Definite Integrals for Expressing Polynomials into Fourier Series

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(Received October 10, 1960)

Synopsis. This article gives several kinds of definite integrals, which will be convenient to express polynomials into Fourier series. Examples are added for illustration.

We take the following polynomial of ρ:

$$w(\rho) = a_0 + a_1\rho + a_2\rho^2 + \cdots + a_r\rho^r, \tag{1}$$

which is defined in the interval $(0, 1)$; a_0, $a_1, \cdots$, a_r being constants independent of ρ. This polynomial may be written in the form

$$w(\rho) = \sum_{n=1}^{\infty} c_n \sin n\pi\rho, \tag{2}$$

or in the form

$$w(\rho) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\pi\rho. \tag{3}$$

The present tables will provide a convenient means for expressing equation (1) into equation (2) or (3).

For instance, we take simple beams such as given in Figs. 1~4. Figs. 1 and 2 are subjected to "continuous" loads throughout the whole interval (1, 0), and hence Tables I~II serve for such cases. Fig. 3 is subjected to two kinds of loads, which is connected at the midpoint of the beam; and hence Tables III~VI serve for such a case. Fig. 4 is subjected to a partial load in a certain range of the beam; and hence Tables VII~XII serve for such a case. Several examples which follow will serve as illustration.

Example 1 (Fig. 1). By the elementary theory of bending of beam, the deflection w at point x of the simple beam subjected to uniform load q is given by the equation

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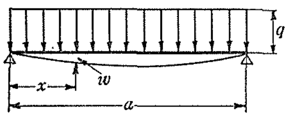

Fig. 1.

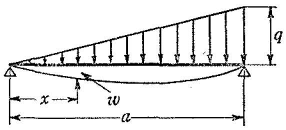

Fig. 2.

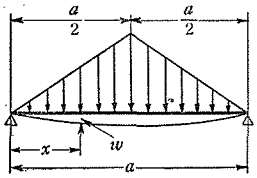

Fig. 3.

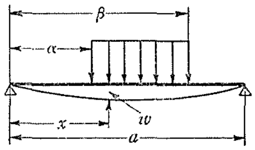

Fig. 4.

$$w = \frac{qa^4}{24EI}\left(\frac{x}{a} - 2\frac{x^3}{a^3} + \frac{x^4}{a^4}\right), \tag{4}$$

where EI is the flexural rigidity, and is supposed to be a constant throughout the length of the beam. To get the Fourier sine series for equation (4), we put

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{a} = \frac{qa^4}{24EI}\left(\frac{x}{a} - 2\frac{x^3}{a^3} + \frac{x^4}{a^4}\right),$$

multiply by $\sin\frac{n\pi x}{a}\,dx$, on both sides, and integrate the result from 0 to a. Putting

$$\frac{x}{a} = \rho, \qquad dx = a\,d\rho,$$

we have

$$(\text{Integration of left-hand side}) = \int_0^a \left(\sum_{m=1}^{\infty} c_m \sin\frac{m\pi x}{a}\right)\sin\frac{n\pi x}{a} = \frac{a}{2}c_n,$$

$$(\text{Integration of right-hand side}) = \frac{qa^4}{24EI}\int_0^a \left(\frac{x}{a} - 2\frac{x^3}{a^3} + \frac{x^4}{a^4}\right)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{qa^4}{24EI}a\int_0^1 (\rho - 2\rho^3 + \rho^4)\sin n\pi\rho\,d\rho,$$

and hence

$$c_n = 2\frac{qa^4}{24EI}\int_0^1 (\rho - 2\rho^3 + \rho^4)\sin n\pi\rho\,d\rho. \tag{5}$$

In virtue of Table I, we have

$$\int_0^1 \rho \sin n\pi\rho \, d\rho = \begin{cases} \dfrac{1}{n\pi} & (n = 1, 3, 5, \cdots), \\ -\dfrac{1}{n\pi} & (n = 2, 4, 6, \cdots); \end{cases}$$

$$\int_0^1 \rho^3 \sin n\pi\rho \, d\rho = \begin{cases} \dfrac{1}{n\pi} - \dfrac{6}{(n\pi)^3} & (n = 1, 3, 5, \cdots), \\ -\dfrac{1}{n\pi} + \dfrac{6}{(n\pi)^3} & (n = 2, 4, 6, \cdots); \end{cases}$$

$$\int_0^1 \rho^4 \sin n\pi\rho \, d\rho = \begin{cases} \dfrac{1}{n\pi} - \dfrac{12}{(n\pi)^3} + \dfrac{48}{(n\pi)^5} & (n = 1, 3, 5, \cdots), \\ -\dfrac{1}{n\pi} + \dfrac{12}{(n\pi)^3} & (n = 2, 4, 6, \cdots). \end{cases}$$

Substituting these values into equation (5), we get

$$c_n = \frac{qa^4}{12EI} \times \begin{cases} \dfrac{1}{n\pi}(1 - 2 + 1) + \dfrac{1}{(n\pi)^3}\{(-2) \times (-6) - 12\} + \dfrac{48}{(n\pi)^5} \\ \qquad\qquad (n = 1, 3, 5, \cdots), \\ \dfrac{1}{n\pi}(-1 + 2 - 1) + \dfrac{1}{(n\pi)^3}\{(-2) \times 6 + 12\} \qquad (n = 2, 4, 6, \cdots) \end{cases}$$

$$= \frac{qa^4}{12EI} \cdot \frac{48}{(n\pi)^5} = \frac{4qa^4}{\pi^5 EI} \cdot \frac{1}{n^5} \qquad (n = 1, 3, 5, \cdots).$$

