.833,3\psi_{x\mathrm{AC}})$, $\quad M_{x\mathrm{HE}} = 1.2(2\varphi_{x\mathrm{H}} + 0.833,3\psi_{x\mathrm{AC}})$,

$M_{y\mathrm{EH}} = -0.28\varphi_{y\mathrm{H}}$, $\quad M_{y\mathrm{HE}} = 0.28\varphi_{y\mathrm{H}}$,

$M_{z\mathrm{EH}} = 1.2(\varphi_{z\mathrm{H}} + \psi_{z\mathrm{EH}})$, $\quad M_{z\mathrm{HE}} = 1.2(2\varphi_{z\mathrm{H}} + \psi_{z\mathrm{EH}})$.

For member FI :

$M_{x\mathrm{FI}} = \varphi_{x\mathrm{I}} + 0.833,3\psi_{x\mathrm{BD}}$, $\quad M_{x\mathrm{IF}} = 2\varphi_{x\mathrm{I}} + 0.833,3\psi_{x\mathrm{BD}}$,

$M_{y\mathrm{FI}} = -0.2\varphi_{y\mathrm{I}}$, $\quad M_{y\mathrm{IF}} = 0.2\varphi_{y\mathrm{I}}$,

$M_{z\mathrm{FI}} = \varphi_{z\mathrm{I}} + \psi_{z\mathrm{EH}}$, $\quad M_{z\mathrm{IF}} = 2\varphi_{z\mathrm{I}} + \psi_{z\mathrm{EH}}$.

For member GJ :

$M_{x\mathrm{GJ}} = 1.1(\varphi_{x\mathrm{J}} + \psi_{x\mathrm{GJ}})$, $\quad M_{x\mathrm{JG}} = 1.1(2\varphi_{x\mathrm{J}} + \psi_{x\mathrm{GJ}})$,

$M_{y\mathrm{GJ}} = -0.26\varphi_{y\mathrm{J}}$, $\quad M_{y\mathrm{JG}} = 0.26\varphi_{y\mathrm{J}}$,

$M_{zGJ} = 0.9(\varphi_{zJ} + 1.5\psi_{zEH})$, $M_{zJG} = 0.9(2\varphi_{zJ} + 1.5\psi_{zEH})$.

For member HI :

$M_{xHI} = 0.24(\varphi_{xH} - \varphi_{xI})$, $M_{xIH} = 0.24(\varphi_{xI} - \varphi_{xH})$,

$M_{yHI} = 2\varphi_{yH} + \varphi_{yI} + 0.625\psi_{xAC} - 0.625\psi_{xBD}$,

$M_{yIH} = \varphi_{yH} + 2\varphi_{yI} + 0.625\psi_{xAC} - 0.625\psi_{xBD}$,

$M_{zHI} = 0.8(2\varphi_{zH} + \varphi_{zI})$, $M_{zIH} = 0.8(\varphi_{zH} + 2\varphi_{zI})$.

For member IJ :

$M_{xIJ} = 0.30(\varphi_{xI} - \varphi_{xJ})$, $M_{xJI} = 0.3(\varphi_{xJ} - \varphi_{xI})$,

$M_{yIJ} = 1.3(2\varphi_{yI} + \varphi_{yJ} + 0.833,3\psi_{xBD} - 0.666,7\psi_{xGJ})$,

$M_{yJI} = 1.3(\varphi_{yI} + 2\varphi_{yJ} + 0.833,3\psi_{xBD} - 0.666,7\psi_{xGJ})$,

$M_{zIJ} = 1.4(2\varphi_{zI} + \varphi_{zJ}) - 3$, $M_{zJI} = 1.4(\varphi_{zI} + 2\varphi_{zJ}) + 3$.

For member HN :

$M_{xHN} = 0.7(2\varphi_{xH} + \varphi_{xN})$, $M_{xNH} = 0.7(\varphi_{xH} + 2\varphi_{xN})$,

$M_{yHN} = 0.9(2\varphi_{yH} + \varphi_{yN} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{yNH} = 0.9(\varphi_{yH} + 2\varphi_{yN} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{zHN} = 0.26(\varphi_{zH} - \varphi_{zN})$, $M_{zNH} = 0.26(\varphi_{zN} - \varphi_{zH})$.

For member IP :

$M_{xIP} = 2\varphi_{xI} + \varphi_{xP}$, $M_{xPI} = \varphi_{xI} + 2\varphi_{xP}$,

$M_{yIP} = 0.8(2\varphi_{yI} + \varphi_{yP} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{yPI} = 0.8(\varphi_{yI} + 2\varphi_{yP} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{zIP} = 0.18(\varphi_{zI} - \varphi_{zP})$, $M_{zPI} = 0.18(\varphi_{zP} - \varphi_{zI})$.

For member JQ :

$M_{xJQ} = 1.6(2\varphi_{xJ} + \varphi_{xQ})$, $M_{xQJ} = 1.6(\varphi_{xJ} + 2\varphi_{xQ})$,

$M_{yJQ} = 1.4(2\varphi_{yJ} + \varphi_{yQ} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{yQJ} = 1.4(\varphi_{yJ} + 2\varphi_{yQ} + 1.5\psi_{zKN} - 1.5\psi_{zEH})$,

$M_{zJQ} = 0.18(\varphi_{zJ} - \varphi_{zQ})$, $M_{zQJ} = 0.18(\varphi_{zQ} - \varphi_{zJ})$.

For member KN :

$M_{xKN} = 1.2(\varphi_{xN} + 0.833,3\psi_{xAC})$, $M_{xNK} = 1.2(2\varphi_{xN} + 0.833,3\psi_{xAC})$,

$M_{y\mathrm{KN}} = -0.16\varphi_{y\mathrm{N}}$, $M_{y\mathrm{NK}} = 0.16\varphi_{y\mathrm{N}}$,

$M_{z\mathrm{KN}} = 1.2(\varphi_{z\mathrm{N}} + \psi_{z\mathrm{KN}})$, $M_{z\mathrm{NK}} = 1.2(2\varphi_{z\mathrm{N}} + \psi_{z\mathrm{KN}})$.

For member LP :

$M_{x\mathrm{LP}} = 1.2(\varphi_{x\mathrm{P}} + 0.833,3\psi_{x\mathrm{BD}})$, $M_{x\mathrm{PL}} = 1.2(2\varphi_{x\mathrm{P}} + 0.833,3\psi_{x\mathrm{BD}})$,

$M_{y\mathrm{LP}} = -0.2\varphi_{y\mathrm{P}}$, $M_{y\mathrm{PL}} = 0.2\varphi_{y\mathrm{P}}$,

$M_{z\mathrm{LP}} = 1.2(\varphi_{z\mathrm{P}} + \psi_{z\mathrm{KN}})$, $M_{z\mathrm{PL}} = 1.2(2\varphi_{z\mathrm{P}} + \psi_{z\mathrm{KN}})$.

For member MQ :

$M_{x\mathrm{MQ}} = 1.3(\varphi_{x\mathrm{Q}} + \psi_{x\mathrm{GJ}})$, $M_{x\mathrm{QM}} = 1.3(2\varphi_{x\mathrm{Q}} + \psi_{x\mathrm{GJ}})$,

$M_{y\mathrm{MQ}} = -0.16\varphi_{y\mathrm{Q}}$, $M_{y\mathrm{QM}} = 0.16\varphi_{y\mathrm{Q}}$,

$M_{z\mathrm{MQ}} = 1.4(\varphi_{z\mathrm{Q}} + 1.5\psi_{z\mathrm{KN}})$, $M_{z\mathrm{QM}} = 1.4(2\varphi_{z\mathrm{Q}} + 1.5\psi_{z\mathrm{KN}})$.

For member NP :

$M_{x\mathrm{NP}} = 0.24(\varphi_{x\mathrm{N}} - \varphi_{x\mathrm{P}})$, $M_{x\mathrm{PN}} = 0.24(\varphi_{x\mathrm{P}} - \varphi_{x\mathrm{N}})$,

$M_{y\mathrm{NP}} = 0.6(2\varphi_{y\mathrm{N}} + \varphi_{y\mathrm{P}} + 0.625\psi_{x\mathrm{AC}} - 0.625\psi_{x\mathrm{BD}})$,

$M_{y\mathrm{PN}} = 0.6(\varphi_{y\mathrm{N}} + 2\varphi_{y\mathrm{P}} + 0.625\psi_{x\mathrm{AC}} - 0.625\psi_{x\mathrm{BD}})$,

$M_{z\mathrm{NP}} = 1.2(2\varphi_{z\mathrm{H}} + \varphi_{z\mathrm{P}}) - 6$, $M_{z\mathrm{PN}} = 1.2(\varphi_{z\mathrm{N}} + 2\varphi_{z\mathrm{P}}) + 6$.

For member PQ :

$M_{x\mathrm{PQ}} = 0.2(\varphi_{x\mathrm{P}} - \varphi_{x\mathrm{Q}})$, $M_{x\mathrm{QP}} = 0.2(\varphi_{x\mathrm{Q}} - \varphi_{x\mathrm{P}})$,

$M_{y\mathrm{PQ}} = 0.6(2\varphi_{y\mathrm{P}} + \varphi_{y\mathrm{Q}} + 0.833,3\psi_{x\mathrm{BD}} - 0.666,7\psi_{x\mathrm{GJ}})$,

$M_{y\mathrm{QP}} = 0.6(\varphi_{y\mathrm{P}} + 2\varphi_{y\mathrm{Q}} + 0.833,3\psi_{x\mathrm{BD}} - 0.666,7\psi_{x\mathrm{GJ}})$,

$M_{z\mathrm{PQ}} = 2\varphi_{z\mathrm{P}} + \varphi_{z\mathrm{Q}}$, $M_{z\mathrm{QP}} = \varphi_{z\mathrm{P}} + 2\varphi_{z\mathrm{Q}}$.

2) Expressions of End-Shears

Observing constituent frames in row:

$X_{x\mathrm{CA}} = -(1/l_{\mathrm{CA}})(M_{z\mathrm{AC}} + M_{z\mathrm{CA}}) = -(1/5)(3\varphi_{z\mathrm{C}} + 2\psi_{z\mathrm{AC}}) = -0.6\varphi_{z\mathrm{C}} - 0.4\psi_{z\mathrm{AC}}$,

$X_{x\mathrm{DB}} = -(1/l_{\mathrm{DB}})(M_{z\mathrm{BD}} + M_{z\mathrm{DB}}) = -(1/5)(3\varphi_{z\mathrm{D}} + 2\psi_{z\mathrm{AC}}) = -0.6\varphi_{z\mathrm{D}} - 0.4\psi_{z\mathrm{AC}}$,

$$X_{x\mathrm{CH}} = -(1/l_{\mathrm{CH}})(M_{y\mathrm{CH}} + M_{y\mathrm{HC}}) = -(1/5)(3.6\varphi_{y\mathrm{C}} + 3.6\varphi_{y\mathrm{H}} + 2.88\psi_{z\mathrm{EH}} - 2.4\psi_{z\mathrm{AC}})$$
$$= -0.72\varphi_{y\mathrm{C}} - 0.72\varphi_{y\mathrm{H}} - 0.576\psi_{z\mathrm{EH}} + 0.48\psi_{z\mathrm{AC}},$$

$$X_{x\mathrm{DI}} = -(1/l_{\mathrm{DI}})(M_{y\mathrm{DI}} + M_{y\mathrm{ID}}) = -(1/5)(2.1\varphi_{y\mathrm{D}} + 2.1\varphi_{y\mathrm{I}} + 1.68\psi_{z\mathrm{EH}} - 1.4\psi_{z\mathrm{AC}})$$
$$= -0.42\varphi_{y\mathrm{D}} - 0.42\varphi_{y\mathrm{I}} - 0.3\,36\psi_{z\mathrm{EH}} + 0.28\psi_{z\mathrm{AC}},$$

$X_{x\mathrm{HE}} = -(1/l_{\mathrm{HE}})(M_{z\mathrm{HE}} + M_{z\mathrm{EH}}) = -(1.2/6)(3\varphi_{z\mathrm{H}} + 2\psi_{z\mathrm{EH}}) = -0.6\varphi_{z\mathrm{H}} - 0.4\psi_{z\mathrm{EH}},$

$X_{x\mathrm{IF}} = -(1/l_{\mathrm{IF}})(M_{z\mathrm{IF}} + M_{z\mathrm{FI}}) = -(1/6)(3\varphi_{z\mathrm{I}} + 2\psi_{z\mathrm{EH}}) = -0.5\varphi_{z\mathrm{I}} - 0.333,3\psi_{z\mathrm{EH}},$

$X_{x\mathrm{JG}} = -(1/l_{\mathrm{JG}})(M_{z\mathrm{JG}} + M_{z\mathrm{GJ}}) = -(0.9/4)(3\varphi_{z\mathrm{J}} + 3\psi_{z\mathrm{EH}}) = -0.675\varphi_{z\mathrm{J}} - 0.675\psi_{z\mathrm{EH}},$

$X_{x\mathrm{HC}} = X_{x\mathrm{CH}} = -0.72\varphi_{y\mathrm{C}} - 0.72\varphi_{y\mathrm{H}} - 0.576\psi_{z\mathrm{EH}} + 0.48\psi_{z\mathrm{AC}},$

$X_{x\mathrm{ID}} = X_{x\mathrm{DI}} = -0.42\varphi_{y\mathrm{D}} - 0.42\varphi_{y\mathrm{I}} - 0.336\psi_{z\mathrm{EH}} + 0.28\psi_{z\mathrm{AC}},$

$$X_{x\mathrm{HN}} = -(1/l_{\mathrm{HN}})(M_{y\mathrm{HN}} + M_{y\mathrm{NH}}) = -(0.9/4)(3\varphi_{y\mathrm{H}} + 3\varphi_{y\mathrm{N}} + 3\psi_{z\mathrm{KN}} - 3\psi_{z\mathrm{EH}})$$
$$= -0.675\varphi_{y\mathrm{H}} - 0.675\varphi_{y\mathrm{N}} - 0.675\psi_{z\mathrm{KN}} + 0.675\psi_{z\mathrm{EH}},$$

$$X_{x\mathrm{IP}} = -(1/l_{\mathrm{IP}})(M_{y\mathrm{IP}} + M_{y\mathrm{PI}}) = -(0.8/4)(3\varphi_{y\mathrm{I}} + 3\varphi_{y\mathrm{P}} + 3\psi_{z\mathrm{KN}} - 3\psi_{z\mathrm{EH}})$$
$$= -0.6\varphi_{y\mathrm{I}} - 0.6\varphi_{y\mathrm{P}} - 0.6\psi_{z\mathrm{KN}} + 0.6\psi_{z\mathrm{EH}},$$

$$X_{x\mathrm{JQ}} = -(1/l_{\mathrm{JQ}})(M_{y\mathrm{JQ}} + M_{y\mathrm{QJ}}) = -(1.4/4)(3\varphi_{y\mathrm{J}} + 3\varphi_{y\mathrm{Q}} + 3\psi_{z\mathrm{KN}} - 3\psi_{z\mathrm{EH}})$$
$$= -1.05\varphi_{y\mathrm{J}} - 1.05\varphi_{y\mathrm{Q}} - 1.05\psi_{z\mathrm{KN}} + 1.05\psi_{z\mathrm{EH}},$$

$X_{x\mathrm{NK}} = -(1/l_{\mathrm{NK}})(M_{z\mathrm{NK}} + M_{z\mathrm{KN}}) = -(1.2/6)(3\varphi_{z\mathrm{N}} + 2\psi_{z\mathrm{KN}}) = -0.6\varphi_{z\mathrm{N}} - 0.4\psi_{z\mathrm{KN}},$

$X_{x\mathrm{PL}} = -(1/l_{\mathrm{PL}})(M_{z\mathrm{PL}} + M_{z\mathrm{LP}}) = -(1.2/6)(3\varphi_{z\mathrm{P}} + 2\psi_{z\mathrm{KN}}) = -0.6\varphi_{z\mathrm{P}} - 0.4\psi_{z\mathrm{KN}},$

$$X_{x\mathrm{QM}} = -(1/l_{\mathrm{QM}})(M_{z\mathrm{QM}} + M_{z\mathrm{MQ}}) = -(1.4/4)(3\varphi_{z\mathrm{Q}} + 3\psi_{z\mathrm{KN}})$$
$$= -1.05\varphi_{z\mathrm{Q}} - 1.05\psi_{z\mathrm{KN}},$$

$X_{x\mathrm{NH}} = X_{x\mathrm{HN}} = -0.675\varphi_{y\mathrm{H}} - 0.675\varphi_{y\mathrm{N}} - 0.675\psi_{z\mathrm{KN}} + 0.675\psi_{z\mathrm{EH}},$

$X_{x\mathrm{PI}} = X_{x\mathrm{IP}} = -0.6\varphi_{y\mathrm{I}} - 0.6\varphi_{y\mathrm{P}} - 0.6\psi_{z\mathrm{KN}} + 0.6\psi_{z\mathrm{EH}},$

$X_{x\mathrm{QJ}} = X_{x\mathrm{JQ}} = -1.05\varphi_{y\mathrm{J}} - 1.05\varphi_{y\mathrm{Q}} - 1.05\psi_{z\mathrm{KN}} + 1.05\psi_{z\mathrm{EH}}.$