Hence the wanted series becomes

$$w = \frac{qa^4}{24EI}\left(\frac{x}{a} - 2\frac{x^3}{a^3} + \frac{x^4}{a^4}\right) = \frac{4qa^4}{\pi^5 EI} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{a}. \tag{6}$$

It will be seen that the Fourier coefficient c_n is in general given by the equation

$$c_n = 2\int_0^1 w(\rho) \sin n\pi\rho \, d\rho. \tag{7}$$

Example 2 (Fig. 2). The deflection w of the beam of Fig. 2 is given by

$$w = \frac{qa^4}{360EI}\left(7\frac{x}{a} - 10\frac{x^3}{a^3} + 3\frac{x^5}{a^5}\right). \tag{8}$$

Referring to Table I, we at once have

$$w = \frac{2qa^4}{\pi^5 EI}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \sin\frac{n\pi x}{a}. \tag{9}$$

Computation work necessary for getting equation (9) is merely

$$(-1)^{n+1}\frac{1}{n\pi}(7 - 10 + 3) = 0, \qquad (-1)^{n+1}\frac{1}{(n\pi)^3}\{(-10)\times(-6) + 3\times(-20)\} = 0,$$

$$(-1)^{n+1}\frac{1}{(n\pi)^5} \times 3 \times 120 \times 2 \times \frac{qa^4}{360EI} = \frac{2qa^4}{\pi^5 EI}\cdot\frac{(-1)^{n+1}}{n^5}, \quad = c_n;$$

and this work is effected efficiently by using small pieces of paper of about 5 mm × 15 mm size, on respective sheets of which the numerals 7, —10, 3 are written.

Example 3 (Fig. 3). The deflection w for the beam indicated in Fig. 3 is given by the equations

$$w_1 = \frac{qa^4}{960EI}\left(25\frac{x}{a} - 40\frac{x^3}{a^3} + 16\frac{x^5}{a^5}\right) \qquad (0 < x < \tfrac{1}{2}a), \tag{10}$$

$$w_2 = \frac{qa^4}{960EI}\left(1 + 15\frac{x}{a} + 40\frac{x^2}{a^2} - 120\frac{x^3}{a^3} + 80\frac{x^4}{a^4} - 16\frac{x^5}{a^5}\right) \qquad (\tfrac{1}{2}a < x < a). \tag{11}$$

In this case equation (7) becomes

$$c_n = 2\int_0^1 w(\rho)\sin n\pi\rho\, d\rho = 2\int_0^{\frac{1}{2}} w_1(\rho)\sin n\pi\rho\, d\rho + 2\int_{\frac{1}{2}}^1 w_2(\rho)\sin n\pi\rho\, d\rho.$$

For two integrals of the right-hand side, reference may be made to Tables III and V respectively. In this way we get

$$c_n = \frac{8qa^4}{\pi^6 EI}\cdot\frac{(-1)^{\frac{n-1}{2}}}{n^6} \qquad (n = 1, 3, 5, \cdots).$$

Thus we arrive at the wanted series

$$w = \frac{8qa^4}{\pi^6 EI}\sum_{n=1,3,5,\cdots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^6}\sin\frac{n\pi x}{a}. \tag{12}$$

Example 4. The present integral tables may be used for converting a given Fourier series into its polynomial. As a simple example, we take the Fourier sine series

$$w(\rho) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{(n\pi)^5} \sin n\pi\rho \qquad (0 < \rho < 1), \tag{13}$$

and shall find its polynomial. Now from Table I, we can put

$$w(\rho) = \alpha_0 + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3 + \alpha_4\rho^4,$$

where α_0, $\alpha_1, \cdots, \alpha_4$ are constants to be determined. Referring to Table I, and collecting same powers of $\frac{1}{(n\pi)^r}$, we have

$$\left.\begin{aligned} &2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0, \\ &4\alpha_2 - 6\alpha_3 - 12\alpha_4 = 0, \\ &48\alpha_4 = \tfrac{1}{2}, \end{aligned}\right\} \quad (n = 1, 3, 5, \cdots);$$

$$\left.\begin{aligned} &-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = 0, \\ &6\alpha_3 + 12\alpha_4 = 0, \end{aligned}\right\} \quad (n = 2, 4, 6, \cdots);$$

from which

$$\alpha_0 = 0, \qquad \alpha_1 = \frac{1}{96}, \qquad \alpha_2 = 0, \qquad \alpha_3 = -\frac{1}{48}, \qquad \alpha_4 = \frac{1}{96}.$$

Hence the wanted polynomial becomes

$$w(\rho) = \frac{1}{96}(\rho - 2\rho^3 + \rho^4) \qquad (0 < \rho < 1). \tag{14}$$

This is the reverse of Example 1.

Contents

$(r = 0, 1, 2, 3, \cdots; \quad n = 1, 2, 3, \cdots)$

I. $\int_0^1 \rho^r \sin n\pi\rho \, d\rho$

r	n	$\int_0^1 \rho^r \sin n\pi\rho \, d\rho$	r	n	$\int_0^1 \rho^r \sin n\pi\rho \, d\rho$
0	$1, 3, 5, \cdots$	$\frac{2}{n\pi}$	1	$1, 3, 5, \cdots$	$\frac{1}{n\pi}$ $\Big\} = (-1)^{n+1}\frac{1}{n\pi}$
	$2, 4, 6, \cdots$	0		$2, 4, 6, \cdots$	$-\frac{1}{n\pi}$
2	$1, 3, 5, \cdots$	$\frac{1}{n\pi} - \frac{4}{(n\pi)^3}$			
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi}$			
3	$1, 3, 5, \cdots$	$\frac{1}{n\pi} - \frac{6}{(n\pi)^3}$ $\Big\} = (-1)^{n+1}\left[\frac{1}{n\pi} - \frac{6}{(n\pi)^3}\right]$			
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi} + \frac{6}{(n\pi)^3}$			
4	$1, 3, 5, \cdots$	$\frac{1}{n\pi} - \frac{12}{(n\pi)^3} + \frac{48}{(n\pi)^5}$			
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi} + \frac{12}{(n\pi)^3}$			
5	$1, 3, 5, \cdots$	$\frac{1}{n\pi} - \frac{20}{(n\pi)^3} + \frac{120}{(n\pi)^5}$ $\Big\} = (-1)^{n+1}\left[\frac{1}{n\pi} - \frac{20}{(n\pi)^3} + \frac{120}{(n\pi)^5}\right]$			
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi} + \frac{20}{(n\pi)^3} - \frac{120}{(n\pi)^5}$			
6	$1, 3, 5, \cdots$	$\frac{1}{n\pi} - \frac{30}{(n\pi)^3} + \frac{360}{(n\pi)^5} - \frac{1\,440}{(n\pi)^7}$			
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi} + \frac{30}{(n\pi)^3} - \frac{360}{(n\pi)^5}$			
$\cdots$	$\cdots\cdots\cdots\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$			
0, 2, 4, $\cdots$ 2s	$1, 3, 5, \cdots$	$(2s)!\left[\frac{1}{(2s)!\,n\pi} - \frac{1}{(2s-2)!\,(n\pi)^3} + \frac{1}{(2s-4)!\,(n\pi)^5} - \cdots + \frac{(-1)^{s-1}}{2!\,(n\pi)^{2s-1}} + \frac{2(-1)^s}{0!\,(n\pi)^{2s+1}}\right]$			
	$2, 4, 6, \cdots$	$(2s)!\left[-\frac{1}{(2s)!\,n\pi} + \frac{1}{(2s-2)!\,(n\pi)^3} - \frac{1}{(2s-4)!\,(n\pi)^5} + \cdots + \frac{(-1)^{s-1}}{4!\,(n\pi)^{2s-3}} + \frac{(-1)^s}{2!\,(n\pi)^{2s-1}}\right]$			

r	n	$\int_0^1 \rho^r \sin n\pi\rho \, d\rho$
1, 3, 5, ⋯ $2s+1$	$1, 3, 5, \cdots$	$(2s+1)!\left[\frac{1}{(2s+1)!\,n\pi} - \frac{1}{(2s-1)!\,(n\pi)^3} + \frac{1}{(2s-3)!\,(n\pi)^5} - \cdots + \frac{(-1)^{s-1}}{3!\,(n\pi)^{2s-1}} + \frac{(-1)^s}{1!\,(n\pi)^{2s+1}}\right]$
	$2, 4, 6, \cdots$	$(2s+1)!\left[-\frac{1}{(2s+1)!\,n\pi} + \frac{1}{(2s-1)!\,(n\pi)^3} - \frac{1}{(2s-3)!\,(n\pi)^5} + \cdots + \frac{(-1)^{s}}{3!\,(n\pi)^{2s-1}} + \frac{(-1)^{s+1}}{1!\,(n\pi)^{2s+1}}\right]$

II. $\int_0^1 \rho^r \cos n\pi\rho \, d\rho$

r	n	$\int_0^1 \rho^r \cos n\pi\rho \, d\rho$		r	n	$\int_0^1 \rho^r \cos n\pi\rho \, d\rho$
0	$1, 3, 5, \cdots$	0		1	$1, 3, 5, \cdots$	$-\frac{2}{(n\pi)^2}$
	$2, 4, 6, \cdots$	0			$2, 4, 6, \cdots$	0
2	$1, 3, 5, \cdots$	$-\frac{2}{(n\pi)^2}$	$= (-1)^n \frac{2}{(n\pi)^2}$	3	$1, 3, 5, \cdots$	$-\frac{3}{(n\pi)^2} + \frac{12}{(n\pi)^4}$
	$2, 4, 6, \cdots$	$\frac{2}{(n\pi)^2}$			$2, 4, 6, \cdots$	$\frac{3}{(n\pi)^2}$
4	$1, 3, 5, \cdots$	$-\frac{4}{(n\pi)^2} + \frac{24}{(n\pi)^4}$	$= (-1)^n\left[\frac{4}{(n\pi)^2} - \frac{24}{(n\pi)^4}\right]$			
	$2, 4, 6, \cdots$	$\frac{4}{(n\pi)^2} - \frac{24}{(n\pi)^4}$				
5	$1, 3, 5, \cdots$	$-\frac{5}{(n\pi)^2} + \frac{60}{(n\pi)^4} - \frac{240}{(n\pi)^6}$				
	$2, 4, 6, \cdots$	$\frac{5}{(n\pi)^2} - \frac{60}{(n\pi)^4}$				
6	$1, 3, 5, \cdots$	$-\frac{6}{(n\pi)^2} + \frac{120}{(n\pi)^4} - \frac{720}{(n\pi)^6}$	$= (-1)^n\left[\frac{6}{(n\pi)^2} - \frac{120}{(n\pi)^4} + \frac{720}{(n\pi)^6}\right]$			
	$2, 4, 6, \cdots$	$\frac{6}{(n\pi)^2} - \frac{120}{(n\pi)^4} + \frac{720}{(n\pi)^6}$				
⋯	…………	…………………………………				