Similarly in column :

$$X_{z\mathrm{CA}} = -(1/l_{\mathrm{CA}})(M_{x\mathrm{AC}} + M_{x\mathrm{CA}}) = -(0.8/5)(3\varphi_{x\mathrm{C}} + 2\psi_{x\mathrm{AC}})$$
$$= -0.48\varphi_{x\mathrm{C}} - 0.32\psi_{x\mathrm{AC}},$$

$$X_{z\mathrm{HE}} = -(1/l_{\mathrm{HE}})(M_{x\mathrm{HE}} + M_{x\mathrm{EH}}) = -(1.2/6)(3\varphi_{x\mathrm{H}} + 1.667\psi_{x\mathrm{AC}})$$
$$= -0.6\varphi_{x\mathrm{H}} - 0.333,3\psi_{x\mathrm{AC}}$$

$$X_{z\mathrm{NK}} = -(1/l_{\mathrm{NK}})(M_{x\mathrm{NK}} + M_{x\mathrm{KN}}) = -(1.2/6)(3\varphi_{x\mathrm{N}} + 1.667\psi_{x\mathrm{AC}})$$
$$= -0.6\varphi_{x\mathrm{N}} - 0.333,3\psi_{x\mathrm{AC}},$$

$$X_{z\mathrm{CD}} = -(1/l_{\mathrm{CD}})(M_{y\mathrm{CD}} + M_{y\mathrm{DC}}) = -(0.9/8)(3\varphi_{y\mathrm{C}} + 3\varphi_{y\mathrm{D}} + 1.25\psi_{x\mathrm{AC}} - 1.25\psi_{x\mathrm{BD}})$$
$$= -0.337,5\varphi_{y\mathrm{C}} - 0.337,5\varphi_{y\mathrm{D}} - 0.140,6\psi_{x\mathrm{AC}} + 0.140,6\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{HI}} = -(1/l_{\mathrm{HI}})(M_{y\mathrm{HI}} + M_{y\mathrm{IH}}) = -(1/8)(3\varphi_{y\mathrm{H}} + 3\varphi_{y\mathrm{I}} + 1.25\psi_{x\mathrm{AC}} - 1.25\psi_{x\mathrm{BD}})$$
$$= -0.375\varphi_{y\mathrm{H}} - 0.375\varphi_{y\mathrm{I}} - 0.156,25\psi_{x\mathrm{AC}} + 0.156,25\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{NP}} = -(1/l_{\mathrm{NP}})(M_{y\mathrm{NP}} + M_{y\mathrm{PN}}) = -(0.6/8)(3\varphi_{y\mathrm{N}} + 3\varphi_{y\mathrm{P}} + 1.25\psi_{x\mathrm{AC}} - 1.25\psi_{x\mathrm{BD}})$$
$$= -0.225\varphi_{y\mathrm{N}} - 0.225\varphi_{y\mathrm{P}} - 0.093,75\psi_{x\mathrm{AC}} + 0.093,75\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{DB}} = -(1/l_{\mathrm{BD}})(M_{x\mathrm{BD}} + M_{x\mathrm{DB}}) = -(0.8/5)(3\varphi_{x\mathrm{D}} + 2\psi_{x\mathrm{BD}})$$
$$= -0.48\varphi_{x\mathrm{D}} - 0.32\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{IF}} = -(1/l_{\mathrm{IF}})(M_{x\mathrm{IF}} + M_{x\mathrm{FI}}) = -(1/6)(3\varphi_{x\mathrm{I}} + 1.667\psi_{x\mathrm{BD}})$$
$$= -0.5\varphi_{x\mathrm{I}} - 0.277,8\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{PL}} = -(1/l_{\mathrm{PL}})(M_{x\mathrm{PL}} + M_{x\mathrm{LP}}) = -(1.2/6)(3\varphi_{x\mathrm{P}} + 1.667\psi_{x\mathrm{BD}})$$
$$= -0.6\varphi_{x\mathrm{P}} - 0.333,3\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{DC}} = X_{z\mathrm{CD}} = -0.337,5\varphi_{y\mathrm{C}} - 0.337,5\varphi_{y\mathrm{D}} - 0.140,6\psi_{x\mathrm{AC}} + 0.140,6\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{IH}} = X_{z\mathrm{HI}} = -0.375\varphi_{y\mathrm{H}} - 0.375\varphi_{y\mathrm{I}} - 0.156,25\psi_{x\mathrm{AC}} + 0.156,25\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{PN}} = X_{z\mathrm{NP}} = -0.225\varphi_{y\mathrm{N}} - 0.225\varphi_{y\mathrm{P}} - 0.093,75\psi_{x\mathrm{AC}} + 0.093,75\psi_{x\mathrm{BD}},$$

$$X_{z\mathrm{IJ}} = -(1/l_{\mathrm{IJ}})(M_{y\mathrm{IJ}} + M_{y\mathrm{JI}}) = -(1.3/6)(3\varphi_{y\mathrm{I}} + 3\varphi_{y\mathrm{J}} + 1.666,7\psi_{x\mathrm{BD}} - 1.333,3\psi_{x\mathrm{GJ}})$$
$$= -0.65\varphi_{y\mathrm{I}} - 0.65_{y\mathrm{J}} - 0.361,1\psi_{x\mathrm{BD}} + 0.288,9\psi_{x\mathrm{GJ}},$$

$$X_{z\mathrm{PQ}} = -(1/l_{\mathrm{PQ}})(M_{y\mathrm{PQ}} + M_{y\mathrm{QP}}) = -(0.6/6)(3\varphi_{y\mathrm{P}} + 3\varphi_{y\mathrm{Q}} + 1.667\psi_{x\mathrm{BD}} - 1.333\psi_{x\mathrm{GJ}})$$
$$= -0.3\varphi_{y\mathrm{P}} - 0.3\varphi_{y\mathrm{Q}} - 0.166,7\psi_{x\mathrm{BD}} + 0.133,3\psi_{x\mathrm{GJ}},$$

$$X_{z\mathrm{JG}} = -(1/l_{\mathrm{JG}})(M_{x\mathrm{JG}} + M_{x\mathrm{GJ}}) = -(1.1/4)(3\varphi_{x\mathrm{J}} + 2\psi_{x\mathrm{GJ}}) = -0.825\varphi_{x\mathrm{J}} - 0.55\psi_{x\mathrm{GJ}},$$

$$X_{z\mathrm{QM}} = -(1/l_{\mathrm{QM}})(M_{x\mathrm{QM}} + M_{x\mathrm{MQ}}) = -(1.3/4)(3\varphi_{x\mathrm{Q}} + 2\psi_{x\mathrm{GJ}})$$
$$= -0.975\varphi_{x\mathrm{Q}} - 0.65\psi_{x\mathrm{GJ}},$$

$$X_{z\mathrm{JI}} = X_{z\mathrm{IJ}} = -0.65\varphi_{y\mathrm{I}} - 0.65\varphi_{y\mathrm{J}} - 0.361,1\psi_{x\mathrm{BD}} + 0.288,9\psi_{x\mathrm{GJ}},$$

$$X_{z\mathrm{QP}} = X_{z\mathrm{PQ}} = -0.3\varphi_{y\mathrm{P}} - 0.3\varphi_{y\mathrm{Q}} - 0.166,7\psi_{x\mathrm{BD}} + 0.133,3\psi_{x\mathrm{GJ}}.$$

3) Elastic Equations

i) Joint Equilibrium Equations

About x axis :

At joint C, $M_{x\mathrm{CA}} + M_{x\mathrm{CD}} + M_{x\mathrm{CH}} = 0.$

〃 D, $M_{x\mathrm{DB}} + M_{x\mathrm{DC}} + M_{x\mathrm{DI}} = 0.$

At joint H, $M_{x\mathrm{HC}} + M_{x\mathrm{HE}} + M_{x\mathrm{HI}} + M_{x\mathrm{HN}} = 0.$

〃 I, $M_{x\mathrm{ID}} + M_{x\mathrm{IF}} + M_{x\mathrm{IH}} + M_{x\mathrm{IJ}} + M_{x\mathrm{IP}} = 0.$

〃 J, $M_{x\mathrm{JG}} + M_{x\mathrm{JI}} + M_{x\mathrm{JQ}} = 0.$

〃 N, $M_{x\mathrm{NH}} + M_{x\mathrm{NK}} + M_{x\mathrm{NP}} = 0.$

〃 P, $M_{x\mathrm{PI}} + M_{x\mathrm{PL}} + M_{x\mathrm{PN}} + M_{x\mathrm{PQ}} = 0.$

〃 Q, $M_{x\mathrm{QJ}} + M_{x\mathrm{QM}} + M_{x\mathrm{QP}} = 0.$

About y axis :

At joint C, $M_{y\mathrm{CA}} + M_{y\mathrm{CD}} + M_{y\mathrm{CH}} = 0.$

〃 D, $M_{y\mathrm{DB}} + M_{y\mathrm{DC}} + M_{y\mathrm{DI}} = 0.$

〃 H, $M_{y\mathrm{HC}} + M_{y\mathrm{HE}} + M_{y\mathrm{HI}} + M_{y\mathrm{HN}} = 0.$

〃 I, $M_{y\mathrm{ID}} + M_{y\mathrm{IF}} + M_{y\mathrm{IH}} + M_{y\mathrm{IJ}} + M_{y\mathrm{IP}} = 0.$

〃 J, $M_{y\mathrm{JG}} + M_{y\mathrm{JI}} + M_{y\mathrm{JQ}} = 0.$

〃 N, $M_{y\mathrm{NH}} + M_{y\mathrm{NK}} + M_{y\mathrm{NP}} = 0.$

〃 P, $M_{y\mathrm{PI}} + M_{y\mathrm{PL}} + M_{y\mathrm{PN}} + M_{y\mathrm{PQ}} = 0.$

〃 Q, $M_{y\mathrm{QJ}} + M_{y\mathrm{QM}} + M_{y\mathrm{QP}} = 0.$

About z axis :

At joint C, $M_{z\mathrm{CA}} + M_{z\mathrm{CD}} + M_{z\mathrm{CH}} = 0.$

〃 D, $M_{z\mathrm{DB}} + M_{z\mathrm{DC}} + M_{z\mathrm{DI}} = 0.$

〃 H, $M_{z\mathrm{HC}} + M_{z\mathrm{HE}} + M_{z\mathrm{HI}} + M_{z\mathrm{HN}} = 0.$

〃 I, $M_{z\mathrm{ID}} + M_{z\mathrm{IF}} + M_{z\mathrm{IH}} + M_{z\mathrm{IJ}} + M_{z\mathrm{IP}} = 0.$

〃 J, $M_{z\mathrm{JG}} + M_{z\mathrm{JI}} + M_{z\mathrm{JQ}} = 0.$

〃 N, $M_{z\mathrm{NH}} + M_{z\mathrm{NK}} + M_{z\mathrm{NP}} = 0.$

〃 P, $M_{z\mathrm{PI}} + M_{z\mathrm{PL}} + M_{z\mathrm{PN}} + M_{z\mathrm{PQ}} = 0.$

〃 Q, $M_{z\mathrm{QJ}} + M_{z\mathrm{QM}} + M_{z\mathrm{QP}} = 0.$

ii) Horizontal Shear Equations

For the 1st column-frame :

$$X_{z\mathrm{CA}} + X_{z\mathrm{HE}} + X_{z\mathrm{NK}} + X_{z\mathrm{CD}} + X_{z\mathrm{HI}} + X_{z\mathrm{NP}} = 0.$$

Table 11 Elastic equations

Eq.	Left-hand side															
	φ_{xC}	φ_{xD}	φ_{xH}	φ_{xI}	φ_{xJ}	φ_{xN}	φ_{xP}	φ_{xQ}	φ_{yC}	φ_{yD}	φ_{yH}	φ_{yI}	φ_{yJ}	φ_{yN}	φ_{yP}	φ_{yQ}
1	4.800	−0.200	1.500													
2	−0.200	3.800		1.000												
3	1.500		7.040	−0.240		0.700										
4		1.000	−0.240	6.540	−0.300		1.000									
5				−0.300	5.700			1.600								
6			0.700			4.040	−0.240									
7				1.000		−0.240	4.840	−0.200								
8					1.600		−0.200	6.000								
9									4.440	0.900	1.200					
10									0.900	3.420		0.700				
11									1.200		6.480	1.000		0.900		
12										0.700	1.000	7.800	1.300		0.800	
13												1.300	5.660			1.400
14											0.900			3.160	0.600	
15												0.800		0.600	4.200	0.600
16													1.400		0.600	4.160
17																
18																
19																
20																
21																
22																
23																
24																
25	0.800		1.000			1.000			0.563	0.563	0.625	0.625		0.375	0.375	
26		0.800		0.833			1.000	1.300	−0.563	−0.563	−0.625	0.458	1.083	−0.375	0.125	0.500
27					1.100							−0.867	−0.867		−0.400	−0.400
28									−1.200	−0.700	−1.200	−0.700				
29									1.440	0.840	0.090	−0.360	−2.100	−1.350	−1.200	−2.100
30											1.350	1.200	2.100	1.350	1.200	2.100

Table 11—continued—

φ_{zC}	φ_{zD}	φ_{zH}	φ_{zI}	φ_{zJ}	φ_{zN}	φ_{zP}	φ_{zQ}	ϕ_{xAC}	ϕ_{xBD}	ϕ_{xGJ}	ϕ_{zAC}	ϕ_{zEH}	ϕ_{zKN}	Right-hand side
								0.800						
									0.800					2.083
								1.000						
									0.833					−2.083
										1.100				
								1.000						
									1.000					
										1.300				
								0.563	−0.563		−1.200	1.440		
								0.563	−0.563		−0.700	0.840		
								0.625	−0.625		−1.200	0.090	1.350	
								0.625	0.458	−0.867	−0.700	−0.360	1.200	
									1.083	−0.867		−2.100	2.100	
								0.375	−0.375			−1.350	1.350	
								0.375	0.125	−0.400		−1.200	1.200	
									0.550	−0.400		−2.100	2.100	
3.560	0.700	−0.160									1.000			
0.700	3.580		−0.180								1.000			
−0.160		4.420	0.800		−0.260							1.200		
	−0.180	0.800	6.760	1.400		−0.180						1.000		3.000
			1.400	4.780			−0.180					1.350		−3.000
		−0.260			5.060	1.200							1.200	6.000
			−0.180		1.200	6.980	1.000						1.200	−6.000
				−0.180		1.000	4.980						2.100	
								2.296	−0.651					
								−0.651	3.083	−0.704				
									−0.704	2.163				
1.000	1.000										2.600	−1.520		
		1.200	1.000	1.350							−1.520	9.290	−4.650	
					1.200	1.200	2.100					−4.650	8.350	

For the 2nd column-frame:

$$X_{z\mathrm{DB}} + X_{z\mathrm{IF}} + X_{z\mathrm{PL}} - X_{z\mathrm{DC}} - X_{z\mathrm{IH}} - X_{z\mathrm{PN}} + X_{z\mathrm{IJ}} + X_{z\mathrm{PQ}} = 0.$$

For the 3rd column-frame:

$$X_{z\mathrm{JG}} + X_{z\mathrm{QM}} - X_{z\mathrm{JI}} - X_{z\mathrm{QP}} = 0.$$

For the 1st row-frame :

$$X_{x\mathrm{CA}} + X_{x\mathrm{DB}} - X_{x\mathrm{CH}} - X_{x\mathrm{DI}} = 0.$$

For the 2nd row frame:

$$X_{x\mathrm{HE}} + X_{x\mathrm{IF}} + X_{x\mathrm{JG}} + X_{x\mathrm{HC}} + X_{x\mathrm{ID}} - X_{x\mathrm{HN}} - X_{x\mathrm{IP}} - X_{x\mathrm{JQ}} = 0.$$

For the 3rd row frame:

$$X_{x\mathrm{NK}} + X_{x\mathrm{PL}} + X_{x\mathrm{QM}} + X_{x\mathrm{NH}} + X_{x\mathrm{PI}} + X_{x\mathrm{QJ}} = 0.$$

Substituting the expressions in 1) and 2), the required elastic equations are obtained as shown in Table 11, in which twenty-four from the beginning are the joint equilibrium equations and the remainders are the horizontal shear equations.

4) Solutions Obtained

Simultaneous equations in Table 11 are solved, and we have the joint rotations;

at joint C,	$\varphi_{x\mathrm{C}} = +0.049,75$,	$\varphi_{y\mathrm{C}} = -0.016,63$,	$\varphi_{z\mathrm{C}} = -0.012,36$,
〃 D,	$\varphi_{x\mathrm{D}} = +0.697,2$,	$\varphi_{y\mathrm{D}} = -0.017,36$,	$\varphi_{z\mathrm{C}} = +0.025,36$,
〃 H,	$\varphi_{x\mathrm{H}} = -0.014,73$,	$\varphi_{y\mathrm{H}} = +0.021,97$,	$\varphi_{z\mathrm{H}} = -0.027,17$,
〃 I,	$\varphi_{x\mathrm{I}} = -0.425,0$,	$\varphi_{y\mathrm{I}} = +0.030,89$,	$\varphi_{z\mathrm{I}} = +0.577,9$,
〃 J,	$\varphi_{x\mathrm{J}} = -0.029,68$,	$\varphi_{y\mathrm{J}} = +0.082,26$,	$\varphi_{z\mathrm{J}} = -0.796,9$,
〃 N,	$\varphi_{x\mathrm{N}} = +0.033,81$,	$\varphi_{y\mathrm{N}} = +0.065,21$,	$\varphi_{z\mathrm{N}} = +1.488,1$,
〃 P,	$\varphi_{x\mathrm{P}} = +0.123,7$,	$\varphi_{y\mathrm{P}} = +0.046,74$,	$\varphi_{z\mathrm{P}} = -1.109,2$,
〃 Q,	$\varphi_{x\mathrm{Q}} = +0.005,594$,	$\varphi_{y\mathrm{Q}} = +0.094,59$,	$\varphi_{z\mathrm{Q}} = +0.266,2$,

and revolutions

$\psi_{x\mathrm{AC}} = -0.096,59$,	$\psi_{x\mathrm{BD}} = -0.164,4$,	$\psi_{x\mathrm{GJ}} = +0.029,73$,
$\psi_{z\mathrm{AC}} = +0.021,90$,	$\psi_{z\mathrm{EH}} = +0.035,56$,	$\psi_{z\mathrm{KN}} = -0.171,3$.

5) Results Obtained

Using these φ's and ψ's, end-moments in 1) are now determined. The results are found in Table 12. In Fig. 18～20 the moment diagrams are shown by full lines. The reactions at supports are also computed and are shown in Fig. 21.

Table 12 Values of end-moments (t-m)

Member	AC		BD		CD		CH	
End	A	C	B	D	C	D	C	H
M_x	−0.038	+0.002	+0.426	+0.984	−0.130	+0.130	+0.127	+0.030
M_y	+0.004	−0.004	+0.004	−0.004	−0.007	−0.008	+0.011	+0.058
M_z	+0.010	−0.003	+0.047	+0.073	+0.000	+0.027	+0.002	−0.002

Member	DI		EH		FI		GJ	
End	D	I	E	H	F	I	G	J
M_x	−1.114	+1.931	−0.114	−0.132	−0.562	−0.987	+0.000	−0.033
M_y	+0.012	+0.046	−0.006	+0.006	−0.006	+0.006	−0.021	+0.021
M_z	−0.100	+0.100	+0.010	−0.023	+0.613	+1.191	−0.669	−1.386

Member	HI		IJ		HN		IP	
End	H	I	I	J	H	N	I	P
M_x	+0.099	−0.099	−0.119	+0.119	+0.003	+0.037	−0.726	−0.178
M_y	+0.117	+0.126	−0.017	+0.050	−0.181	−0.142	−0.161	−0.149
M_z	+0.419	+0.903	−2.500	+1.578	−0.394	+0.394	+0.304	−0.304

Member	JQ		KN		LP		MQ	
End	J	Q	K	N	L	P	M	Q
M_x	−0.086	−0.030	−0.056	−0.015	−0.016	+0.133	+0.046	+0.053
M_y	−0.072	−0.056	−0.010	+0.010	−0.009	+0.009	−0.015	+0.015
M_z	−0.191	+0.191	+1.580	+3.370	−1.537	−2.870	+0.013	+0.386

Table 12—continued—

Member	N P		P Q	
End	N	P	P	Q
M_x	−0.022	+0.022	+0.024	−0.024
M_y	+0.132	+0.121	+0.019	+0.046
M_z	−3.760	+5.120	−1.952	−0.577

6) Comparison with the Current Solutions

The frame is analysed two-dimensionally and the results are compared with the values above obtained, see Table 13. Also, they are shown by broken lines in Fig. 18∼20. Serious errors produced by the two-dimensional analysis are found in Table 13. Especially, the percentages enclosed by the brackets are noticeable. In the 2nd-and the 3rd-row-frames, the differences between exact values (M_z) and those obtained by the conventional solution (M) are remarkable. The reason will be such that while the 2nd-row-frame tends to deflect towards negative x and the 3rd-row-frame tends to positive x, the connecting girders tend to prevent these swayings. The existence of the connecting girders are not taken into account in the two-dimensional analysis.