r	n	$\int_0^1 \rho^r \cos n\pi\rho \, d\rho$
0, 2, 4, … 2s	1, 3, 5, …	$(2s)!\left[-\frac{1}{(2s-1)!\,(n\pi)^2}+\frac{1}{(2s-3)!\,(n\pi)^4}-\frac{1}{(2s-5)!\,(n\pi)^6}+\cdots+\frac{(-1)^{s-1}}{3!\,(n\pi)^{2s-2}}+\frac{(-1)^s}{1!\,(n\pi)^{2s}}\right]$
	2, 4, 6, …	$(2s)!\left[\frac{1}{(2s-1)!\,(n\pi)^2}-\frac{1}{(2s-3)!\,(n\pi)^4}+\frac{1}{(2s-5)!\,(n\pi)^6}-\cdots+\frac{(-1)^s}{3!\,(n\pi)^{2s-2}}+\frac{(-1)^{s+1}}{1!\,(n\pi)^{2s}}\right]$
1, 3, 5, … 2s + 1	1, 3, 5, …	$(2s+1)!\left[-\frac{1}{(2s)!\,(n\pi)^2}+\frac{1}{(2s-2)!\,(n\pi)^4}-\frac{1}{(2s-4)!\,(n\pi)^6}+\cdots+\frac{(-1)^s}{2!\,(n\pi)^{2s}}+\frac{2(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\right]$
	2, 4, 6, …	$(2s+1)!\left[\frac{1}{(2s)!\,(n\pi)^2}-\frac{1}{(2s-2)!\,(n\pi)^4}+\frac{1}{(2s-4)!\,(n\pi)^6}-\cdots+\frac{(-1)^s}{4!\,(n\pi)^{2s-2}}+\frac{(-1)^{s+1}}{2!\,(n\pi)^{2s}}\right]$

III. $\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$

r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$	r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$
0	1, 5, 9, …	$\frac{1}{n\pi}$	0	3, 7, 11, …	$\frac{1}{n\pi}$
	2, 6, 10, …	$\frac{2}{n\pi}$		4, 8, 12, …	0
1	1, 5, 9, …	$\frac{1}{(n\pi)^2}$	1	3, 7, 11, …	$-\frac{1}{(n\pi)^2}$
	2, 6, 10, …	$\frac{1}{2n\pi}$		4, 8, 12, …	$-\frac{1}{2n\pi}$
2	1, 5, 9, …	$\frac{1}{(n\pi)^2}-\frac{2}{(n\pi)^3}$	2	3, 7, 11, …	$-\frac{1}{(n\pi)^2}-\frac{2}{(n\pi)^3}$
	2, 6, 10, …	$\frac{1}{4n\pi}-\frac{4}{(n\pi)^3}$		4, 8, 12, …	$-\frac{1}{4n\pi}$

r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$	r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$
3	$1, 5, 9, \cdots$	$\frac{3}{4(n\pi)^2} - \frac{6}{(n\pi)^4}$	3	$3, 7, 11, \cdots$	$-\frac{3}{4(n\pi)^2} + \frac{6}{(n\pi)^4}$
	$2, 6, 10, \cdots$	$\frac{1}{8n\pi} - \frac{3}{(n\pi)^3}$		$4, 8, 12, \cdots$	$-\frac{1}{8n\pi} + \frac{3}{(n\pi)^3}$
4	$1, 5, 9, \cdots$	$\frac{1}{2(n\pi)^2} - \frac{12}{(n\pi)^4} + \frac{24}{(n\pi)^5}$	4	$3, 7, 11, \cdots$	$-\frac{1}{2(n\pi)^2} + \frac{12}{(n\pi)^4} + \frac{24}{(n\pi)^5}$
	$2, 6, 10, \cdots$	$\frac{1}{16n\pi} - \frac{3}{(n\pi)^3} + \frac{48}{(n\pi)^5}$		$4, 8, 12, \cdots$	$-\frac{1}{16n\pi} + \frac{3}{(n\pi)^3}$
5	$1, 5, 9, \cdots$	$\frac{5}{16(n\pi)^2} - \frac{15}{(n\pi)^4} + \frac{120}{(n\pi)^6}$	5	$3, 7, 11, \cdots$	$-\frac{5}{16(n\pi)^2} + \frac{15}{(n\pi)^4} - \frac{120}{(n\pi)^6}$
	$2, 6, 10, \cdots$	$\frac{1}{32n\pi} - \frac{5}{2(n\pi)^3} + \frac{60}{(n\pi)^5}$		$4, 8, 12, \cdots$	$-\frac{1}{32n\pi} + \frac{5}{2(n\pi)^3} - \frac{60}{(n\pi)^5}$
$\cdots$	…………	…………………………	$\cdots$	…………	…………………………

r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$
0, 2, 4, …, $2s$	$1, 5, 9, \cdots$	$(2s)!\left[\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s-1}}{2\cdot 1!\,(n\pi)^{2s}} + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\right]$
	$3, 7, 11, \cdots$	$(2s)!\left[-\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} + \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} - \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^s}{2\cdot 1!\,(n\pi)^{2s}} + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\right]$
	$2, 6, 10, \cdots$	$(2s)!\left[\frac{1}{2^{2s}(2s)!\,n\pi} - \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} + \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{2(-1)^s}{0!\,(n\pi)^{2s+1}}\right]$
	$4, 8, 12, \cdots$	$(2s)!\left[-\frac{1}{2^{2s}(2s)!\,n\pi} + \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} - \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{(-1)^s}{2^2\cdot 2!\,(n\pi)^{2s-1}}\right]$
1, 3, 5, …, $2s+1$	$1, 5, 9, \cdots$	$(2s+1)!\left[\frac{1}{2^{2s}(2s)!\,(n\pi)^2} - \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4} + \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^s}{0!\,(n\pi)^{2s+2}}\right]$

r	n	$\int_0^{\frac{1}{2}} \rho^r \sin n\pi\rho \, d\rho$
1, 3, 5, ⋯ 2s + 1	3, 7, 11, ⋯	$(2s+1)!\left[-\frac{1}{2^{2s}(2s)!\,(n\pi)^2}+\frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4}-\frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6}+\cdots+\frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\right]$
	2, 6, 10, ⋯	$(2s+1)!\left[\frac{1}{2^{2s+1}(2s+1)!\,n\pi}-\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3}+\frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5}+\cdots+\frac{(-1)^{s}}{2\cdot 1!\,(n\pi)^{2s+1}}\right]$
	4, 8, 12, ⋯	$(2s+1)!\left[-\frac{1}{2^{2s+1}(2s+1)!\,n\pi}+\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3}-\frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5}+\cdots+\frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s+1}}\right]$

IV. $\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho \, d\rho$

r	n	$\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho \, d\rho$	r	n	$\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho \, d\rho$
0	1, 5, 9, ⋯	$\frac{1}{n\pi}$	0	3, 7, 11, ⋯	$-\frac{1}{n\pi}$
	2, 6, 10, ⋯	0		4, 8, 12, ⋯	0
1	1, 5, 9, ⋯	$\frac{1}{2n\pi}-\frac{1}{(n\pi)^2}$	1	3, 7, 11, ⋯	$-\frac{1}{2n\pi}-\frac{1}{(n\pi)^2}$
	2, 6, 10, ⋯	$-\frac{2}{(n\pi)^2}$		4, 8, 12, ⋯	0
2	1, 5, 9, ⋯	$\frac{1}{4n\pi}-\frac{2}{(n\pi)^3}$	2	3, 7, 11, ⋯	$-\frac{1}{4n\pi}+\frac{2}{(n\pi)^3}$
	2, 6, 10, ⋯	$-\frac{1}{(n\pi)^2}$		4, 8, 12, ⋯	$\frac{1}{(n\pi)^2}$
3	1, 5, 9, ⋯	$\frac{1}{8n\pi}-\frac{3}{(n\pi)^3}+\frac{6}{(n\pi)^4}$	3	3, 7, 11, ⋯	$-\frac{1}{8n\pi}+\frac{3}{(n\pi)^3}+\frac{6}{(n\pi)^4}$
	2, 6, 10, ⋯	$-\frac{3}{4(n\pi)^2}+\frac{12}{(n\pi)^4}$		4, 8, 12, ⋯	$\frac{3}{4(n\pi)^2}$

r	n	$\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho \, d\rho$	r	n	$\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho \, d\rho$
4	$1, 5, 9, \cdots$	$\frac{1}{16n\pi} - \frac{3}{(n\pi)^3} + \frac{24}{(n\pi)^5}$	4	$3, 7, 11, \cdots$	$-\frac{1}{16n\pi} + \frac{3}{(n\pi)^3} - \frac{24}{(n\pi)^5}$
	$2, 6, 10, \cdots$	$-\frac{1}{2(n\pi)^2} + \frac{12}{(n\pi)^4}$		$4, 8, 12, \cdots$	$\frac{1}{2(n\pi)^2} - \frac{12}{(n\pi)^4}$
$\cdots$	…………	……………………	$\cdots$	…………	……………………
0, 2, 4, $\cdots$ $2s$	$1, 5, 9, \cdots$	$(2s)!\Big[\frac{1}{2^{2s}(2s)!\,n\pi} - \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} + \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\Big]$			
	$3, 7, 11, \cdots$	$(2s)!\Big[-\frac{1}{2^{2s}(2s)!\,n\pi} + \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} - \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+1}}\Big]$			
	$2, 6, 10, \cdots$	$(2s)!\Big[-\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} + \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} - \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^s}{2\cdot 1!\,(n\pi)^{2s}}\Big]$			
	$4, 8, 12, \cdots$	$(2s)!\Big[\frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s}}\Big]$			
1, 3, 5, $\cdots$ $2s+1$	$1, 5, 9, \cdots$	$(2s+1)!\Big[\frac{1}{2^{2s+1}(2s+1)!\,n\pi} - \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3} + \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5} + \cdots + \frac{(-1)^s}{2\cdot 1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$			
	$3, 7, 11, \cdots$	$(2s+1)!\Big[-\frac{1}{2^{2s+1}(2s+1)!\,n\pi} + \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3} - \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5} + \cdots + \frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$			
	$2, 6, 10, \cdots$	$(2s+1)!\Big[-\frac{1}{2^{2s}(2s)!\,(n\pi)^2} + \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4} - \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^s}{2^2\cdot 2!\,(n\pi)^{2s}} + \frac{2(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$			