Table 13 Percentage errors

(a) The 2nd-row-frame

Member	E H		F I		G J		H I		I J		
End	E	H	F	I	G	J	H	I	I	J	
M	+0.109	−0.162	+0.971	+1.625	−0.422	−1.272	+0.162	+0.866	−2.491	+1.272	
M_z	+0.010	−0.023	+0.613	+1.191	−0.669	−1.386	+0.419	+0.903	−2.500	+1.578	
$(M-M_z)/M_z$	(+990)	(+604)	+58.4	+36.4	−36.9	−8.2	−61.3	−4.1	−0.4	−19.4	(%)
H	−0.009		+0.433		−0.424						
H_x	−0.002		+0.300		−0.514						
$(H-H_x)/H_x$	(+350)		+33.3		−17.5						(%)
V	−0.129		+5.238		+1.797						
V_y	−0.144		+5.208		+1.875						
$(V-V_y)/V_y$	−10.4		+0.6		−4.2						(%)

(b) The 3rd-row-frame

Member	KN		LP		MQ		NP		PQ		
End	K	N	L	P	M	Q	N	P	P	Q	
M	+1.500	+3.465	−1.861	−3.258	−0.237	+0.339	−3.465	+5.173	−1.915	−0.340	
M_z	+1.580	+3.370	−1.537	−2.870	+0.013	+0.386	−3.760	+5.120	−1.952	−0.577	
$(M-M_z)/M_z$	−5.1	+2.8	+21.1	+13.5	—	−12.2	−7.8	+1.0	−1.9	−41.1	(%)
H	+0.828		−0.853		+0.026						
H_x	+0.825		−0.735		+0.100						
$(H-H_x)/H_x$	+0.4		+16.1		−74.0						(%)
V	+3.214		+2.933		−0.376						
V_y	+2.840		+3.366		−0.450						
$(V-V_y)/V_y$	+13.2		−12.9		−16.4						(%)

(c) The 2nd-column-frame

Member	BD		FI		LP		DI		IP		
End	B	D	F	I	L	P	D	I	I	P	
M	+0.413	+1.024	−0.673	−1.141	−0.052	+0.143	−1.024	+1.912	−0.772	−0.142	
M_x	+0.426	+0.984	−0.562	−0.987	−0.016	+0.133	−1.114	+1.931	−0.726	−0.178	
$(M-M_x)/M_x$	−3.1	+4.1	+19.8	+15.6	(+225)	+7.5	−8.1	−1.0	+6.3	−25.4	(%)
H	−0.287		+0.302		−0.015						
H_z	−0.282		+0.258		−0.019						
$(H-H_z)/H_z$	+1.8		+17.1		−21.1						(%)
V	+2.322		+5.238		+2.933						
V_y	+2.340		+5.208		+3.366						
$(V-V_y)/V_y$	−0.8		+0.6		−12.9						(%)

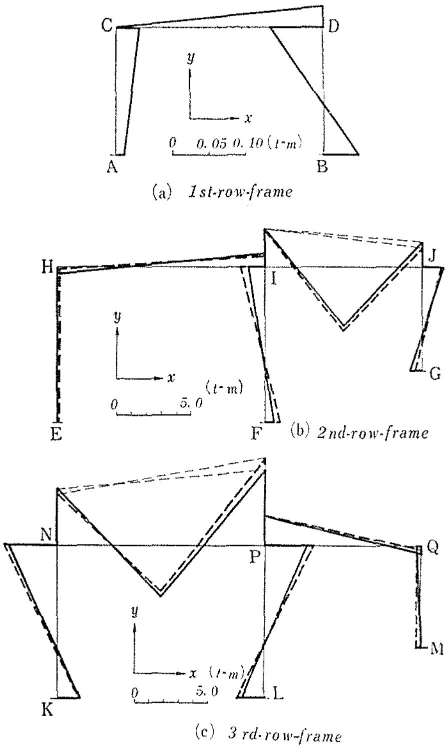


Fig. 18 M_z diagram

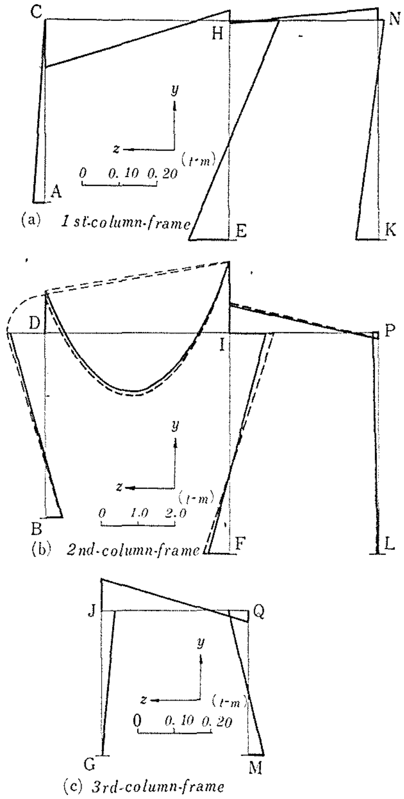


Fig. 19 M_x diagram

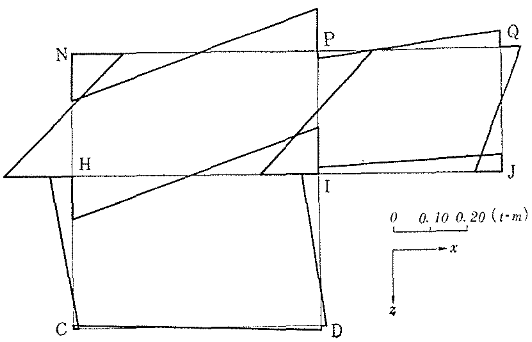


Fig. 20 M_y diagram

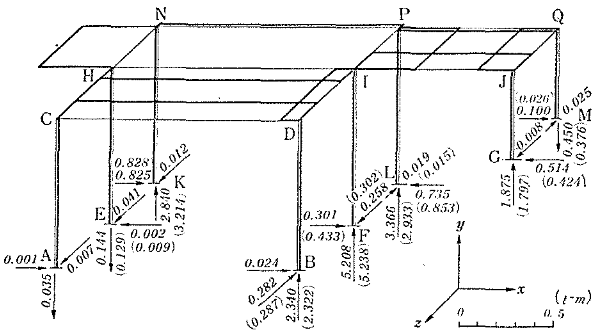


Fig. 21 Torsional moment diagram and reactions, values in ( ) are two-dimensional ones

18. Temperature Effects

To show the analysis of temperature effects, take, for example, the frame in Fig. 22. Let the girder BC and the column CG be subjected to the temperature rises of $t_1 = 10°\text{C}$ and $t_2 = 15°\text{C}$ respectively. The coefficient of the thermal expansion is ε. The member-lengths and their relative stiffnesses are as shown in Table 14.

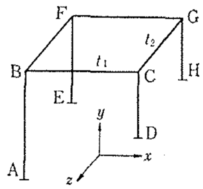


Fig. 22

Table 14

Member	AB	BC	CD	BF	CG	EF	FG	GH	
Length	5.4	4.0	3.0	5.0	5.0	4.2	4.0	3.0	(m)
k_x	1.2	1.4	0.8	1.5	1.5	1.2	1.4	0.8	
k_y	1.0	1.2	1.0	1.0	1.0	1.2	1.2	0.8	
k_z	1.2	1.0	0.8	1.4	1.4	1.0	0.8	1.0	

To analyse this frame, we have to find at first the relations among deflection angles by applying the conditions of compatibility described in §*12*, which are to be read from Table 15, and we have

$$R_{z\mathrm{CD}} = 1.8R_{z\mathrm{AB}} + 0.000,16, \quad R_{z\mathrm{BC}} = 0,$$

$$R_{x\mathrm{GH}} = R_{x\mathrm{CD}} + 0.000,30, \quad R_{x\mathrm{CG}} = 0,$$

$$R_{y\mathrm{CG}} = R_{y\mathrm{BF}} - 0.000,096, \quad R_{y\mathrm{FG}} = R_{y\mathrm{BC}} - 0.000,225.$$

Table 15 Compatibility conditions

Space	Member	Length (s) x	y	z	Direction angle (α)	$\sin\alpha$	$\cos\alpha$	Revolution angle	Change in length (Δs)	$\Delta s \cos\alpha$	$Rs \sin\alpha$	$\Delta s \sin\alpha$	$Rs \cos\alpha$
ABCD (xy plane)	A B	0	5.4	0	$\alpha_z=\pi/2$	1	0	R_{zAB}	0	0	$5.4R_{zAB}$	0	0
	B C	4.0	0	0	$\alpha_z=0$	0	1	R_{zBC}	48×10^{-5}	48×10^{-5}	0	0	$4.0R_{zBC}$
	C D	0	3.0	0	$\alpha_z=3\pi/2$	-1	0	R_{zCD}	0	0	$-3.0R_{zCD}$	0	0
									Σ	$48\times10^{-5}+5.4R_{zAB}-3.0R_{zCD}=\Delta l=0$		$0-4.0R_{zBC}=\Delta h=0$	
DCGH (yz plane)	D C	0	3.0	0	$\alpha_x=\pi/2$	1	0	R_{xDC}	0	0	$3.0R_{xDC}$	0	0
	C G	0	0	5.0	$\alpha_x=0$	0	1	R_{xCG}	90×10^{-5}	90×10^{-5}	0	0	$5.0R_{xCG}$
	G H	0	3.0	0	$\alpha_x=3\pi/2$	-1	0	R_{xGH}	0	0	$-3.0R_{xGH}$	0	0
									Σ	$90\times10^{-5}+3.0R_{xDC}-3.0R_{xGH}=\Delta l=0$		$0-5.0R_{xCG}=\Delta h=0$	
BFCG (xz plane)	B F	0	0	5.0	$\alpha_y=\pi/2$	1	0	R_{yBF}	0	0	$5.0R_{yBF}$	0	0
	F G	4.0	0	0	$\alpha_y=0$	0	1	R_{yFG}	0	0	0	0	$4.0R_{yFG}$
	G C	0	0	5.0	$\alpha_y=3\pi/2$	-1	0	R_{yGC}	90×10^{-5}	0	$-5.0R_{yGC}$	-90×10^{-5}	0
	C B	4.0	0	0	$\alpha_y=\pi$	0	-1	R_{yCB}	48×10^{-5}	-48×10^{-5}	0	0	$-4.0R_{yCB}$
									Σ	$-48\times10^{-5}+5.0R_{yBF}-5.0R_{yGC}=\Delta l=0$		$-90\times10^{-5}-4.0R_{yFG}+4.0R_{yCB}=\Delta h=0$	

For remaining spaces ABFE and EFGH, the relations among deflection angles can easily be known without using the compatibility conditions. The relations thus found, taking R_{xAB}, R_{xCD}, R_{zAB} and R_{zEF} as the independent unknowns will be summarized in the following :

$$R_{yBC} = 1.35R_{xAB} - 0.75R_{xCD}, \qquad R_{yBF} = 0.84R_{zEF} - 1.08\,R_{zAB},$$

$$R_{xEF} = 1.286R_{xAB}, \qquad R_{zGH} = 1.4R_{zEF}.$$

Multiplying $\mu = 6E\overline{K}/1000$ and transforming we have :

For member AB : ψ_{xAB}, ψ_{zAB}.

〃 BC : $\psi_{yBC} = 1.35\psi_{xAB} - 0.75\psi_{xCD}$.

〃 CD : ψ_{xCD}, $\psi_{zCD} = 1.8\psi_{zAB} - 0.16\mu$.

〃 BF : $\psi_{yBF} = 0.84\psi_{zEF} - 1.08\psi_{zAB}$.

〃 CG : $\psi_{yCG} = 0.84\psi_{zEF} - 1.08\psi_{zAB} + 0.096\mu$.

〃 EF : $\psi_{xEF} = 1.286\psi_{xAB}$, ψ_{zEF}.

〃 FG : $\psi_{yFG} = 1.35\,\psi_{xAB} - 0.75\psi_{xCD} + 0.225\mu$.

〃 GH : $\psi_{xGH} = \psi_{xCD} - 0.30\mu$, $\psi_{zGH} = 1.4\psi_{zEF}$.

Now the analysis goes on as before.

1) Expressions of End-Moments

For member AB :

$$M_{xAB} = 1.2(\varphi_{xB} + \psi_{xAB}), \qquad M_{xBA} = 1.2(2\varphi_{xB} + \psi_{xAB}),$$

$$M_{yAB} = -0.2\varphi_{yB}, \qquad M_{yBA} = 0.2\varphi_{yB},$$

$$M_{zAB} = 1.2(\varphi_{zB} + \psi_{zAB}), \qquad M_{zBA} = 1.2(2\varphi_{zB} + \psi_{zAB}).$$

For member BC :

$$M_{xBC} = 0.28(\varphi_{xB} - \varphi_{xC}), \qquad M_{xCB} = 0.28(\varphi_{xC} - \varphi_{xB}),$$

$$M_{yBC} = 1.2(2\varphi_{yB} + \varphi_{yC} + 1.35\psi_{xAB} - 0.75\psi_{xCD}),$$

$$M_{yCB} = 1.2(\varphi_{yB} + 2\varphi_{yC} + 1.35\psi_{xAB} - 0.75\psi_{xCD}),$$

$$M_{zBC} = 2\varphi_{zB} + \varphi_{zC}, \qquad M_{zCB} = \varphi_{zB} + 2\varphi_{zC}.$$

For member CD :

$$M_{xCD} = 0.8(2\varphi_{xC} + \psi_{xCD}), \qquad M_{xDC} = 0.8(\varphi_{xC} + \psi_{xCD}),$$

$M_{y\mathrm{CD}} = 0.2\varphi_{y\mathrm{C}}$, $\quad M_{y\mathrm{DC}} = -0.2\varphi_{y\mathrm{C}}$,

$M_{z\mathrm{CD}} = 0.8(2\varphi_{z\mathrm{C}} + 1.8\psi_{z\mathrm{AB}} - 0.16\mu)$, $\quad M_{z\mathrm{DC}} = 0.8(\varphi_{z\mathrm{C}} + 1.8\psi_{z\mathrm{AB}} - 0.16\mu)$.

For member BF :

$M_{x\mathrm{BF}} = 1.5(2\varphi_{x\mathrm{B}} + \varphi_{x\mathrm{F}})$, $\quad M_{x\mathrm{FB}} = 1.5(\varphi_{x\mathrm{B}} + 2\varphi_{x\mathrm{F}})$,

$M_{y\mathrm{BF}} = 2\varphi_{y\mathrm{B}} + \varphi_{y\mathrm{F}} + 0.84\psi_{z\mathrm{EF}} - 1.08\psi_{z\mathrm{AB}}$,

$M_{y\mathrm{FB}} = \varphi_{y\mathrm{B}} + 2\varphi_{y\mathrm{F}} + 0.84\psi_{z\mathrm{EF}} - 1.08\psi_{z\mathrm{AB}}$,

$M_{z\mathrm{BF}} = 0.28(\varphi_{z\mathrm{B}} - \varphi_{z\mathrm{F}})$, $\quad M_{z\mathrm{FB}} = 0.28(\varphi_{z\mathrm{F}} - \varphi_{z\mathrm{B}})$.

For member CG :

$M_{x\mathrm{CG}} = 1.5(2\varphi_{x\mathrm{C}} + \varphi_{x\mathrm{G}})$, $\quad M_{x\mathrm{GC}} = 1.5(\varphi_{x\mathrm{C}} + 2\varphi_{x\mathrm{G}})$,

$M_{y\mathrm{CG}} = 2\varphi_{y\mathrm{C}} + \varphi_{y\mathrm{G}} + 0.84\psi_{z\mathrm{EF}} - 1.08\psi_{z\mathrm{AB}} + 0.096\mu$,

$M_{y\mathrm{GC}} = \varphi_{y\mathrm{C}} + 2\varphi_{y\mathrm{G}} + 0.84\psi_{z\mathrm{EF}} - 1.08\psi_{z\mathrm{AB}} + 0.096\mu$,

$M_{z\mathrm{CG}} = 0.28(\varphi_{z\mathrm{C}} - \varphi_{z\mathrm{G}})$, $\quad M_{z\mathrm{GC}} = 0.28(\varphi_{z\mathrm{G}} - \varphi_{z\mathrm{C}})$.

For member EF :

$M_{x\mathrm{EF}} = 1.2(\varphi_{x\mathrm{F}} + 1.286\psi_{x\mathrm{AB}})$, $\quad M_{x\mathrm{FE}} = 1.2(2\varphi_{x\mathrm{F}} + 1.286\psi_{x\mathrm{AB}})$,

$M_{y\mathrm{EF}} = -0.24\varphi_{y\mathrm{F}}$, $\quad M_{y\mathrm{FE}} = 0.24\varphi_{y\mathrm{F}}$,

$M_{z\mathrm{EF}} = \varphi_{z\mathrm{F}} + \psi_{z\mathrm{EF}}$, $\quad M_{z\mathrm{FE}} = 2\varphi_{z\mathrm{F}} + \psi_{z\mathrm{EF}}$.

For member FG :

$M_{x\mathrm{FG}} = 0.28(\varphi_{x\mathrm{F}} - \varphi_{x\mathrm{G}})$, $\quad M_{x\mathrm{GF}} = 0.28(\varphi_{x\mathrm{G}} - \varphi_{x\mathrm{F}})$,

$M_{y\mathrm{FG}} = 1.2(2\varphi_{y\mathrm{F}} + \varphi_{y\mathrm{G}} + 1.35\psi_{x\mathrm{AB}} - 0.75\psi_{x\mathrm{CD}} + 0.225\mu)$,

$M_{y\mathrm{GF}} = 1.2(\varphi_{y\mathrm{F}} + 2\varphi_{y\mathrm{G}} + 1.35\psi_{x\mathrm{AB}} - 0.75\psi_{x\mathrm{CD}} + 0.225\mu)$,

$M_{z\mathrm{FG}} = 0.8(2\varphi_{z\mathrm{F}} + \varphi_{z\mathrm{G}})$, $\quad M_{z\mathrm{GF}} = 0.8(\varphi_{z\mathrm{F}} + 2\varphi_{z\mathrm{G}})$.

For member GH :

$M_{x\mathrm{GH}} = 0.8(2\varphi_{x\mathrm{G}} + \psi_{x\mathrm{CD}} - 0.30\mu)$, $\quad M_{x\mathrm{HG}} = 0.8(\varphi_{x\mathrm{G}} + \psi_{x\mathrm{CD}} - 0.30\mu)$,

$M_{y\mathrm{GH}} = 0.16\varphi_{y\mathrm{G}}$, $\quad M_{y\mathrm{HG}} = -0.16\varphi_{y\mathrm{G}}$

$M_{z\mathrm{GH}} = 2\varphi_{z\mathrm{G}} + 1.4\psi_{z\mathrm{EF}}$, $\quad M_{z\mathrm{HG}} = \varphi_{z\mathrm{G}} + 1.4\psi_{z\mathrm{EF}}$.