r	n	$\int_0^{\frac{1}{2}} \rho^r \cos n\pi\rho\, d\rho$
1, 3, 5, $\cdots$ $2s+1$	$4, 8, 12, \cdots$	$(2s+1)!\left[-\frac{1}{2^{2s}(2s)!\,(n\pi)^2} - \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4} + \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s+1}}{2^2\cdot 2!\,(n\pi)^{2s}}\right]$

V. $\int_{\frac{1}{2}}^1 \rho^r \sin n\pi\rho\, d\rho$

r	n	$\int_{\frac{1}{2}}^1 \rho^r \sin n\pi\rho\, d\rho$	r	n	$\int_{\frac{1}{2}}^1 \rho^r \sin n\pi\rho\, d\rho$
0	$1, 5, 9, \cdots$	$\frac{1}{n\pi}$	0	$3, 7, 11, \cdots$	$\frac{1}{n\pi}$
	$2, 6, 10, \cdots$	$-\frac{2}{n\pi}$		$4, 8, 12, \cdots$	0
1	$1, 5, 9, \cdots$	$\frac{1}{n\pi} - \frac{1}{(n\pi)^2}$	1	$3, 7, 11, \cdots$	$\frac{1}{n\pi} + \frac{1}{(n\pi)^2}$
	$2, 6, 10, \cdots$	$-\frac{3}{2n\pi}$		$4, 8, 12, \cdots$	$-\frac{1}{2n\pi}$
2	$1, 5, 9, \cdots$	$\frac{1}{n\pi} - \frac{1}{(n\pi)^2} - \frac{2}{(n\pi)^3}$	2	$3, 7, 11, \cdots$	$\frac{1}{n\pi} + \frac{1}{(n\pi)^2} - \frac{2}{(n\pi)^3}$
	$2, 6, 10, \cdots$	$-\frac{5}{4n\pi} + \frac{4}{(n\pi)^3}$		$4, 8, 12, \cdots$	$-\frac{3}{4n\pi}$
3	$1, 5, 9, \cdots$	$\frac{1}{n\pi} - \frac{3}{4(n\pi)^2} - \frac{6}{(n\pi)^3} + \frac{6}{(n\pi)^4}$	3	$3, 7, 11, \cdots$	$\frac{1}{n\pi} + \frac{3}{4(n\pi)^2} - \frac{6}{(n\pi)^3} - \frac{6}{(n\pi)^4}$
	$2, 6, 10, \cdots$	$-\frac{9}{8n\pi} + \frac{9}{(n\pi)^3}$		$4, 8, 12, \cdots$	$-\frac{7}{8n\pi} + \frac{3}{(n\pi)^3}$
4	$1, 5, 9, \cdots$	$\frac{1}{n\pi} - \frac{1}{2(n\pi)^2} - \frac{12}{(n\pi)^3} + \frac{12}{(n\pi)^4} + \frac{24}{(n\pi)^5}$	4	$3, 7, 11, \cdots$	$\frac{1}{n\pi} + \frac{1}{2(n\pi)^2} - \frac{12}{(n\pi)^3} - \frac{12}{(n\pi)^4} + \frac{24}{(n\pi)^5}$
	$2, 6, 10, \cdots$	$-\frac{17}{16n\pi} + \frac{15}{(n\pi)^3} - \frac{48}{(n\pi)^5}$		$4, 8, 12, \cdots$	$-\frac{15}{16n\pi} + \frac{9}{(n\pi)^3}$