2) Expressions of End-Shears

In x direction:

$$X_{x\mathrm{BA}} = -(1/l_{\mathrm{AB}})(M_{z\mathrm{AB}} + M_{z\mathrm{BA}}) = -(1.2/5.4)(3\varphi_{z\mathrm{B}} + 2\psi_{z\mathrm{AB}})$$
$$= -0.667\varphi_{z\mathrm{B}} - 0.444\psi_{z\mathrm{AB}},$$

$$X_{x\mathrm{CD}} = -(1/l_{\mathrm{CD}})(M_{z\mathrm{CD}} + M_{z\mathrm{DC}}) = -(0.8/3.0)(3\varphi_{z\mathrm{C}} + 3.6\psi_{z\mathrm{AB}} - 0.32\mu)$$
$$= -0.8\varphi_{z\mathrm{C}} - 0.96\psi_{z\mathrm{AB}} + 0.085{,}3\mu,$$

$$X_{x\mathrm{BF}} = -(1/l_{\mathrm{BF}})(M_{y\mathrm{BF}} + M_{y\mathrm{FB}}) = -(1/5.0)(3\varphi_{y\mathrm{B}} + 3\varphi_{y\mathrm{F}} + 1.68\psi_{z\mathrm{EF}} - 2.16\psi_{z\mathrm{AB}})$$
$$= -0.6\varphi_{y\mathrm{B}} - 0.6\varphi_{y\mathrm{F}} - 0.336\psi_{z\mathrm{EF}} + 0.432\psi_{z\mathrm{AB}},$$

$$X_{x\mathrm{CG}} = -(1/l_{\mathrm{CG}})(M_{y\mathrm{CG}} + M_{y\mathrm{GC}}) = -(1/5.0)(3\varphi_{y\mathrm{C}} + 3\varphi_{y\mathrm{G}} + 1.68\psi_{z\mathrm{EF}} - 2.16\psi_{z\mathrm{AB}}$$
$$+0.192\mu) = -0.6\varphi_{y\mathrm{C}} - 0.6\varphi_{y\mathrm{G}} - 0.336\psi_{z\mathrm{EF}} + 0.432\psi_{z\mathrm{AB}} - 0.038{,}4\mu,$$

$$X_{x\mathrm{FE}} = -(1/l_{\mathrm{EF}})(M_{z\mathrm{EF}} + \mathrm{M}_{z\mathrm{FE}}) = -(1/4.2)(3\varphi_{z\mathrm{F}} + 2\psi_{z\mathrm{EF}}) = -0.714\varphi_{z\mathrm{F}} - 0.476\psi_{z\mathrm{EF}},$$

$$X_{x\mathrm{GH}} = -(1/l_{\mathrm{GH}})(M_{z\mathrm{GH}} + M_{z\mathrm{HG}}) = -(1/3.0)(3\varphi_{z\mathrm{G}} + 2.8\psi_{z\mathrm{EF}}) = -\varphi_{z\mathrm{G}} - 0.933\psi_{z\mathrm{EF}},$$

$$X_{x\mathrm{FB}} = X_{x\mathrm{BF}} = -0.6\varphi_{y\mathrm{B}} - 0.6\varphi_{y\mathrm{F}} - 0.336\psi_{z\mathrm{EF}} + 0.432\psi_{z\mathrm{AB}},$$

$$X_{x\mathrm{GC}} = X_{x\mathrm{CG}} = -0.6\varphi_{y\mathrm{C}} - 0.6\varphi_{y\mathrm{G}} - 0.336\psi_{z\mathrm{EF}} + 0.432\psi_{z\mathrm{AB}} - 0.038{,}4\mu.$$

In z direction :

$$X_{z\mathrm{BA}} = -(1/l_{\mathrm{AB}})(M_{x\mathrm{AB}} + M_{x\mathrm{BA}}) = -(1.2/5.4)(3\varphi_{x\mathrm{B}} + 2\psi_{x\mathrm{AB}})$$
$$= -0.667\varphi_{x\mathrm{B}} - 0.444\psi_{x\mathrm{AB}},$$

$$X_{z\mathrm{FE}} = -(1/l_{\mathrm{EF}})(M_{x\mathrm{EF}} + M_{x\mathrm{FE}}) = -(1.2/4.2)(3\varphi_{x\mathrm{F}} + 2.571\psi_{x\mathrm{AB}})$$
$$= -0.857\varphi_{x\mathrm{F}} - 0.735\psi_{x\mathrm{AB}},$$

$$X_{z\mathrm{BC}} = -(1/l_{\mathrm{BC}})(M_{y\mathrm{BC}} + M_{y\mathrm{CB}}) = -(1.2/4.0)(3\varphi_{y\mathrm{B}} + 3\varphi_{y\mathrm{C}} + 2.7\psi_{x\mathrm{AB}} - 1.5\psi_{x\mathrm{CD}})$$
$$= -0.9\varphi_{y\mathrm{B}} - 0.9\varphi_{y\mathrm{C}} - 0.81\psi_{x\mathrm{AB}} + 0.45\psi_{x\mathrm{CD}},$$

$$X_{z\mathrm{FG}} = -(1/l_{\mathrm{FG}})(M_{y\mathrm{FG}} + M_{y\mathrm{GF}}) = -(1.2/4.0)(3\varphi_{y\mathrm{F}} + 3\varphi_{y\mathrm{G}} + 2.7\psi_{x\mathrm{AB}} - 1.5\psi_{x\mathrm{CD}}$$
$$+ 0.45\mu) = -0.9\varphi_{y\mathrm{F}} - 0.9\varphi_{y\mathrm{G}} - 0.81\psi_{x\mathrm{AB}} + 0.45\psi_{x\mathrm{CD}} - 0.135\mu.$$

$$X_{z\mathrm{CD}} = -(1/l_{\mathrm{CD}})(M_{x\mathrm{CD}} + M_{x\mathrm{DC}}) = -(0.8/3.0)(3\varphi_{x\mathrm{C}} + 2\psi_{x\mathrm{CD}})$$
$$= -0.8\varphi_{x\mathrm{C}} - 0.533\psi_{x\mathrm{CD}},$$

$$X_{z\mathrm{GH}} = -(1/l_{\mathrm{GH}})(M_{x\mathrm{GH}} + M_{x\mathrm{HG}}) = -(0.8/3.0)(3\varphi_{x\mathrm{G}} + 2\psi_{x\mathrm{CD}} - 0.6\mu)$$
$$= -0.8\varphi_{x\mathrm{G}} - 0.533\psi_{x\mathrm{CD}} + 0.16\mu,$$

$$X_{z\mathrm{CB}} = X_{z\mathrm{BC}} = -0.9\varphi_{y\mathrm{B}} - 0.9\varphi_{y\mathrm{C}} - 0.81\psi_{x\mathrm{AB}} + 0.45\psi_{x\mathrm{CD}},$$

$$X_{z\mathrm{GF}} = X_{z\mathrm{FG}} = -0.9\varphi_{y\mathrm{F}} - 0.9\varphi_{y\mathrm{G}} - 0.81\psi_{x\mathrm{AB}} + 0.45\psi_{x\mathrm{CD}} - 0.135\mu.$$

3) Elastic Equations

i) Joint Equilibrium Equations

About x axis :

At joint B, $M_{xBA}+M_{xBC}+M_{xBF}=0.$

〃 C, $M_{xCB}+M_{xCD}+M_{xCG}=0.$

〃 F, $M_{xFB}+M_{xFE}+M_{xFG}=0.$

〃 G, $M_{xGC}+M_{xGF}+M_{xGH}=0.$

About y axis :

At joint B, $M_{yBA}+M_{yBC}+M_{yBF}=0.$

〃 C, $M_{yCB}+M_{yCD}+M_{yCG}=0.$

〃 F, $M_{yFB}+M_{yFE}+M_{yFG}=0.$

〃 G, $M_{yGC}+M_{yGF}+M_{yGH}=0.$

About z axis :

Table 16 Elastic equations

Eq.	Left-hand side							
	φ_{xB}	φ_{xC}	φ_{xF}	φ_{xG}	φ_{yB}	φ_{yC}	φ_{yF}	φ_{yG}
1	5.680	−0.280	1.500					
2	−0.280	4.880		1.500				
3	1.500		5.680	−0.280				
4		1.500	−0.280	4.880				
5					4.600	1.200	1.000	
6					1.200	4.600		1.000
7					1.000		4.640	1.200
8						1.000	1.200	4.560
9								
10								
11								
12								
13	1.200		1.543		1.620	1.620	1.620	1.620
14		0.800		0.800	−0.900	−0.900	−0.900	−0.900
15					−1.080	−1.080	−1.080	−1.080
16					0.840	0.840	0.840	0.840

At joint B, $M_{zBA} + M_{zBC} + M_{zBF} = 0.$

〃 C, $M_{zCB} + M_{zCD} + M_{zCG} = 0.$

〃 F, $M_{zFB} + M_{zFE} + M_{zFG} = 0.$

〃 G, $M_{zGC} + M_{zGF} + M_{zGH} = 0.$

ii) Horizontal Shear Equations

In z direction :

For the 1st column-frame : $X_{zBA} + X_{zFE} + X_{zBC} + X_{zFG} = 0.$

〃 2nd 〃 : $X_{zCD} + X_{zGH} - X_{zCB} - X_{zGF} = 0.$

In x direction :

For the 1st row-frame : $- X_{xBA} - X_{xCD} + X_{xBF} + X_{xCG} = 0.$

〃 2nd 〃 : $- X_{xFE} - X_{xGH} - X_{xFB} - X_{xGC} = 0.$

These are summarized in Table 16.

Table 16—continued—

φ_{zB}	φ_{zC}	φ_{zF}	φ_{zG}	ψ_{xAB}	ψ_{xCD}	ψ_{zAB}	ψ_{zEF}	Right-hand side (multiplier: μ)
				1.200				
					0.800			
				1.543				
					0.800			0.240
				1.620	−0.900	−1.080	0.840	
				1.620	−0.900	−1.080	0.840	−0.096
				1.620	−0.900	−1.080	0.840	−0.270
				1.620	−0.900	−1.080	0.840	−0.366
4.680	1.000	−0.280				1.200		
1.000	3.880		−0.280			1.440		0.128
−0.280		3.880	0.800				1.000	
	−0.280	0.800	3.880				1.400	
				5.038	−1.620			−0.243
				−1.620	1.967			0.295
1.200	1.440					4.082	−1.210	0.223
		1.000	1.400			−1.210	2.913	−0.054

Solving these simultaneously, we obtain:

At joint B, $\varphi_{xB} = -0.001,865$, $\varphi_{yB} = +0.042,13$, $\varphi_{zB} = -0.018,57$,

〃 C, $\varphi_{xC} = -0.036,07$, $\varphi_{yC} = +0.020,51$, $\varphi_{zC} = +0.016,12$,

〃 F, $\varphi_{xF} = +0.003,216$, $\varphi_{yF} = -0.013,79$, $\varphi_{zF} = -0.002,133$,

〃 G, $\varphi_{xG} = +0.035,36$, $\varphi_{yG} = -0.036,25$, $\varphi_{zG} = +0.000,693,3$,

and

$\psi_{xAB} = -0.003,609$, $\psi_{xCD} = +0.153,1$, $\psi_{zAB} = +0.058,48$,

$\psi_{zEF} = +0.002,521$.

Replacing these φ's and ψ's in the expressions of end-moments 2), we find the solutions shown in Table 17.

Table 17 Values of end-moments (t-m)

Member	AB		BC		CD		BF	
End	A	B	B	C	C	D	B	F
M_x	−6.57	−8.81	+9.58	−9.58	+64.77	+93.62	−0.77	+6.85
M_y	−8.43	+8.43	−17.91	−43.86	+4.10	−4.10	+9.43	−46.49
M_z	+47.89	+25.61	−21.02	+13.67	−18.00	−30.89	−4.60	+4.60

Member	CG		EF		FG		GH	
End	C	G	E	F	F	G	G	H
M_x	−55.17	+51.98	−1.71	+2.15	−9.00	+9.00	−60.94	−89.23
M_y	+39.73	−17.03	+3.31	−3.31	+49.77	+22.82	−5.80	+5.80
M_z	+4.32	−4.32	+0.39	−1.75	−2.86	−0.60	+4.92	+4.22

4) Comparison with the Current Solutions

In the two-dimensional analysis, we treat only the two plane frames shown in Fig. 23 and the results obtained are compared with the exact values as shown in Table 18 and in Fig. 24~27. See that the two-dimensional analysis does not give the moments about y axis in Fig. 26 and that the magnitudes of them are as large as those of moments about the axes x and z, so that they can never been neglected. The deflection diagram is shown in Fig. 28.

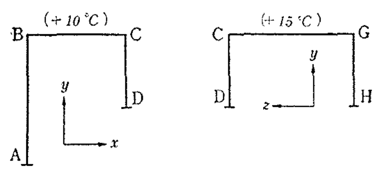
Table 18 Percentage errors

(a) The 1st-row-frame Multiplier: μ'

Member	AB		BC		CD		
End	A	B	B	C	C	D	
M	+48.8	+24.4	−24.4	+13.2	−13.2	−26.6	
M_z	+47.9	+25.6	−21.0	+13.7	−18.0	−30.9	
$(M-M_z)/M_z$	+1.9	−4.7	+14.3	−3.6	−26.7	−13.9	(%)
H	+13.4					−13.3	
H_x	+13.6					−16.3	
$(H-H_x)/H_x$	−1.5					−18.4	(%)
V	+2.7					−2.7	
V_y	+0.6					−1.2	
$(V-V_y)/V_y$	(+350)					(+125)	(%)

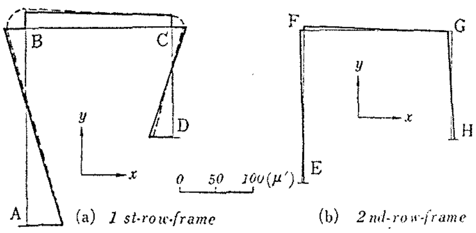
(b) The 2nd-column-frame Multiplier: μ'

Member	DC		CG		GH		
End	D	C	C	G	G	H	
M	+89.0	+58.1	−58.1	+58.1	−58.1	−89.0	
M_x	+93.6	+64.8	−55.2	+52.0	−60.9	−89.2	
$(M-M_x)/M_x$	−4.9	−10.3	+5.3	+11.7	−4.6	−0.2	(%)
H	−49.0					+49.0	
H_z	−52.8					+50.1	
$(H-H_z)/H_z$	−7.2					−2.2	(%)
V	−2.7					0.0	
V_y	−1.2					−1.5	
$(V-V_y)/V_y$	(+125)					(−100)	(%)



(a) 1 st-row-frame (b) 2 nd-column frame

Fig. 23



(a) 1 st-row-frame (b) 2 nd-row-frame

Fig. 24

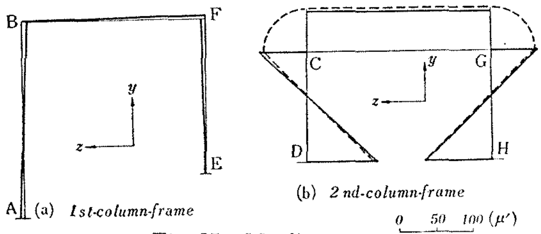


Fig. 25 M_x diagram

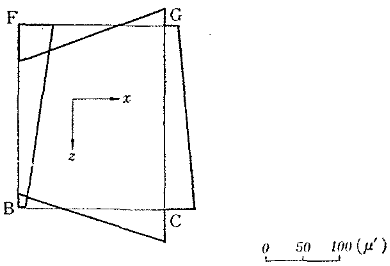


Fig. 26 M_y diagram

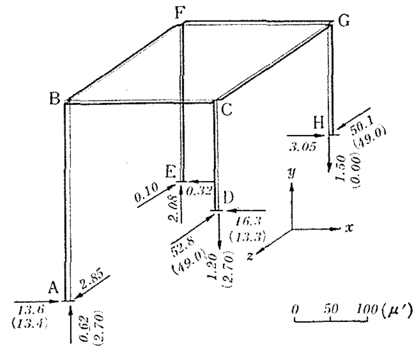
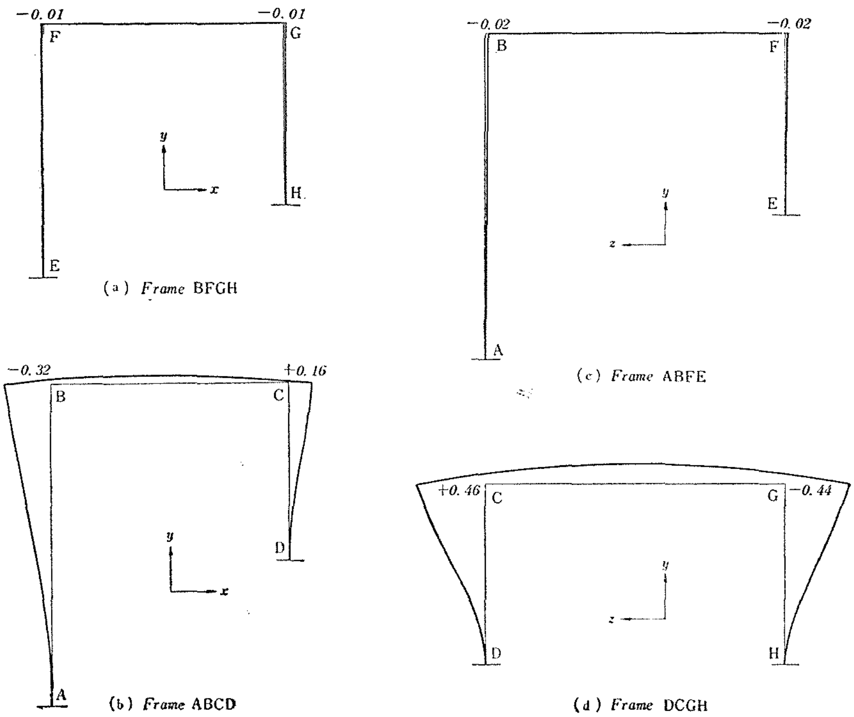


Fig. 27 Torsional moment diagram and reactions



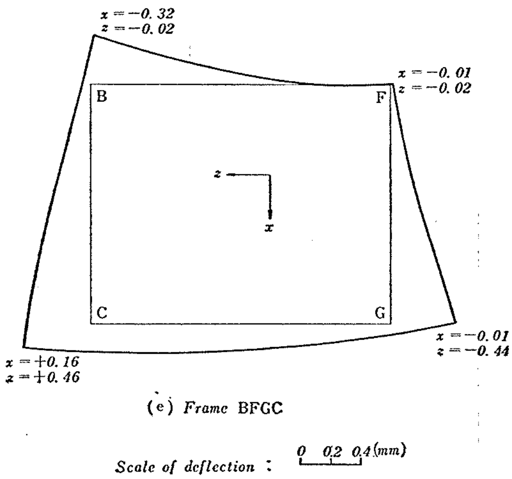


Fig. 28 Deflection diagram

19. Settlements of Supports

The frame in Fig. 29 undergoes settlements at supports; they are 12 cm at C in negative z direction horizontally, and 6 cm at I vertically upward. The dimensions and stiffnesses are to be read from Table 19.

Table 19

Member	AD	BE	CF	DE	EF	DJ	EK	FL	JK	KL	GJ	HK	IL	
Length	4.2	5.4	5.4	4.0	6.0	5.0	5.0	5.0	4.0	6.0	3.6	4.2	4.2	(m)
k_x	1.0	1.2	1.2	1.4	1.2	1.5	1.5	1.5	1.4	1.2	1.0	1.2	0.8	
k_y	1.0	1.0	1.0	1.2	1.0	1.0	1.0	1.0	1.2	1.0	1.0	1.2	0.8	
k_z	1.0	1.2	1.2	1.0	1.0	1.4	1.4	1.4	1.0	1.0	1.0	1.0	1.0	

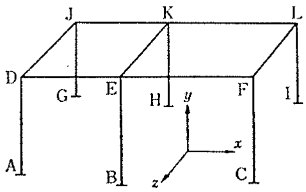


Fig. 29

As in the manner previously shown, the compatibility equations, Table 20, give the following relations :

$$\psi_{xIL} = 1.29\psi_{xCF} - 28.6\mu, \qquad \psi_{xFL} = 12\mu,$$

$$\psi_{zIL} = 0.857\psi_{zGJ}, \qquad \psi_{zKL} = 10\mu,$$

where $\mu = 6E\overline{K}/1000$.

Table 20 Compatibility conditions

Space	Member	Length (s) x	y	z	Direction angle (α)	sin α	cos α	Revolution angle	Rs sin α	Rs cos α
HKLI (xy plane)	HK	0	4.2	0	$\alpha_z=\pi/2$	1	0	R_{zHK}	$4.2R_{zHK}$	0
	KL	6.0	0	0	$\alpha_z=0$	0	1	R_{zKL}	0	$6.0R_{zKL}$
	LI	0	4.2	0	$\alpha_z=3\pi/2$	-1	0	R_{zLI}	$-4.2R_{zLI}$	0
								Σ	$4.2R_{zHK}-4.2R_{zLI}=\Delta l=0$	$-6.0R_{zKL}=\Delta h=0.06$
CFLI (yz plane)	CF	0	5.4	0	$\alpha_x=\pi/2$	1	0	R_{xCF}	$5.4R_{xCF}$	0
	FL	0	0	5.0	$\alpha_x=0$	0	1	R_{xFL}	0	$5.0R_{xFL}$
	LI	0	4.2	0	$\alpha_x=3\pi/2$	-1	0	R_{xLI}	$-4.2R_{xLI}$	0
								Σ	$5.4R_{xCF}-4.2R_{xLI}=\Delta l=-0.12$	$-5.0R_{xFL}=\Delta h=0.06$

The relations below are readily found by inspecting the remaining spaces:

$$\psi_{zBE}=0.778\psi_{zAD},\qquad \psi_{zCF}=\psi_{zBE},\qquad \psi_{yDE}=1.05\psi_{xAD}-1.35\psi_{xBE},$$
$$\psi_{yEF}=0.9\psi_{xBE}-0.9\psi_{xCF},\qquad \psi_{yDJ}=-0.84\psi_{zAD}+0.72\psi_{zGJ},$$
$$\psi_{yEK}=\psi_{yDJ},\qquad \psi_{yJK}=\psi_{yDE},\qquad \psi_{yKL}=\psi_{yEF},$$
$$\psi_{xGJ}=1.167\psi_{xAD},\qquad \psi_{xHK}=1.286\psi_{xBE},\qquad \psi_{zHK}=0.857\psi_{zGJ}.$$

1) Expressions of End-Moments

For member AD :

$$M_{xAD}=\varphi_{xD}+\psi_{xAD},\qquad M_{xDA}=2\varphi_{xD}+\psi_{xAD},$$
$$M_{yAD}=-0.2\varphi_{yD},\qquad M_{yDA}=0.2\varphi_{yD},$$
$$M_{zAD}=\varphi_{zD}+\psi_{zAD},\qquad M_{zDA}=2\varphi_{zD}+\psi_{zAD}.$$

For member BE :

$$M_{xBE}=1.2(\varphi_{xE}+\psi_{xBE}),\qquad M_{xEB}=1.2(2\varphi_{xE}+\psi_{xBE}),$$
$$M_{yBE}=-0.2\varphi_{yE},\qquad M_{yEB}=0.2\varphi_{yE},$$
$$M_{zBE}=1.2(\varphi_{zE}+0.778\psi_{zAD}),\qquad M_{zEB}=1.2(2\varphi_{zE}+0.778\psi_{zAD}).$$

For member CF :

$$M_{xCF}=1.2(\varphi_{xF}+\psi_{xCF}),\qquad M_{xFC}=1.2(2\varphi_{xF}+\psi_{xCF}),$$
$$M_{yCF}=0.2\varphi_{yF},\qquad M_{yFC}=0.2\varphi_{yF},$$

$M_{z\mathrm{CF}} = 1.2(\varphi_{z\mathrm{F}} + 0.778\psi_{z\mathrm{AD}})$, $M_{z\mathrm{FC}} = 1.2(2\varphi_{z\mathrm{F}} + 0.778\psi_{z\mathrm{AD}})$,

For member DE :

$M_{x\mathrm{DE}} = 0.28(\psi_{x\mathrm{D}} - \varphi_{x\mathrm{E}})$, $M_{x\mathrm{ED}} = 0.28(\varphi_{x\mathrm{E}} - \varphi_{x\mathrm{D}})$,

$M_{y\mathrm{DE}} = 1.2(2\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{E}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}})$,

$M_{y\mathrm{ED}} = 1.2(\varphi_{y\mathrm{D}} + 2\varphi_{y\mathrm{E}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}})$,

$M_{z\mathrm{DE}} = 2\varphi_{z\mathrm{D}} + \varphi_{z\mathrm{E}}$, $M_{z\mathrm{ED}} = \varphi_{z\mathrm{D}} + 2\varphi_{z\mathrm{E}}$.

For member EF :

$M_{x\mathrm{EF}} = 0.24(\varphi_{x\mathrm{E}} - \varphi_{x\mathrm{F}})$, $M_{x\mathrm{FE}} = 0.24(\varphi_{x\mathrm{F}} - \varphi_{x\mathrm{E}})$,

$M_{y\mathrm{EF}} = 2\varphi_{y\mathrm{E}} + \varphi_{y\mathrm{F}} + 0.9\psi_{x\mathrm{BE}} - 0.9\psi_{x\mathrm{CF}}$,

$M_{y\mathrm{FE}} = \varphi_{y\mathrm{E}} + 2\varphi_{y\mathrm{F}} + 0.9\psi_{x\mathrm{BE}} - 0.9\psi_{x\mathrm{CF}}$,

$M_{z\mathrm{EF}} = 2\varphi_{z\mathrm{E}} + \varphi_{z\mathrm{F}}$, $M_{z\mathrm{FE}} = \varphi_{z\mathrm{E}} + 2\varphi_{z\mathrm{F}}$.

For member DJ :

$M_{x\mathrm{DJ}} = 1.5(2\varphi_{x\mathrm{D}} + \varphi_{x\mathrm{J}})$, $M_{x\mathrm{JD}} = 1.5(\varphi_{x\mathrm{D}} + 2\varphi_{x\mathrm{J}})$,

$M_{y\mathrm{DJ}} = 2\varphi_{y\mathrm{D}} + \varphi_{y\mathrm{J}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{y\mathrm{JD}} = \varphi_{y\mathrm{D}} + 2\varphi_{y\mathrm{J}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{z\mathrm{DJ}} = 0.28(\varphi_{z\mathrm{D}} - \varphi_{z\mathrm{J}})$, $M_{z\mathrm{JD}} = 0.28(\varphi_{z\mathrm{J}} - \varphi_{z\mathrm{D}})$.

For member EK :

$M_{x\mathrm{EK}} = 1.5(2\varphi_{x\mathrm{E}} + \varphi_{x\mathrm{K}})$, $M_{x\mathrm{KE}} = 1.5(\varphi_{x\mathrm{E}} + 2\varphi_{x\mathrm{K}})$,

$M_{y\mathrm{EK}} = 2\varphi_{y\mathrm{E}} + \varphi_{y\mathrm{K}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{y\mathrm{KE}} = \varphi_{y\mathrm{E}} + 2\varphi_{y\mathrm{K}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{z\mathrm{EK}} = 0.28(\varphi_{z\mathrm{E}} - \varphi_{z\mathrm{K}})$, $M_{z\mathrm{KE}} = 0.28(\varphi_{z\mathrm{K}} - \varphi_{z\mathrm{E}})$.

For member FL :

$M_{x\mathrm{FL}} = 1.5(2\varphi_{x\mathrm{F}} + \varphi_{x\mathrm{L}} + 12\mu)$, $M_{x\mathrm{LF}} = 1.5(\varphi_{x\mathrm{F}} + 2\varphi_{x\mathrm{L}} + 12\mu)$,

$M_{y\mathrm{FL}} = 2\varphi_{y\mathrm{F}} + \varphi_{y\mathrm{L}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{y\mathrm{LF}} = \varphi_{y\mathrm{F}} + 2\varphi_{y\mathrm{L}} - 0.84\psi_{z\mathrm{AD}} + 0.72\psi_{z\mathrm{GJ}}$,

$M_{z\mathrm{FL}} = 0.28(\varphi_{z\mathrm{F}} - \varphi_{z\mathrm{L}})$, $M_{z\mathrm{LF}} = 0.28(\varphi_{z\mathrm{L}} - \varphi_{z\mathrm{F}})$.

For member JK :

$M_{x\mathrm{JK}} = 0.28(\varphi_{x\mathrm{J}} - \varphi_{x\mathrm{K}})$, $M_{x\mathrm{KJ}} = 0.28(\varphi_{x\mathrm{K}} - \varphi_{x\mathrm{J}})$,

$M_{y\mathrm{JK}} = 1.2(2\varphi_{y\mathrm{J}} + \varphi_{y\mathrm{K}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}}),$

$M_{y\mathrm{KJ}} = 1.2(\varphi_{y\mathrm{J}} + 2\varphi_{y\mathrm{K}} + 1.05\psi_{x\mathrm{AD}} - 1.35\psi_{x\mathrm{BE}}),$

$M_{z\mathrm{JK}} = 2\varphi_{z\mathrm{J}} + \varphi_{z\mathrm{K}},$ $M_{z\mathrm{KJ}} = \varphi_{z\mathrm{J}} + 2\varphi_{z\mathrm{K}}.$

For member KL :

$M_{x\mathrm{KL}} = 0.24(\varphi_{x\mathrm{K}} - \varphi_{x\mathrm{L}}),$ $M_{x\mathrm{LK}} = 0.24(\varphi_{x\mathrm{L}} - \varphi_{x\mathrm{K}}),$

$M_{y\mathrm{KL}} = 2\varphi_{y\mathrm{K}} + \varphi_{y\mathrm{L}} + 0.9\psi_{x\mathrm{BE}} - 0.9\psi_{x\mathrm{CF}},$

$M_{y\mathrm{LK}} = \varphi_{y\mathrm{K}} + 2\varphi_{y\mathrm{L}} + 0.9\psi_{x\mathrm{BE}} - 0.9\psi_{x\mathrm{CF}},$

$M_{z\mathrm{KL}} = 2\varphi_{z\mathrm{K}} + \varphi_{z\mathrm{L}} + 10\mu,$ $M_{z\mathrm{LK}} = \varphi_{z\mathrm{K}} + 2\varphi_{z\mathrm{L}} + 10\mu.$

For member GJ :

$M_{x\mathrm{GJ}} = \varphi_{x\mathrm{J}} + 1.167\psi_{x\mathrm{AD}},$ $M_{x\mathrm{JG}} = 2\varphi_{x\mathrm{J}} + 1.167\psi_{x\mathrm{AD}},$

$M_{y\mathrm{GJ}} = -0.2\varphi_{y\mathrm{J}},$ $M_{y\mathrm{JG}} = 0.2\varphi_{y\mathrm{J}},$

$M_{z\mathrm{GJ}} = \varphi_{z\mathrm{J}} + \psi_{z\mathrm{GJ}},$ $M_{z\mathrm{JG}} = 2\varphi_{z\mathrm{J}} + \psi_{z\mathrm{GJ}}.$

For member HK :

$M_{x\mathrm{HK}} = 1.2(\varphi_{x\mathrm{K}} + 1.286\psi_{x\mathrm{BE}}),$ $M_{x\mathrm{KH}} = 1.2(2\varphi_{x\mathrm{K}} + 1.286\psi_{x\mathrm{BE}}),$

$M_{y\mathrm{HK}} = -0.24\varphi_{y\mathrm{K}},$ $M_{y\mathrm{KH}} = 0.24\varphi_{y\mathrm{K}},$

$M_{z\mathrm{HK}} = \varphi_{z\mathrm{K}} + 0.857\psi_{z\mathrm{GJ}},$ $M_{z\mathrm{KH}} = 2\varphi_{z\mathrm{K}} + 0.857\psi_{z\mathrm{GJ}}.$

For member IL :

$M_{x\mathrm{IL}} = 0.8(\varphi_{x\mathrm{L}} + 1.286\psi_{x\mathrm{CF}} - 28.6\mu),$ $M_{x\mathrm{LI}} = 0.8(2\varphi_{x\mathrm{L}} + 1.286\psi_{x\mathrm{CF}} - 28.6\mu),$

$M_{y\mathrm{IL}} = -0.16\varphi_{y\mathrm{L}},$ $M_{y\mathrm{LI}} = 0.16\varphi_{y\mathrm{L}}$

$M_{z\mathrm{IL}} = \varphi_{z\mathrm{L}} + 0.857\psi_{z\mathrm{GJ}},$ $M_{z\mathrm{LI}} = 2\varphi_{z\mathrm{L}} + 0.857\psi_{z\mathrm{GJ}}.$

2) Expressions of End-Shears

In x direction :

$$\begin{aligned} X_{x\mathrm{DA}} &= -(1/l_{\mathrm{AD}})(M_{z\mathrm{AD}} + M_{z\mathrm{DA}}) = -(1.0/4.2)(3\varphi_{z\mathrm{D}} + 2\psi_{z\mathrm{AD}}) \\ &= -0.714\varphi_{z\mathrm{D}} - 0.476\psi_{z\mathrm{AD}}, \end{aligned}$$

$$\begin{aligned} X_{x\mathrm{EB}} &= -(1/l_{\mathrm{BE}})(M_{z\mathrm{BE}} + M_{z\mathrm{EB}}) = -(1.2/5.4)(3\varphi_{z\mathrm{E}} + 1.556\psi_{z\mathrm{AD}}) \\ &= -0.667\varphi_{z\mathrm{E}} - 0.346\psi_{z\mathrm{AD}}, \end{aligned}$$

$$\begin{aligned} X_{x\mathrm{FC}} &= -(1/l_{\mathrm{CF}})(M_{z\mathrm{CF}} + M_{z\mathrm{FC}}) = -(1.2/5.4)(3\varphi_{z\mathrm{F}} + 1.556\psi_{z\mathrm{AD}}) \\ &= -0.667\varphi_{z\mathrm{F}} - 0.346\psi_{z\mathrm{AD}}, \end{aligned}$$

$$X_{x\mathrm{DJ}} = -(1/l_{\mathrm{DJ}})(M_{y\mathrm{DJ}} + M_{y\mathrm{JD}}) = -(1/5.0)(3\varphi_{y\mathrm{D}} + 3\varphi_{y\mathrm{J}} - 1.68\psi_{z\mathrm{AD}} + 1.44\psi_{z\mathrm{GJ}})$$
$$= -0.6\varphi_{y\mathrm{D}} - 0.6\varphi_{y\mathrm{J}} + 0.336\psi_{z\mathrm{AD}} - 0.288\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{EK}} = -(1/l_{\mathrm{EK}})(M_{y\mathrm{EK}} + M_{y\mathrm{KE}}) = -(1/5.0)(3\varphi_{y\mathrm{E}} + 3\varphi_{y\mathrm{K}} - 1.68\psi_{z\mathrm{AD}} + 1.44\psi_{z\mathrm{GJ}})$$
$$= -0.6\varphi_{y\mathrm{E}} - 0.6\varphi_{y\mathrm{K}} + 0.336\psi_{z\mathrm{AD}} - 0.288\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{FL}} = -(1/l_{\mathrm{FL}})(M_{y\mathrm{FL}} + M_{y\mathrm{LF}}) = -(1/5.0)(3\varphi_{y\mathrm{F}} + 3\varphi_{y\mathrm{L}} - 1.68\psi_{z\mathrm{AD}} + 1.44\psi_{z\mathrm{GJ}})$$
$$= -0.6\varphi_{y\mathrm{F}} - 0.6\varphi_{y\mathrm{L}} + 0.336\psi_{z\mathrm{AD}} - 0.288\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{JG}} = -(1/l_{\mathrm{GJ}})(M_{z\mathrm{GJ}} + M_{z\mathrm{JG}}) = -(1/3.6)(3\varphi_{z\mathrm{J}} + 2\psi_{z\mathrm{GJ}}) = -0.833\varphi_{z\mathrm{J}} - 0.556\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{KH}} = -(1/l_{\mathrm{HK}})(M_{z\mathrm{HK}} + M_{z\mathrm{KH}}) = -(1/4.2)(3\varphi_{z\mathrm{K}} + 1.714\psi_{z\mathrm{GJ}})$$
$$= -0.714\varphi_{z\mathrm{K}} - 0.408\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{LI}} = -(1/l_{\mathrm{IL}})(M_{z\mathrm{IL}} + M_{z\mathrm{LI}}) = -(1/4.2)(3\varphi_{z\mathrm{L}} + 1.714\psi_{z\mathrm{GJ}})$$
$$= -0.714\varphi_{z\mathrm{L}} - 0.408\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{JD}} = X_{x\mathrm{DJ}} = -0.6\varphi_{y\mathrm{D}} - 0.6\varphi_{y\mathrm{J}} + 0.336\psi_{z\mathrm{AD}} - 0.288\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{KE}} = X_{x\mathrm{EK}} = -0.6\varphi_{y\mathrm{E}} - 0.6\varphi_{y\mathrm{K}} + 0.336\psi_{z\mathrm{AD}} - 0.228\psi_{z\mathrm{GJ}},$$

$$X_{x\mathrm{LF}} = X_{x\mathrm{FL}} = -0.6\varphi_{y\mathrm{F}} - 0.6\varphi_{y\mathrm{L}} + 0.336\psi_{z\mathrm{AD}} - 0.288\psi_{z\mathrm{GJ}}.$$

In z direction :

$$X_{z\mathrm{DA}} = -(1/l_{\mathrm{DA}})(M_{x\mathrm{DA}} + M_{x\mathrm{AD}}) = -(1/4.2)(3\varphi_{x\mathrm{D}} + 2\psi_{x\mathrm{AD}})$$
$$= -0.714\varphi_{x\mathrm{D}} - 0.476\psi_{x\mathrm{AD}},$$

$$X_{z\mathrm{JG}} = -(1/l_{\mathrm{JG}})(M_{x\mathrm{JG}} + M_{x\mathrm{GJ}}) = -(1/3.6)(3\varphi_{x\mathrm{J}} + 2.333\psi_{x\mathrm{AD}})$$
$$= -0.833\varphi_{x\mathrm{J}} - 0.648\psi_{x\mathrm{AD}},$$

$$X_{z\mathrm{DE}} = -(1/l_{\mathrm{DE}})(M_{y\mathrm{DE}} + M_{y\mathrm{ED}}) = -(1.2/4.0)(3\varphi_{y\mathrm{D}} + 3\varphi_{y\mathrm{E}} + 2.10\psi_{x\mathrm{AD}} - 2.70\psi_{x\mathrm{BE}})$$
$$= -0.9\varphi_{y\mathrm{D}} - 0.9\varphi_{y\mathrm{E}} - 0.63\psi_{x\mathrm{AD}} + 0.81\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{JK}} = -(1/l_{\mathrm{JK}})(M_{y\mathrm{JK}} + M_{y\mathrm{KJ}}) = -(1.2/4.0)(3\varphi_{y\mathrm{J}} + 3\varphi_{y\mathrm{K}} + 2.10\psi_{x\mathrm{AD}} - 2.70\psi_{x\mathrm{BE}})$$
$$= -0.9\varphi_{y\mathrm{J}} - 0.9\varphi_{y\mathrm{K}} - 0.63\psi_{x\mathrm{AD}} + 0.81\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{EB}} = -(1/l_{\mathrm{BE}})(M_{x\mathrm{BE}} + M_{x\mathrm{EB}}) = -(1.2/5.4)(3\varphi_{x\mathrm{E}} + 2\psi_{x\mathrm{BE}})$$
$$= -0.667\varphi_{x\mathrm{E}} - 0.444\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{KH}} = -(1/l_{\mathrm{HK}})(M_{x\mathrm{HK}} + M_{x\mathrm{KH}}) = -(1.2/4.2)(3\varphi_{x\mathrm{K}} + 2.572\psi_{x\mathrm{BE}})$$
$$= -0.857\varphi_{x\mathrm{K}} - 0.735\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{ED}} = X_{z\mathrm{DE}} = -0.9\varphi_{y\mathrm{D}} - 0.9\varphi_{y\mathrm{E}} - 0.63\psi_{x\mathrm{AD}} + 0.81\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{KJ}} = X_{z\mathrm{JK}} = -0.9\varphi_{y\mathrm{J}} - 0.9\varphi_{y\mathrm{K}} - 0.63\psi_{x\mathrm{AD}} + 0.81\psi_{x\mathrm{BE}},$$

$$X_{z\mathrm{EF}} = -(1/l_{\mathrm{EF}})(M_{y\mathrm{EF}} + M_{y\mathrm{FE}}) = -(1/6.0)(3\varphi_{y\mathrm{E}} + 3\varphi_{y\mathrm{F}} + 1.8\psi_{x\mathrm{BF}} - 1.8\psi_{x\mathrm{CF}})$$
$$= -0.5\varphi_{y\mathrm{E}} - 0.5\varphi_{y\mathrm{F}} - 0.3\psi_{x\mathrm{BE}} + 0.3\psi_{x\mathrm{CF}},$$

$$X_{z\mathrm{KL}} = -(1/l_{\mathrm{KL}})(M_{y\mathrm{KL}} + M_{y\mathrm{LK}}) = -(1/6.0)(3\varphi_{y\mathrm{K}} + 3\varphi_{y\mathrm{L}} + 1.8\psi_{x\mathrm{BE}} - 1.8\psi_{x\mathrm{CF}})$$
$$= -0.5\varphi_{y\mathrm{K}} - 0.5\varphi_{y\mathrm{L}} - 0.3\psi_{x\mathrm{BE}} + 0.3\psi_{x\mathrm{CF}},$$

$$X_{z\mathrm{FC}} = -(1/l_{\mathrm{CF}})(M_{x\mathrm{CF}} + M_{x\mathrm{FC}}) = -(1.2/5.4)(3\varphi_{x\mathrm{F}} + 2\psi_{x\mathrm{CF}})$$
$$= -0.667\varphi_{x\mathrm{F}} - 0.444\psi_{x\mathrm{CF}},$$

$$X_{z\mathrm{LI}} = -(1/l_{\mathrm{IL}})(M_{x\mathrm{IL}} + M_{x\mathrm{LI}}) = -(0.8/4.2)(3\varphi_{x\mathrm{L}} + 2.572\psi_{x\mathrm{CF}} - 57.2\mu)$$
$$= -0.571\varphi_{x\mathrm{L}} - 0.490\psi_{x\mathrm{CF}} + 10.895\mu,$$

$$X_{z\mathrm{FE}} = X_{z\mathrm{EF}} = -0.5\varphi_{y\mathrm{E}} - 0.5\varphi_{y\mathrm{F}} - 0.3\psi_{x\mathrm{BE}} + 0.3\psi_{x\mathrm{CF}},$$

$$X_{z\mathrm{LK}} = X_{z\mathrm{KL}} = -0.5\varphi_{y\mathrm{K}} - 0.5\varphi_{y\mathrm{L}} - 0.3\psi_{x\mathrm{BE}} + 0.3\psi_{x\mathrm{CF}}.$$