r	n	$\int_{\frac{1}{2}}^{1} \rho^r \sin n\pi\rho\, d\rho$	r	n	$\int_{\frac{1}{2}}^{1} \rho^r \sin n\pi\rho\, d\rho$
5	$1, 5, 9, \cdots$	$\frac{1}{n\pi} - \frac{5}{16(n\pi)^2} - \frac{20}{(n\pi)^3} + \frac{15}{(n\pi)^4} + \frac{120}{(n\pi)^5} - \frac{120}{(n\pi)^6}$	5	$3, 7, 11, \cdots$	$\frac{1}{n\pi} + \frac{5}{16(n\pi)^2} - \frac{20}{(n\pi)^3} - \frac{15}{(n\pi)^4} + \frac{120}{(n\pi)^5} + \frac{120}{(n\pi)^6}$
	$2, 6, 10, \cdots$	$-\frac{33}{32n\pi} + \frac{45}{2(n\pi)^3} - \frac{180}{(n\pi)^5}$		$4, 8, 12, \cdots$	$-\frac{31}{32n\pi} + \frac{35}{2(n\pi)^3} - \frac{60}{(n\pi)^5}$
$\cdots$	....	....	$\cdots$	....	....
$0, 2, 4, \cdots 2s$	$1, 5, 9, \cdots$	$(2s)!\Big[\frac{1}{(2s)!\,n\pi} - \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{1}{(2s-2)!\,(n\pi)^3} + \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{1}{(2s-4)!\,(n\pi)^5} - \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^s}{2\cdot 1!\,(n\pi)^{2s}} + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\Big]$			
	$3, 7, 11, \cdots$	$(2s)!\Big[\frac{1}{(2s)!\,n\pi} + \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{1}{(2s-2)!\,(n\pi)^3} - \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{1}{(2s-4)!\,(n\pi)^5} + \frac{1}{2^{2s-5}(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s}} + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\Big]$			
	$2, 6, 10, \cdots$	$(2s)!\Big[-\frac{2^{2s}+1}{2^{2s}(2s)!\,n\pi} + \frac{2^{2s-2}+1}{2^{2s-2}(2s-2)!\,(n\pi)^3} - \frac{2^{2s-4}+1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{2(-1)^{s+1}}{0!\,(n\pi)^{2s+1}}\Big]$			
	$4, 8, 12, \cdots$	$(2s)!\Big[-\frac{2^{2s}-1}{2^{2s}(2s)!\,n\pi} + \frac{2^{2s-2}-1}{2^{2s-2}(2s-2)!\,(n\pi)^3} - \frac{2^{2s-4}-1}{2^{2s-4}(2s-4)!\,(n\pi)^5} + \cdots + \frac{(-1)^s(2^2-1)}{2^2\cdot 2!\,(n\pi)^{2s-1}}\Big]$			
$1, 3, 5, \cdots 2s+1$	$1, 5, 9, \cdots$	$(2s+1)!\Big[\frac{1}{(2s+1)!\,n\pi} - \frac{1}{2^{2s}(2s)!\,(n\pi)^2} - \frac{1}{(2s-1)!\,(n\pi)^3} + \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4} + \frac{1}{(2s-3)!\,(n\pi)^5} - \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6} - \cdots + \frac{(-1)^s}{1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$			

r	n	$\int_{\frac{1}{2}}^{1} \rho^r \sin n\pi\rho \, d\rho$
1, 3, 5, ⋯ $2s+1$	3, 7, 11, ⋯	$(2s+1)!\Big[\frac{1}{(2s+1)!\,n\pi}+\frac{1}{2^{2s}(2s)!\,(n\pi)^2}-\frac{1}{(2s-1)!\,(n\pi)^3}-\frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^4}+\frac{1}{(2s-3)!\,(n\pi)^5}+\frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^6}-\cdots+\frac{(-1)^s}{1!\,(n\pi)^{2s+1}}+\frac{(-1)^s}{0!\,(n\pi)^{2s+2}}\Big]$
	2, 6, 10, ⋯	$(2s+1)!\Big[-\frac{2^{2s+1}+1}{2^{2s+1}(2s+1)!\,n\pi}+\frac{2^{2s-1}+1}{2^{2s-1}(2s-1)!\,(n\pi)^3}-\frac{2^{2s-3}+1}{2^{2s-3}(2s-3)!\,(n\pi)^5}+\cdots+\frac{3(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s+1}}\Big]$
	4, 8, 12, ⋯	$(2s+1)!\Big[-\frac{2^{2s+1}-1}{2^{2s+1}(2s+1)!\,n\pi}+\frac{2^{2s-1}-1}{2^{2s-1}(2s-1)!\,(n\pi)^3}-\frac{2^{2s-3}-1}{2^{2s-3}(2s-3)!\,(n\pi)^5}+\cdots+\frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s+1}}\Big]$

VI. $\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$

r	n	$\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$	r	n	$\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$
0	1, 5, 9, ⋯	$-\frac{1}{n\pi}$	0	3, 7, 11, ⋯	$\frac{1}{n\pi}$
	2, 6, 10, ⋯	0		4, 8, 12, ⋯	0
1	1, 5, 9, ⋯	$-\frac{1}{2n\pi}-\frac{1}{(n\pi)^2}$	1	3, 7, 11, ⋯	$\frac{1}{2n\pi}-\frac{1}{(n\pi)^2}$
	2, 6, 10, ⋯	$\frac{2}{(n\pi)^2}$		4, 8, 12, ⋯	0
2	1, 5, 9, ⋯	$-\frac{1}{4n\pi}-\frac{2}{(n\pi)^2}+\frac{2}{(n\pi)^3}$	2	3, 7, 11, ⋯	$\frac{1}{4n\pi}-\frac{2}{(n\pi)^2}-\frac{2}{(n\pi)^3}$
	2, 6, 10, ⋯	$\frac{3}{(n\pi)^2}$		4, 8, 12, ⋯	$\frac{1}{(n\pi)^2}$
3	1, 5, 9, ⋯	$-\frac{1}{8n\pi}-\frac{3}{(n\pi)^2}+\frac{3}{(n\pi)^3}+\frac{6}{(n\pi)^4}$	3	3, 7, 11, ⋯	$\frac{1}{8n\pi}-\frac{3}{(n\pi)^2}-\frac{3}{(n\pi)^3}+\frac{6}{(n\pi)^4}$