3) Elastic Equations, see Table 21

i) Joint Equilibrium Equations

About x axis :

At joint D, $M_{x\mathrm{DA}} + M_{x\mathrm{DE}} + M_{x\mathrm{DJ}} = 0.$

〃 E, $M_{x\mathrm{EB}} + M_{x\mathrm{ED}} + M_{x\mathrm{EF}} + M_{x\mathrm{EK}} = 0.$

〃 F, $M_{x\mathrm{FC}} + M_{x\mathrm{FE}} + M_{x\mathrm{FL}} = 0.$

〃 J, $M_{x\mathrm{JD}} + M_{x\mathrm{JG}} + M_{x\mathrm{JK}} = 0.$

〃 K, $M_{x\mathrm{KE}} + M_{x\mathrm{KH}} + M_{x\mathrm{KJ}} + M_{x\mathrm{KL}} = 0.$

〃 L, $M_{x\mathrm{LF}} + M_{x\mathrm{LI}} + M_{x\mathrm{LK}} = 0.$

About y axis :

At joint D, $M_{y\mathrm{DA}} + M_{y\mathrm{DE}} + M_{y\mathrm{DJ}} = 0.$

〃 E, $M_{y\mathrm{EB}} + M_{y\mathrm{ED}} + M_{y\mathrm{EF}} + M_{y\mathrm{EK}} = 0.$

〃 F, $M_{y\mathrm{FC}} + M_{y\mathrm{FE}} + M_{y\mathrm{FL}} = 0.$

〃 J, $M_{y\mathrm{JD}} + M_{y\mathrm{JG}} + M_{y\mathrm{JK}} = 0.$

〃 K, $M_{y\mathrm{KE}} + M_{y\mathrm{KH}} + M_{y\mathrm{KJ}} + M_{y\mathrm{KL}} = 0.$

〃 L, $M_{y\mathrm{LF}} + M_{y\mathrm{LI}} + M_{y\mathrm{LK}} = 0.$

About z axis :

At joint D, $M_{zDA}+M_{zDE}+M_{zDJ}=0.$

〃 E, $M_{zEB}+M_{zED}+M_{zEF}+M_{zEK}=0.$

〃 F, $M_{zFC}+M_{zFE}+M_{zFL}=0.$

〃 J, $M_{zJD}+M_{zJG}+M_{zJK}=0.$

〃 K, $M_{zKE}+M_{zKH}+M_{zKJ}+M_{zKL}=0.$

〃 L, $M_{zLF}+M_{zLI}+M_{zLK}=0.$

ii) Horizontal Shear Equations

In z direction :

For the 1st column-frame : $X_{zDA}+X_{zJG}+X_{zDE}+X_{zJK}=0.$

〃 2nd 〃 : $X_{zEB}+X_{zKH}-X_{zED}-X_{zKJ}+X_{zEF}+X_{zKL}=0.$

〃 3rd 〃 : $X_{zFC}+X_{zLI}-X_{zFE}-X_{zLK}=0.$

In x direction :

For the 1st row-frame : $-X_{xDA}-X_{xEB}-X_{xFC}+X_{xDJ}+X_{xEK}+X_{xFL}=0.$

〃 2nd 〃 : $X_{xJG}+X_{xKH}+X_{xLH}+X_{xJD}+X_{xKE}+X_{xLF}=0.$

The simultaneous equations in Table 21 are solved, and we have the rotations;

at joint D, $\varphi_{xD}=-0.020,71,$ $\varphi_{yD}=+0.974,7,$ $\varphi_{zD}=-0.619,4,$

〃 E, $\varphi_{xE}=-0.700,5,$ $\varphi_{yE}=+1.829,$ $\varphi_{zE}=-0.285,1,$

〃 F, $\varphi_{xF}=-6.073,$ $\varphi_{yF}=+1.966,$ $\varphi_{zF}=-0.663,6,$

〃 J, $\varphi_{xJ}=-0.010,19,$ $\varphi_{yJ}=+0.981,3,$ $\varphi_{zJ}=+0.323,4,$

〃 K, $\varphi_{xK}=-0.601,2,$ $\varphi_{yK}=+1.809,$ $\varphi_{zK}=-1.299,$

〃 L, $\varphi_{xL}=+0.015,21,$ $\varphi_{yL}=+1.997,$ $\varphi_{zL}=-2.025,$

and revolutions

$\psi_{xAD}=-0.071,53,$ $\psi_{xBE}=+2.988,$ $\psi_{xCF}=+13.383,$

$\psi_{zAD}=+3.027,$ $\psi_{zGJ}=-0.258,7.$

Using these φ's and ψ's, end-moments are determined. Results are read from Table 22.

Table 21 Elastic equations

Eq.	Left-hand side											
	φ_{xD}	φ_{xE}	φ_{xF}	φ_{xJ}	φ_{xK}	φ_{xL}	φ_{yD}	φ_{yE}	φ_{yF}	φ_{yJ}	φ_{yK}	φ_{yL}
1	5.280	−0.280		1.500								
2	−0.280	5.920	−0.240		1.500							
3		−0.240	5.640			1.500						
4	1.500			5.280	−0.280							
5		1.500		−0.280	5.920	−0.240						
6			1.500		−0.240	4.840						
7							4.600	1.200		1.000		
8							1.200	6.600	1.000		1.000	
9								1.000	4.200			1.000
10							1.000			4.600	1.200	
11								1.000		1.200	6.640	1.000
12									1.000		1.000	4.160
13												
14												
15												
16												
17												
18												
19	1.000			1.167			1.260	1.260		1.260	1.260	
20		1.200			1.543		−1.620	−0.720	0.900	−1.620	−0.720	0.900
21			1.200			1.029		−0.900	−0.900		−0.900	−0.900
22							−0.840	−0.840	−0.840	−0.840	−0.840	−0.840
23							0.720	0.720	0.720	0.720	0.720	0.720

Table 21—continued—

φ_{zD}	φ_{zE}	φ_{zF}	φ_{zJ}	φ_{zK}	φ_{zL}	ψ_{xAD}	ψ_{xBE}	ψ_{xCF}	ψ_{zAD}	ψ_{zGJ}	Right-hand side (multiplier : μ)
						1.000					
							1.200				
								1.200			−18.000
						1.167					
							1.543				
								1.029			4.880
						1.260	−1.620		−0.840	0.720	
						1.260	−0.720	−0.900	−0.840	0.720	
							0.900	−0.900	−0.840	0.720	
						1.260	−1.620		−0.840	0.720	
						1.260	−0.720	−0.900	−0.840	0.720	
							0.900	−0.900	−0.840	0.720	
4.280	1.000		−0.280						1.000		
1.000	6.680	1.000		−0.280					0.933		
	1.000	4.680			−0.280				0.933		
−0.280			4.280	1.000						1.000	
	−0.280		1.000	1.280	1.000					0.857	−10.000
		−0.280		1.000	4.280					0.857	−10.000
						3.339	−2.268				
						−2.268	6.118	−1.080			
							−1.080	2.761			19.611
1.000	0.933	0.933							3.046	−1.210	
			1.000	0.857	0.857				−1.210	2.683	

Table 22 Values of end-moments (multiplier : μ)

Member	AD		BE		CF		DE	
End	A	D	B	E	C	F	D	E
M_x	−0.092	−0.113	+2.745	+1.904	+8.772	+1.484	+0.190	−0.190
M_y	−0.195	+0.195	−0.366	+0.366	−0.393	+0.393	−0.397	+0.629
M_z	+2.408	+1.788	+2.484	+2.142	+2.030	+1.233	−1.524	−1.190

Member	EF		DJ		EK		FL	
End	E	F	D	J	E	K	F	L
M_x	+1.289	−1.289	−0.077	−0.062	−3.003	−2.854	−0.196	+8.936
M_y	−3.732	−3.595	−0.202	+0.208	+2.738	+2.718	+3.200	+3.231
M_z	−1.234	−1.612	−0.264	+0.264	+0.284	−0.284	+0.381	−0.381

Member	JK		KL		GJ		HK	
End	J	K	K	L	G	J	H	K
M_x	+0.165	−0.165	−0.148	+0.148	−0.094	−0.104	+3.890	+3.168
M_y	−0.405	+0.588	−3.741	−3.553	−0.196	+0.196	−0.434	+0.434
M_z	−0.652	−2.275	+5.377	+4.651	+0.065	+0.388	−1.521	−2.820

Member	IL	
End	I	L
M_x	−9.099	−9.087
M_y	−0.320	+0.320
M_z	−2.247	−4.272

4) Comparison with the Current Solutions

To solve this frame by the current two-dimensional method, we treat two constituent plane frames in Fig. 30 separately and the results and the errors are shown in Table 23. The moment diagrams are drawn by broken lines in Figs. 31, 32, 33 and 34.

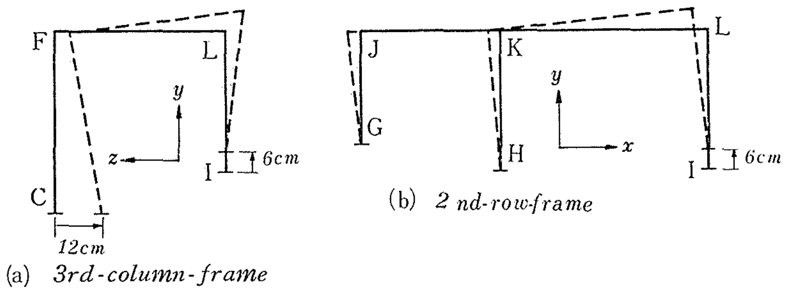


Fig. 30 Deflected skeleton

Table 23 Percentage errors of end-moments and reactions

(a) The 2nd-row-frame (multiplier : μ)

Member	GJ		JK		HK		KL		IL		
End	G	J	J	K	H	K	K	L	I	L	
M	+2.077	+1.893	−1.893	−3.234	+0.413	−1.112	+4.347	+3.269	−0.665	−3.269	
M_z	+0.065	+0.388	−0.652	−2.275	−1.521	−2.820	+5.377	+4.651	−2.247	−4.272	
$(M-M_z)/M_z$	(+3095)	(+388)	(+190)	+42.2	——	−60.6	−19.2	−29.7	−70.4	−23.5	(%)
H	+1.103				−0.166				−0.937		
H_x	+0.126				−1.034				−1.552		
$(H-H_x)/H_x$	(+775)				−83.9				−39.6		(%)
V	+1.282				−2.551				+1.796		
V_y	+0.705				−3.573				+3.420		
$(V-V_y)/V_y$	+81.8				−28.6				−47.5		(%)

(b) The 3rd-column-frame (multiplier : μ)

Member	CF		FL		IL		
End	C	F	F	L	I	L	
M	+11.916	+3.563	−3.563	+6.200	−5.841	−6.203	
M_x	+8.772	+1.484	−0.196	+8.936	−9.099	−9.087	
$(M-M_x)/M_x$	+35.8				−35.8		(%)
H	−2.866				+2.867		
H_z	−1.898				+4.331		
$(H-H_z)/H_z$	+51.0				−33.4		(%)
V	−0.527				+1.796		
V_y	−2.222				+3.420		
$(V-V_y)/V_y$	−76.3				−47.5		(%)

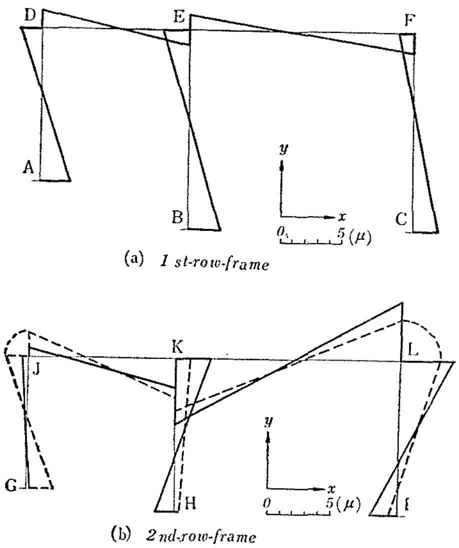


Fig. 31 M_z diagram

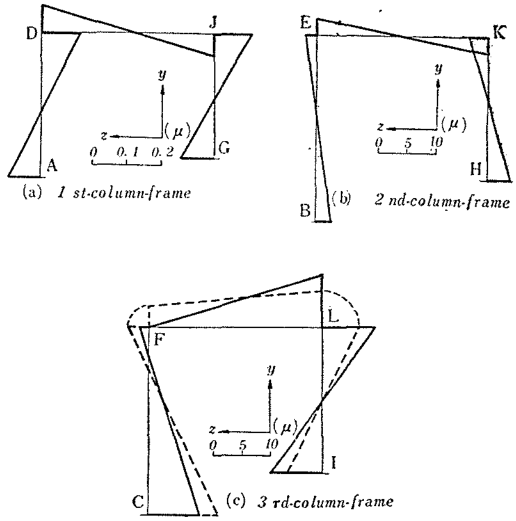


Fig. 32 M_x diagram

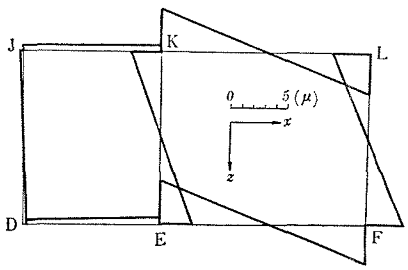


Fig. 33 M_y diagram

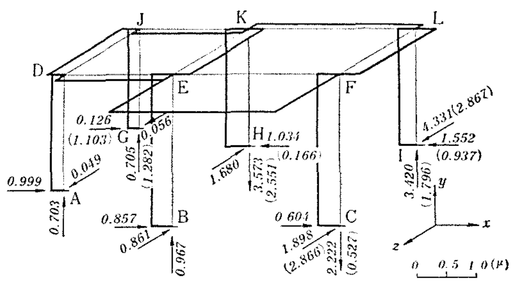


Fig. 34 Torsional moments and reactions

See, for example, Fig. 31. The effect of the vertical settlement at support I does not reach far away beyond the bay considered, and we find that the column at the left in the adjacent bay is affected very little as shown by full line. This tendency does not appear at all in the results obtained by the conventional solution which are shown by broken lines.

It should be especially noticed that the twisting moment of the remarkable magnitude is acting upon the member EF, see Fig. 34.

Chapter IV. Analysis of Grid Works

20. A Square Grid Supported Simply at the Corners, Concentrated Load on the Center

A grid work[6), 8)] can be regarded as a rigid frame in space from which the columns are taken off. Therefore, the author's method is equally applicable to its analysis as before. The loads are assumed here, as usual, to act perpendicularly to the structural plane, hence the horizontal shear equations are not required.

The analyses of the grid works have been presented by many investigators. Among these, Prof. T. Fukuda's exact method[7)] is the most famous in this country. On this account, the illustrative examples treated below are taken from his paper mainly to show the accuracies of the author's solutions. Fig. 35[*)] shows the grid work to be considered. The condition of symmetry gives the following relations, that is, the grid is dealt with four unknowns.

$$\varphi : \begin{cases} \varphi_{x\mathrm{A}} = \varphi_{x\mathrm{B}} = \varphi_{z\mathrm{A}} = \varphi_{z\mathrm{C}} \equiv \varphi_1, \\ \varphi_{x\mathrm{C}} = \varphi_{x\mathrm{D}} = \varphi_{z\mathrm{B}} = \varphi_{z\mathrm{D}} = -\varphi_1, \\ \varphi_{x\mathrm{E}} = \varphi_{x\mathrm{F}} \equiv \varphi_2, \\ \varphi_{x\mathrm{I}} = \varphi_{z\mathrm{H}} = -\varphi_2. \end{cases}$$

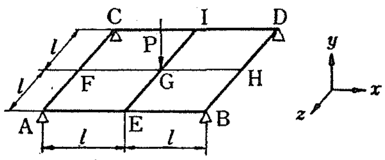


Fig. 35

$$\psi : \begin{cases} \psi_{x\mathrm{AF}} = \psi_{x\mathrm{BH}} = \psi_{z\mathrm{AE}} = \psi_{z\mathrm{CI}} \equiv \psi_1, \\ \psi_{x\mathrm{FC}} = \psi_{x\mathrm{HD}} = \psi_{z\mathrm{EB}} = \psi_{z\mathrm{ID}} = -\psi_1, \\ \psi_{z\mathrm{EG}} = \psi_{z\mathrm{FG}} \equiv \psi_2, \\ \psi_{x\mathrm{GI}} = \psi_{z\mathrm{GH}} = -\psi_2. \end{cases}$$

1) Expressions of End-Moments

Due to the condition of symmetry, only the following end-moment expressions are required, where k_b and k_t mean the stiffness ratios for bending and for torsion respectively.

$$M_{z\mathrm{AE}} = k_b(2\varphi_1 + \psi_1), \qquad M_{z\mathrm{AF}} = 2\beta k_t(\varphi_1 - \varphi_2),$$

$$M_{x\mathrm{EG}} = k_b(2\varphi_2 + \psi_2), \qquad M_{z\mathrm{EA}} = k_b(\varphi_1 + \psi_1), \qquad M_{x\mathrm{GE}} = k_b(\varphi_2 + \psi_2).$$

The remainders are expressed by these:

$$M_{x\mathrm{AE}} = M_{z\mathrm{AF}}, \qquad M_{x\mathrm{AF}} = M_{z\mathrm{AE}}, \qquad M_{x\mathrm{BE}} = M_{z\mathrm{AF}}, \text{ etc.}$$

---

*) Fukuda: II Abschnitt, §2, 2)

2) Elastic Equations

i) Joint Equilibrium Equations

$$\text{At joint A,}\quad \sum M_{xA} = M_{xAE} + M_{xAF} = 0.$$

$$〃\quad \text{E,}\quad \sum M_{xE} = M_{xEA} + M_{xEB} + M_{xEG} = 0.$$

ii) Vertical Shear Equations

$$\text{At joint E,}\quad \sum X_{yE} = X_{yEA} - X_{yEB} - X_{yEG} = 0.$$

$$〃\quad \text{G,}\quad \sum X_{yG} = X_{yGE} + X_{yGF} - X_{yGH} - X_{yGI} = P,$$

where,

$$X_{yEA} = -(1/l)(M_{xEA} + M_{xAE}),\qquad X_{yEB} = -(1/l)(M_{xEB} + M_{xBE}),$$

$$X_{yEG} = -(1/l)(M_{xEG} + M_{xGE}),\qquad X_{yGE} = -(1/l)(M_{xGE} + M_{xEG}),$$

$$X_{yGF} = -(1/l)(M_{xGF} + M_{xFG}),\qquad X_{yGH} = -(1/l)(M_{xGH} + M_{xHG}),$$

$$X_{yGI} = -(1/l)(M_{xGI} + M_{xIG}).$$

Substituting the end-moments in 1), we have the simultaneous equations:

$$(2k_b + 2\beta k_t)\varphi_1 - 2\beta k_t\varphi_2 + k_b\psi_1 = 0,$$

$$4\beta k_t\varphi_1 - (2k_b + 4\beta k_t)\varphi_2 - k_b\psi_2 = 0,$$

$$6\varphi_1 - 3\varphi_2 + 4\psi_1 - 2\psi_2 = 0,$$

$$12\varphi_2 + 8\psi_2 = -(l/k_b)P.$$

Solution gives :

$$\varphi_1 = \frac{Pl}{8k_b}\cdot\frac{k_b + 16\beta k_t}{k_b + 12\beta k_t},\qquad \varphi_2 = \frac{Pl}{4k_b}\cdot\frac{k_b + 8\beta k_t}{k_b + 12\beta k_t},$$

$$\psi_1 = -\frac{Pl}{4k_b}\cdot\frac{k_b + 15\beta k_t}{k_b + 12\beta k_t},\qquad \psi_2 = -\frac{Pl}{2k_b}\cdot\frac{k_b + 9\beta k_t}{k_b + 12\beta k_t}.$$

3) Values of End-Moments

Using these, we have from 1) :

$$M_{xAE} = \frac{Pl}{4}\cdot\frac{1}{k_b + 12\beta k_t}(k_b + 16\beta k_t - k_b - 15\beta k_t) = \frac{Pl}{4}\cdot\frac{\beta k_t}{k_b + 12\beta k_t},$$

$$M_{xEA} = \frac{Pl}{8}\cdot\frac{1}{k_b + 12\beta k_t}(k_b + 16\beta k_t - 2k_b - 30\beta k_t) = -\frac{Pl}{8}\cdot\frac{k_b + 14\beta k_t}{k_b + 12\beta k_t},$$

$$M_{xGE} = \frac{-Pl}{4}\cdot\frac{1}{k_b + 12\beta k_t}(-k_b - 8\beta k_t + 2k_b + 18\beta k_t) = -\frac{Pl}{4}\cdot\frac{k_b + 10\beta k_t}{k_b + 12\beta k_t},$$

$$M_{x\mathrm{EG}} = \frac{-Pl}{2} \cdot \frac{1}{k_b + 12\beta k_t}(-k_b - 8\beta k_t + k_b + 9\beta k_t) = -\frac{Pl}{2} \cdot \frac{\beta k_t}{k_b + 12\beta k_t},$$

$$M_{z\mathrm{AF}} = \frac{Pl}{4} \cdot \frac{\beta k_t}{k_b(k_b + 12\beta k_t)} \cdot (k_b + 16\beta k_t - 2k_b - 16\beta k_t) = -\frac{Pl}{4} \cdot \frac{\beta k_t}{k_b + 12\beta k_t}.$$

Prof. FUKUDA's values are as follows, where $\alpha = EI/GH$ and H represents the coefficient of twisting resistance; note that in his analysis eight unknowns being employed.

$$M_{z\mathrm{AE}} = \frac{Pl}{16(\alpha+3)}, \qquad M_{z\mathrm{EA}} = -\frac{Pl(2\alpha+7)}{16(\alpha+3)},$$

$$M_{x\mathrm{GE}} = -\frac{Pl(2\alpha+5)}{8(\alpha+3)}, \qquad M_{x\mathrm{EG}} = -\frac{Pl}{8(\alpha+3)},$$

$$M_{z\mathrm{AF}} = -\frac{Pl}{16(\alpha+3)}.$$

The relation between α and the author's factor β is:

$$\alpha = \frac{EI}{GH} = \frac{2(m+1)}{m} \cdot \frac{I}{H} = \frac{2(m+1)}{m} \cdot \frac{\frac{I}{l}}{\frac{H}{l}} = \frac{2(m+1)}{m} \cdot \frac{k_b}{k_t}$$

$$= \frac{1}{4} \cdot \frac{8(m+1)}{m} \cdot \frac{k_b}{k_t} = \frac{1}{4}\frac{k_b}{\beta k_t}.$$

$$\therefore \quad \alpha = \frac{k_b}{4\beta k_t}.$$

Referring to this relation, it is seen that Prof. FUKUDA's solutions just come to the authors' except signs. It is to be noticed that the author's method needs only four independent unknowns, half of the FUKUDA's method, and that the slope-deflection method is a powerful means also in dealing with the grid works. The

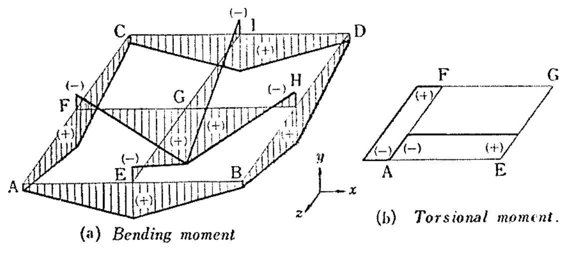


Fig. 36 Moment diagram

moment diagrams are shown in Fig. 36.

21. A Square Grid Supported Simply at the Corners, Concentrated Load on an Edge Joint. Fig. 37.[*)]

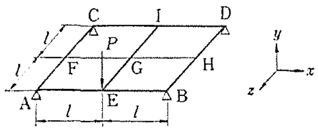


Fig. 37

From symmetry, we have the following nine φ's and four ψ's as unknowns:

$$\varphi : \left\{ \begin{array}{lll} \text{At joint} & \text{A}, & \varphi_{xA},\ \varphi_{zA} \\ \quad〃 & \text{E}, & \varphi_{xE} \\ \quad〃 & \text{F}, & \varphi_{xF},\ \varphi_{zF} \\ \quad〃 & \text{G}, & \varphi_{xG} \\ \quad〃 & \text{C}, & \varphi_{xC},\ \varphi_{zC} \\ \quad〃 & \text{I}, & \varphi_{xI} \end{array} \right. \qquad \psi : \left\{ \begin{array}{lll} \text{For member} & \text{AE}, & \psi_{xAE} \\ \quad〃 & \text{FG}, & \psi_{zFG} \\ \quad〃 & \text{CI}, & \psi_{zCI} \\ \quad〃 & \text{AF}, & \psi_{xAF} \end{array} \right.$$

The remainders are expressed by these :

$$\varphi_{xH} = \varphi_{xF}, \qquad \varphi_{zD} = -\varphi_{zC}, \quad \text{etc.},$$

and

$$\psi_{xEG} = \psi_{xAF} + \psi_{zFG} - \psi_{zAE}, \quad \psi_{xGI} = -\psi_{zFG} - \psi_{xAF} + \psi_{zCI}, \quad \text{etc.}$$

1) Expressions of End-Moments

$$\text{At joint A:} \left\{ \begin{array}{ll} M_{xAF} = k_b(2\varphi_{xA} + \varphi_{xF} + \psi_{xAF}), & M_{xAB} = 2\beta k_t(\varphi_{xA} - \varphi_{xE}), \\ M_{zAF} = 2\beta k_t(\varphi_{zA} - \varphi_{zF}), & M_{zAE} = k_b(2\varphi_{zA} + \psi_{zAE}). \end{array} \right.$$

$$〃 \quad \text{B:} \left\{ \begin{array}{ll} M_{xBH} = M_{xAF}, & M_{xBE} = M_{xAE}, \\ M_{zBH} = -M_{zAF}, & M_{zBE} = -M_{zAE}. \end{array} \right.$$

$$〃 \quad \text{C:} \left\{ \begin{array}{ll} M_{xCF} = k_b(2\varphi_{xC} + \varphi_{xF} - \psi_{xAF}), & M_{xCI} = 2\beta k_t(\varphi_{xC} - \varphi_{xI}), \\ M_{zCF} = -M_{zFC}, & M_{zCI} = k_b(2\varphi_{zC} + \psi_{zCI}). \end{array} \right.$$

$$〃 \quad \text{D:} \left\{ \begin{array}{ll} M_{xDH} = M_{xCF}, & M_{xDI} = M_{xCI}, \\ M_{zDH} = -M_{zCF}, & M_{zDI} = -M_{zCI}. \end{array} \right.$$

[*)] FUKUDA : II Abschnitt, §2, 4)

At joint E:
$$\begin{cases} M_{x\mathrm{EA}} = -M_{x\mathrm{AE}}, \quad M_{x\mathrm{EB}} = -M_{x\mathrm{AE}}, \\ M_{x\mathrm{EG}} = k_b(2\varphi_{x\mathrm{E}} + \varphi_{x\mathrm{G}} + \psi_{x\mathrm{AF}} + \psi_{z\mathrm{FG}} - \psi_{z\mathrm{AE}}), \\ M_{z\mathrm{EA}} = k_b(\varphi_{z\mathrm{A}} + \psi_{z\mathrm{AE}}), \quad M_{z\mathrm{EB}} = -M_{z\mathrm{EA}}, \quad M_{z\mathrm{EG}} = 0. \end{cases}$$

〃 F:
$$\begin{cases} M_{x\mathrm{FA}} = k_b(2\varphi_{x\mathrm{F}} + \varphi_{x\mathrm{A}} + \psi_{x\mathrm{AF}}), \quad M_{x\mathrm{FC}} = k_b(2\varphi_{x\mathrm{F}} + \varphi_{x\mathrm{C}} - \psi_{x\mathrm{AF}}), \\ M_{x\mathrm{FG}} = 2\beta k_t(\varphi_{x\mathrm{F}} - \varphi_{x\mathrm{G}}), \\ M_{z\mathrm{FA}} = -M_{z\mathrm{AF}}, \quad M_{z\mathrm{FC}} = 2\beta k_t(\varphi_{z\mathrm{F}} - \varphi_{z\mathrm{C}}), \quad M_{z\mathrm{FG}} = k_b(2\varphi_{z\mathrm{F}} + \psi_{z\mathrm{FG}}). \end{cases}$$

〃 G:
$$\begin{cases} M_{x\mathrm{GE}} = k_b(2\varphi_{x\mathrm{G}} + \varphi_{x\mathrm{E}} + \psi_{x\mathrm{AF}} + \psi_{z\mathrm{FG}} - \psi_{z\mathrm{AF}}), \\ M_{x\mathrm{GF}} = -M_{x\mathrm{FG}}, \quad M_{x\mathrm{GH}} = -M_{x\mathrm{FG}}, \\ M_{x\mathrm{GI}} = k_b(2\varphi_{x\mathrm{G}} + \varphi_{x\mathrm{I}} - \psi_{z\mathrm{FG}} - \psi_{x\mathrm{AF}} + \psi_{z\mathrm{CI}}), \\ M_{z\mathrm{GE}} = 0, \quad M_{z\mathrm{GF}} = k_b(\varphi_{z\mathrm{F}} + \psi_{z\mathrm{FG}}), \\ M_{z\mathrm{GH}} = -M_{z\mathrm{GF}}, \quad M_{z\mathrm{GI}} = 0. \end{cases}$$

〃 H:
$$\begin{cases} M_{x\mathrm{HB}} = M_{x\mathrm{FA}}, \quad M_{x\mathrm{HD}} = M_{x\mathrm{FC}}, \quad M_{x\mathrm{HG}} = M_{x\mathrm{FG}}, \\ M_{z\mathrm{HB}} = -M_{z\mathrm{FA}}, \quad M_{z\mathrm{HD}} = -M_{z\mathrm{FC}}, \quad M_{z\mathrm{HG}} = -M_{z\mathrm{FG}}. \end{cases}$$

〃 I:
$$\begin{cases} M_{x\mathrm{IC}} = -M_{x\mathrm{CI}}, \quad M_{x\mathrm{ID}} = -M_{x\mathrm{CI}}, \\ M_{x\mathrm{IG}} = k_b(2\varphi_{x\mathrm{I}} + \varphi_{x\mathrm{G}} - \psi_{z\mathrm{FG}} - \psi_{x\mathrm{AF}} + \psi_{z\mathrm{CI}}), \\ M_{z\mathrm{IC}} = k_b(\varphi_{z\mathrm{C}} + \psi_{z\mathrm{CI}}), \quad M_{z\mathrm{ID}} = -M_{z\mathrm{IC}}, \; M_{z\mathrm{IG}} = 0. \end{cases}$$

2) Elastic Equations

i) Joint Equilibrium Equations

About x axis:
$$\begin{cases} \sum M_{x\mathrm{A}} = M_{x\mathrm{AF}} + M_{x\mathrm{AE}} = 0, \\ \sum M_{x\mathrm{C}} = M_{x\mathrm{CF}} + M_{x\mathrm{CI}} = 0, \\ \sum M_{x\mathrm{E}} = M_{x\mathrm{EA}} + M_{x\mathrm{EB}} + M_{x\mathrm{EG}} = 0, \\ \sum M_{x\mathrm{F}} = M_{x\mathrm{FA}} + M_{x\mathrm{FC}} + M_{x\mathrm{FG}} = 0, \\ \sum M_{x\mathrm{G}} = M_{x\mathrm{GF}} + M_{x\mathrm{GH}} + M_{x\mathrm{GE}} + M_{x\mathrm{GI}} = 0, \\ \sum M_{x\mathrm{I}} = M_{x\mathrm{IC}} + M_{x\mathrm{ID}} + M_{x\mathrm{IG}} = 0. \end{cases}$$

About z axis:
$$\begin{cases} \sum M_{z\mathrm{A}} = M_{z\mathrm{AE}} + M_{z\mathrm{AF}} = 0, \\ \sum M_{z\mathrm{C}} = M_{z\mathrm{CF}} + M_{z\mathrm{CI}} = 0, \\ \sum M_{z\mathrm{F}} = M_{z\mathrm{FA}} + M_{z\mathrm{FC}} + M_{z\mathrm{FG}} = 0. \end{cases}$$

ii) Vertical Shear Equations

$$
\begin{aligned}
&\text{At joint E}: && X_{y\mathrm{EA}} - X_{y\mathrm{EB}} - X_{y\mathrm{EG}} - P = 0.\\
&\quad〃\quad \text{F}: && X_{y\mathrm{FA}} - X_{y\mathrm{FC}} - X_{y\mathrm{FG}} = 0.\\
&\quad〃\quad \text{G}: && X_{y\mathrm{GF}} - X_{y\mathrm{GH}} + X_{y\mathrm{GE}} - X_{y\mathrm{GI}} = 0.\\
&\quad〃\quad \text{I}: && X_{y\mathrm{IC}} - X_{y\mathrm{ID}} + X_{y\mathrm{IG}} = 0.
\end{aligned}
$$

where,

$$
\begin{aligned}
X_{y\mathrm{EA}} &= -(1/l)(M_{z\mathrm{AE}} + M_{z\mathrm{EA}}), & X_{y\mathrm{EB}} &= -(1/l)(M_{z\mathrm{EB}} + M_{z\mathrm{BE}}),\\
X_{y\mathrm{EG}} &= -(1/l)(M_{x\mathrm{EG}} + M_{x\mathrm{GE}}), & X_{y\mathrm{FA}} &= -(1/l)(M_{x\mathrm{AF}} + M_{x\mathrm{FA}}),\\
X_{y\mathrm{FC}} &= -(1/l)(M_{x\mathrm{FC}} + M_{x\mathrm{CF}}), & X_{y\mathrm{FG}} &= -(1/l)(M_{z\mathrm{FG}} + M_{z\mathrm{GF}}),\\
X_{y\mathrm{GF}} &= -(1/l)(M_{z\mathrm{FG}} + M_{z\mathrm{GF}}), & X_{y\mathrm{GH}} &= -(1/l)(M_{z\mathrm{GH}} + M_{z\mathrm{HG}}),\\
X_{y\mathrm{GE}} &= -(1/l)(M_{x\mathrm{EG}} + M_{x\mathrm{GE}}), & X_{y\mathrm{GI}} &= -(1/l)(M_{x\mathrm{GI}} + M_{x\mathrm{IG}}),\\
X_{y\mathrm{IC}} &= -(1/l)(M_{z\mathrm{CI}} + M_{z\mathrm{IC}}), & X_{y\mathrm{ID}} &= -(1/l)(M_{z\mathrm{ID}} + M_{z\mathrm{DI}}),\\
X_{y\mathrm{IG}} &= -(1/l)(M_{x\mathrm{GI}} + M_{x\mathrm{IG}}).
\end{aligned}
$$

Using expressions in 1), we have the simultaneous equations shown in Table 24, which produces the following solutions :

Table 24 Elastic equations

Eq.	Left-hand side													Right-hand side
	$\varphi_{x\mathrm{A}}$	$\varphi_{x\mathrm{C}}$	$\varphi_{x\mathrm{E}}$	$\varphi_{x\mathrm{F}}$	$\varphi_{x\mathrm{G}}$	$\varphi_{x\mathrm{I}}$	$\varphi_{z\mathrm{A}}$	$\varphi_{z\mathrm{C}}$	$\varphi_{z\mathrm{F}}$	$\phi_{x\mathrm{AF}}$	$\phi_{z\mathrm{AE}}$	$\phi_{z\mathrm{CI}}$	$\phi_{z\mathrm{FG}}$	
1	$2a+1$		-1	a						a				0
2		$2a+1$		a		-1				$-a$				0
3	-2		$2a+2$		a					a	$-a$		a	0
4	a	a		$4a+1$	-1									0
5			a	-2	$4a+2$	a					$-a$	a		0
6		-2			a	$2a+2$				$-a$		a	$-a$	0
7							$2a+1$		-1		a			0
8								$2a+1$	-1			a		0
9							-1	-1	$2a+2$				a	0
10			3		3		-6			2	-6		2	Pl/k_b
11	3	3							3	-4			2	0
12			3			3			6	-4	2	2	-8	0
13					3	3		6		-2		6	-2	0

$$\varphi_{xA} = n(-6 - 15a - 5a^2 + 4a^3 + a^4),$$
$$\varphi_{xC} = -n(6 + 23a + 25a^2 + 8a^3 + a^4),$$
$$\varphi_{xE} = -n(6 + 45a + 86a^2 + 46a^3 + 6a^4),$$
$$\varphi_{xF} = -n(6 + 13a + 2a^2),$$
$$\varphi_{xG} = -n(6 + 49 + 92a^2 + 38a^3 + 4a^4),$$
$$\varphi_{xI} = -n(6 + 29a + 46a^2 + 22a^3 + 2a^4),$$
$$\varphi_{zA} = n(96 + 314a + 311a^2 + 108a^3 + 11a^4),$$
$$\varphi_{zC} = n(96 + 206a + 77a^2 - a^4),$$
$$\varphi_{zF} = n(96 + 248a + 164a^2 + 36a^3 + 2a^4),$$
$$\psi_{xAF} = -n(12 + 38a + 38a^2 + 15a^3 + 2a^4),$$
$$\psi_{zAE} = -n(258 + 775a + 694a^2 + 225a^3 + 22a^4),$$
$$\psi_{zCI} = -n(150 + 325a + 118a^2 - 3a^3 - 2a^4),$$
$$\psi_{zFG} = -n(168 + 436a + 292a^2 + 66a^3 + 4a^4),$$

where,

$$\mu = Pl/24(2 + a)(6 + a)(2 + 4a + a^2),$$
$$n = \mu/k_b.$$

3) Moments and Reactions

From eqs. (a) and (f), end-moments are determined.

$$\text{At joint A:} \left\{ \begin{aligned} M_{xAF} &= -\mu(30 + 81a + 50a^2 + 7a^3), \\ M_{xAE} &= \mu(30 + 81a + 50a^2 + 7a^3), \\ M_{zAF} &= 3\mu(22 + 49a + 24a^2 + 3a^3), \\ M_{zAE} &= -3\mu(22 + 49a + 24a^2 + 3a^3). \end{aligned} \right.$$

$$〃 \quad \text{C:} \left\{ \begin{aligned} M_{xCF} &= -\mu(6 + 21a + 14a^2 + a^3), \\ M_{xCI} &= \mu(6 + 21a + 14a^2 + a^3), \\ M_{zCI} &= 3\mu(14 + 29a + 12a^2 + a^3). \end{aligned} \right.$$

$$〃 \quad \text{E:} \left\{ \begin{aligned} M_{xEG} &= 2\mu(30 + 81a + 50a^2 + 7a^3), \\ M_{zEA} &= -\mu(162 + 461a + 383a^2 + 117a^3 + 11a^4). \end{aligned} \right.$$

$$\text{At joint F:}\quad \begin{cases} M_{x\mathrm{FA}} = -\mu(30 + 79a + 47a^2 + 11a^3 + a^4), \\ M_{x\mathrm{FC}} = \mu(-6 - 11a + 9a^2 + 7a^3 + a^4), \\ M_{x\mathrm{FG}} = 2\mu(18 + 45a + 19a^2 + 2a^3), \\ M_{z\mathrm{FC}} = 3\mu(14 + 29a + 12a^2 + a^3), \\ M_{z\mathrm{FG}} = 6\mu(4 + 10a + 6a^2 + a^3). \end{cases}$$

$$\text{〃}\quad \text{G:}\quad \begin{cases} M_{x\mathrm{GE}} = 2\mu(30 + 79a + 47a^2 + 11a^3 + a^4), \\ M_{x\mathrm{GI}} = -2\mu(-6 - 11a + 9a^2 + 7a^3 + a^4), \\ M_{z\mathrm{GF}} = -2\mu(36 + 94a + 64a^2 + 15a^3 + a^4). \end{cases}$$

$$\text{〃}\quad \text{I:}\quad \begin{cases} M_{x\mathrm{IG}} = 2\mu(6 + 21a + 14a^2 + a^3), \\ M_{z\mathrm{IC}} = -\mu(54 + 119a + 41a^2 - 3a^3 - a^4). \end{cases}$$

Putting $a = 2\alpha$, we see that these are in agreement with Prof. FUKUDA's solutions which are obtained from twenty-five equations. Consider now the case where all members are of steel ($m = 4$) with square sections, i.e., $a = 2.992,6$ and $\mu = 0.000,042,670\ Pl$. Then we have:

$$\text{At joint A:}\quad \begin{cases} M_{x\mathrm{AF}} = -0.037,06Pl & (\mathrm{B}) \\ M_{x\mathrm{AE}} = +0.037,06Pl & (\mathrm{T}) \\ M_{z\mathrm{AF}} = +0.056,98Pl & (\mathrm{T}) \\ M_{z\mathrm{AE}} = -0.056,98Pl & (\mathrm{B}) \end{cases}$$

$$\text{〃}\quad \text{C:}\quad \begin{cases} M_{x\mathrm{CF}} = -0.009,04Pl & (\mathrm{B}) \\ M_{x\mathrm{CI}} = +0.009,04Pl & (\mathrm{T}) \\ M_{z\mathrm{CI}} = +0.028,96Pl & (\mathrm{B}) \end{cases}$$

$$\text{〃}\quad \text{E:}\quad \begin{cases} M_{x\mathrm{EG}} = +0.074,12Pl & (\mathrm{B}) \\ M_{z\mathrm{EA}} = -0.362,87Pl & (\mathrm{B}) \end{cases}$$

$$\text{〃}\quad \text{F:}\quad \begin{cases} M_{x\mathrm{FA}} = -0.043,09Pl & (\mathrm{B}) \\ M_{x\mathrm{FC}} = +0.012,22Pl & (\mathrm{B}) \\ M_{x\mathrm{FG}} = +0.030,87Pl & (\mathrm{T}) \\ M_{z\mathrm{FC}} = +0.028,96Pl & (\mathrm{T}) \\ M_{z\mathrm{FG}} = +0.028,02Pl & (\mathrm{B}) \end{cases}$$

$$
\text{〃} \quad G: \begin{cases} M_{xGE} = +0.086,18Pl & (B) \\ M_{xGI} = -0.024,44Pl & (B) \\ M_{zGF} = -0.111,35Pl & (B) \end{cases}
$$

$$
\text{〃} \quad I: \begin{cases} M_{xIG} = +0.018,09Pl & (B) \\ M_{zIC} = -0.025,78Pl & (B) \end{cases}
$$

Note: (B): Bending moment, (T): Torsional moment.

These coincide with the values shown by Prof. FUKUDA except signs. Fig. 38 shows the moment diagram.

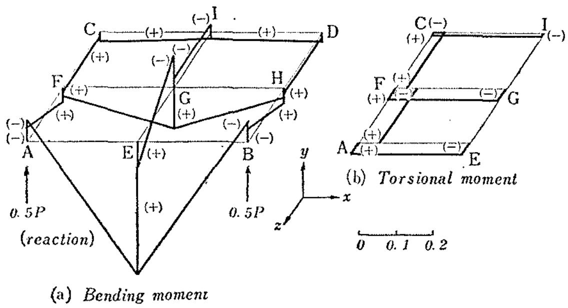


Fig. 38 Moment diagram (multiplier : Pl)

Vertical reactions are calculated as follows :

$$
A = -(1/l)(M_{xAF} + M_{xFA}) - (1/l)(M_{zAE} + M_{zEA})
$$
$$
= -(1/l)(-0.037,06 - 0.043,09)Pl - (1/l)(-0.056,98 - 0.362,87)Pl = 0.5P,
$$

$$
C = (1/l)(M_{xFC} + M_{xCF}) - (1/l)(M_{zCI} + M_{zIC})
$$
$$
= (1/l)(0.012,22 - 0.009,04)Pl - (1/l)(0.028,96 - 0.025,78)Pl = 0.
$$

4) Deflections

Obtaining end-moments and ψ's, we readily determine the deformation of the structure, which are shown in Fig. 39. It will be worth while to mention that this diagram gives the influence diagram for the deflection at the joint considered if we put $P = 1$.

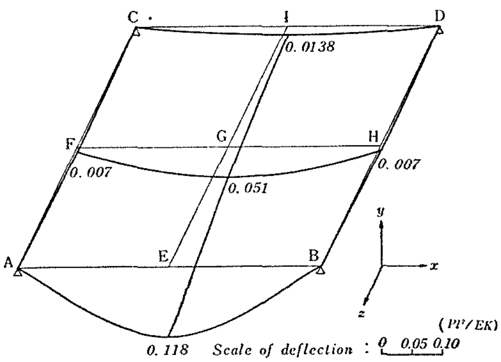


Fig. 39 Deflection diagram

22. *A Square Grid of 3×3 Panels Supported Simply at Corners, Concentrated Loads on the Inner Joints. Fig. 40.* [*)]

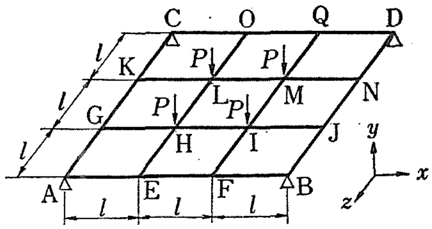


Fig. 40

By symmetry, the following six unknowns are enough.

$$\varphi : \begin{cases} \varphi_{zA} = \varphi_{xA} \equiv \varphi_1, \\ \varphi_{zE} = \varphi_{xG} \equiv \varphi_2, \\ \varphi_{xE} = \varphi_{zG} \equiv \varphi_3, \\ \varphi_{zH} = \varphi_{xH} \equiv \varphi_4. \end{cases} \qquad \phi : \begin{cases} \phi_{zAE} = \phi_{xAG} \equiv \phi_1, \\ \phi_{xEH} = \phi_{zGH} \equiv \phi_2, \\ (\phi_{zEF} = \phi_{xGK} = \phi_{zHI} = \phi_{zHL} = 0). \end{cases}$$

1) Expressions of End-Moments

$$\left.\begin{array}{lll} \text{At joint A}: & & M_{zAE} = k_b(2\varphi_1 + \varphi_2 + \phi_1), \quad M_{zAG} = 2\beta k_t(\varphi_1 - \varphi_3). \\ \text{〃} & \text{E}: & \begin{cases} M_{zEA} = k_b(\varphi_1 + 2\varphi_2 + \phi_1), \quad M_{xEA} = 2\beta k_t(\varphi_3 - \varphi_1), \\ M_{zEF} = k_b\varphi_2, \qquad M_{zEH} = 2\beta k_t(\varphi_2 - \varphi_4), \\ M_{xEH} = k_b(2\varphi_3 + \varphi_4 + \phi_2), \quad M_{xEF} = 0. \end{cases} \\ \text{〃} & \text{H}: & \begin{cases} M_{zHG} = k_b(\varphi_3 + 2\varphi_4 + \phi_2), \quad M_{zHE} = 2\beta k_t(\varphi_4 - \varphi_2), \\ M_{zHI} = k_b\varphi_4, \quad M_{zHL} = 0, \quad M_{xHE} = k_b(\varphi_3 + 2\varphi_4 + \phi_2). \end{cases} \\ \text{〃} & \text{F}: & M_{zFE} = -k_b\varphi_2. \\ \text{〃} & \text{I}: & M_{zIH} = -k_b\varphi_4. \end{array}\right\} \quad \text{(a)}$$

2) Elastic Equations

i) Joint Equilibrium Equations

$$\left.\begin{array}{lll} \text{At joint A}, & & \sum M_{zA} = M_{zAE} + M_{zAG} = 0. \\ \text{〃} & \text{E}, & \sum M_{zE} = M_{zEA} + M_{zEF} + M_{zEH} = 0, \\ & & \sum M_{xE} = M_{xEA} + M_{xEF} + M_{xEH} = 0. \\ \text{〃} & \text{H}, & \sum M_{zH} = M_{zHG} + M_{zHE} + M_{zHI} + M_{zHL} = 0. \end{array}\right\} \quad \text{(b)}$$

*) Fukuda: II Abschnitt, §3, 2)

ii) Vertical Shear Equations

$$\left.\begin{aligned} &\text{At joint E,} \quad \sum X_{yE} = X_{yEA} - X_{yEH} - X_{yEF} = 0, \\ &\quad〃 \quad \text{H,} \quad \sum X_{yH} = X_{yHG} + X_{yHE} - X_{yHI} - X_{yHL} = P, \end{aligned}\right\} \quad \text{(c)}$$

where,

$$\left.\begin{aligned} &X_{yEA} = -(1/l)(M_{zAE} + M_{zEA}), && X_{yEH} = -(1/l)(M_{xEH} + M_{xHE}), \\ &X_{yEF} = -(1/l)(M_{zEF} + M_{zFE}), && X_{yHG} = -(1/l)(M_{zGH} + M_{zHG}), \\ &X_{yHE} = -(1/l)(M_{xHE} + M_{xEH}), && X_{yHI} = -(1/l)(M_{zHI} + M_{zIH}), \\ &X_{yHL} = -(1/l)(M_{xHL} + M_{xLH}). \end{aligned}\right\} \quad \text{(d)}$$

Here, we have the equations shown in Table 25, which produces the following results :

Table 25 Elastic equations

Eq.	Left-hand side						Right-hand side
	φ_1	φ_2	φ_3	φ_4	ψ_1	ψ_2	
(1)	$2(k_b+\beta k_t)$	k_b	$-2\beta k_t$		k_b		0
(2)	k_b	$3k_b+2\beta k_t$		$-2\beta k_t$	k_b		0
(3)	$-2\beta k_t$		$2(k_b+\beta k_t)$	k_b		k_b	0
(4)		$-2\beta k_t$	k_b	$3k_b+2\beta k_t$		k_b	0
(5)	3	3	-3	-3	2	-2	0
(6)			3	3		2	$-(Pl/2k_b)$

$$\left.\begin{aligned} &\varphi_1 = \frac{Pl}{k_b}, && \varphi_2 = \frac{Pl}{2k_b}, && \varphi_3 = \frac{Pl}{k_b}, \\ &\varphi_4 = \frac{Pl}{2k_b}, && \psi_1 = \frac{-5Pl}{2k_b}, && \psi_2 = \frac{-5Pl}{2k_b}. \end{aligned}\right\} \quad \text{(e)}$$

3) End-Moments

Eqs. (a) and (e) determine the end-moments;

$$M_{zAE} = k_b\left(\frac{2Pl}{k_b} + \frac{Pl}{2k_b} - \frac{5Pl}{2k_b}\right) = 0,$$

$$M_{zEA} = k_b\left(\frac{Pl}{k_b} + \frac{2Pl}{2k_b} - \frac{5}{2}\frac{Pl}{k_b}\right) = -\frac{1}{2}Pl,$$

$$\left.\begin{aligned}
M_{z\mathrm{EF}} &= k_b\left(\frac{Pl}{2k_b}\right) = \frac{1}{2}Pl, \\
M_{x\mathrm{EH}} &= k_b\left(\frac{2Pl}{k_b} + \frac{Pl}{2k_b} - \frac{5}{2}\frac{Pl}{k_b}\right) = 0, \\
M_{z\mathrm{HI}} &= k_b\frac{Pl}{2k_b} = \frac{1}{2}Pl, \\
M_{z\mathrm{AG}} &= 2\beta k_t\left(\frac{Pl}{k_b} - \frac{Pl}{k_b}\right) = 0, \\
M_{z\mathrm{EH}} &= 2\beta k_t\left(\frac{Pl}{2k_b} - \frac{Pl}{2k_b}\right) = 0.
\end{aligned}\right\} \qquad \text{(f)}$$

These coincide again with Prof. FUKUDA's solutions which he found from thirteen equations.

In this case, as Prof. FUKUDA points out, all the torsional moments vanish and hence the girders act as if they were simply supported at their extreme ends; refer to Fig. 41.

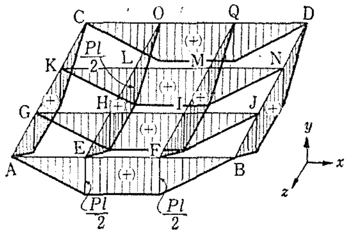


Fig. 41 Moment diagram

23. *A Square Grid of 3×3 Panels Fixed at the Periphery, Concentrated Loads on the Inner Joints. Fig. 42.*[*)]

This problem is rather simple to analyse than the preceding, because the independent unknowns are only two, i.e.,

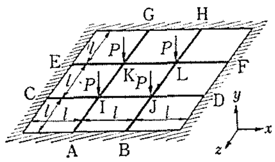


Fig. 42

$$\varphi_{z\mathrm{I}} = \varphi_{z\mathrm{J}} = \varphi_{z\mathrm{K}} = \varphi_{z\mathrm{L}} = \varphi_{x\mathrm{I}} = \varphi_{x\mathrm{J}} = \varphi_{x\mathrm{K}} = \varphi_{x\mathrm{L}} \equiv \varphi,$$

$$\phi_{x\mathrm{AI}} = \phi_{z\mathrm{CI}} \equiv \phi.$$

Thus we have the end-moment expressions:

$$M_{z\mathrm{CI}} = k_b(\varphi + \phi), \quad M_{z\mathrm{IC}} = k_b(2\varphi + \phi),$$

$$M_{z\mathrm{IA}} = 2\beta k_t\varphi, \quad M_{z\mathrm{IJ}} = k_b\varphi, \quad M_{z\mathrm{IK}} = 0.$$

[*)] FUKUDA: II Abschnitt, §4, 2)

These must satisfy the equilibrium conditions below:

$$\sum M_{z\text{I}} = M_{z\text{IC}} + M_{z\text{IA}} + M_{z\text{IJ}} + M_{z\text{IK}} = 0,$$

$$\sum X_{y\text{I}} = X_{y\text{IC}} + X_{y\text{IA}} - X_{y\text{IJ}} - X_{y\text{IK}} = P,$$

which give the following simultaneous equations:

$$(3k_b + 2\beta k_t)\varphi + k_b\phi = 0,$$

$$3k_b\varphi + 2k_t\phi + \frac{Pl}{2} = 0.$$

Solution gives

$$\varphi = \frac{Pl}{2k_b}\cdot\frac{\alpha}{3\alpha+1}, \qquad \phi = \frac{Pl}{4k_b}\cdot\frac{6\alpha+1}{3\alpha+1},$$

where

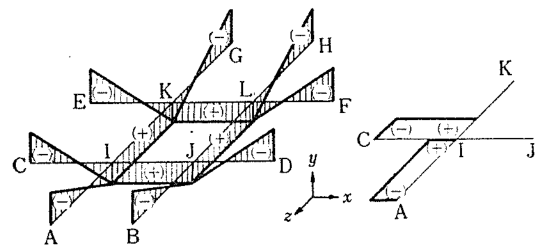
$$\alpha = k_b/4\beta k_t.$$

Introducing these in eqs. (a), the end-moments are determined:

$$M_{z\text{CI}} = -\frac{Pl}{4}\cdot\frac{4\alpha+1}{3\alpha+1}, \qquad M_{z\text{IC}} = -\frac{Pl}{4}\cdot\frac{2\alpha+1}{3\alpha+1},$$

$$M_{z\text{IJ}} = \frac{Pl}{2}\cdot\frac{\alpha}{3\alpha+1}, \qquad M_{z\text{IA}} = \frac{Pl}{4}\cdot\frac{1}{3\alpha+1}.$$

These results agree with those of Prof. FUKUDA which are obtained by solving seven equations simultaneously. Fig. 43 shows the moment diagram.



(a) *Bending moment* (b) *Torsional moment*

Fig. 43 Moment diagram

Summary

Some of the features of the foregoing investigations may be summarized as follows:

(1) To analyse the rigid frames in space, the so-called classical methods which take the stress functions for unknowns are considered not to be useful in practice, because they require numerous condition equations. In addition, we are very often confused in drawing moment diagrams, because those methods require to pay constant attensions in reading the sign conventions adopted. To this, the author's method, depending upon the slope-deflection theory, requires far smaller number of condition equations, about half of those of the classical methods, because it takes end deformations for unknowns, and the moment diagrams are mechanically drawn without confusion.

(2) For the frames without sways, the current two-dimensional analysis may be applied with good accuracy except the cases in which there exist large unbalances among the stiffnesses of members; for such exceptional cases we must treat the frame three-dimensionally.

(3) For the frames with sways, which is the usual case, the conventional two-dimensional treatment produces results containing large errors; the errors will appear either on the safe side or on the dangerous side due to the arrangement of members and loading conditions. Hence, these frames must be analysed three-dimensionally.

(4) The author's method is successfully applied to the analysis of the grid works.

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