r	n	$\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$	r	n	$\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$
3	$2, 6, 10, \cdots$	$\frac{15}{4(n\pi)^2} - \frac{12}{(n\pi)^4}$	3	$4, 8, 12, \cdots$	$\frac{9}{4(n\pi)^2}$
4	$1, 5, 9, \cdots$	$-\frac{1}{16n\pi} - \frac{4}{(n\pi)^2} + \frac{3}{(n\pi)^3} + \frac{24}{(n\pi)^4} - \frac{24}{(n\pi)^5}$	4	$3, 7, 11, \cdots$	$\frac{1}{16n\pi} - \frac{4}{(n\pi)^2} - \frac{3}{(n\pi)^3} + \frac{24}{(n\pi)^4} + \frac{24}{(n\pi)^5}$
	$2, 6, 10, \cdots$	$\frac{9}{2(n\pi)^2} - \frac{36}{(n\pi)^4}$		$4, 8, 12, \cdots$	$\frac{7}{2(n\pi)^2} - \frac{12}{(n\pi)^4}$
$\cdots$	…………	……………………	$\cdots$	…………	……………………
$0, 2, 4, \cdots, 2s$	$1, 5, 9, \cdots$	$(2s)!\Big[-\frac{1}{2^{2s}(2s)!\,n\pi} - \frac{1}{(2s-1)!\,(n\pi)^2} + \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} + \frac{1}{(2s-3)!\,(n\pi)^4} - \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} - \frac{1}{(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^s}{1!\,(n\pi)^{2s}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+1}}\Big]$			
	$3, 7, 11, \cdots$	$(2s)!\Big[\frac{1}{2^{2s}(2s)!\,n\pi} - \frac{1}{(2s-1)!\,(n\pi)^2} - \frac{1}{2^{2s-2}(2s-2)!\,(n\pi)^3} + \frac{1}{(2s-3)!\,(n\pi)^4} + \frac{1}{2^{2s-4}(2s-4)!\,(n\pi)^5} - \frac{1}{(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^s}{1!\,(n\pi)^{2s}} + \frac{(-1)^s}{0!\,(n\pi)^{2s+1}}\Big]$			
	$2, 6, 10, \cdots$	$(2s)!\Big[\frac{2^{2s-1}+1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{2^{2s-3}+1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{2^{2s-5}+1}{2^{2s-5}(2s-5)!\,(n\pi)^6} - \cdots + \frac{3(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s}}\Big]$			
	$4, 8, 12, \cdots$	$(2s)!\Big[\frac{2^{2s-1}-1}{2^{2s-1}(2s-1)!\,(n\pi)^2} - \frac{2^{2s-3}-1}{2^{2s-3}(2s-3)!\,(n\pi)^4} + \frac{2^{2s-5}-1}{2^{2s-5}(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s}}\Big]$			
$1, 3, 5, \cdots, 2s+1$	$1, 5, 9, \cdots$	$(2s+1)!\Big[-\frac{1}{2^{2s+1}(2s+1)!\,n\pi} - \frac{1}{(2s)!\,(n\pi)^2} + \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3} + \frac{1}{(2s-2)!\,(n\pi)^4} - \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5} - \frac{1}{(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s+1}}{2\cdot 1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$			

r	n	$\int_{\frac{1}{2}}^{1} \rho^r \cos n\pi\rho \, d\rho$
1, 3, 5, $\cdots$ $2s+1$	$3, 7, 11, \cdots$	$(2s+1)!\Big[\frac{1}{2^{2s+1}(2s+1)!\,n\pi} - \frac{1}{(2s)!\,(n\pi)^2} - \frac{1}{2^{2s-1}(2s-1)!\,(n\pi)^3} + \frac{1}{(2s-2)!\,(n\pi)^4} + \frac{1}{2^{2s-3}(2s-3)!\,(n\pi)^5} - \frac{1}{(2s-4)!\,(n\pi)^6} - \cdots + \frac{(-1)^s}{2\cdot 1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}}{0!\,(n\pi)^{2s+2}}\Big]$
	$2, 6, 10, \cdots$	$(2s+1)!\Big[\frac{2^{2s}+1}{2^{2s}(2s)!\,(n\pi)^2} - \frac{2^{2s-2}+1}{2^{2s-2}(2s-2)!\,(n\pi)^4} + \frac{2^{2s-4}+1}{2^{2s-4}(2s-4)!\,(n\pi)^6} - \cdots + \frac{2(-1)^s}{0!\,(n\pi)^{2s+2}}\Big]$
	$4, 8, 12, \cdots$	$(2s+1)!\Big[\frac{2^{2s}-1}{2^{2s}(2s)!\,(n\pi)^2} - \frac{2^{2s-2}-1}{2^{2s-2}(2s-2)!\,(\pi n)^4} + \frac{2^{2s-4}-1}{2^{2s-4}(2s-4)!\,(n\pi)^6} - \cdots + \frac{3(-1)^{s+1}}{2^2\cdot 2!\,(n\pi)^{2s}}\Big]$

VII. $\int_0^{\xi} \rho^r \sin n\pi\rho \, d\rho$

r	n	$\int_0^{\xi} \rho^r \sin n\pi\rho \, d\rho$
0	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(1-\cos n\pi\xi)$
1	$1, 2, 3, \cdots$	$-\frac{1}{n\pi}\xi\cos n\pi\xi + \frac{1}{(n\pi)^2}\sin n\pi\xi$
2	$1, 2, 3, \cdots$	$-\frac{1}{n\pi}\xi^2\cos n\pi\xi + \frac{2}{(n\pi)^2}\xi\sin n\pi\xi - \frac{2}{(n\pi)^3}(1-\cos n\pi\xi)$
3	$1, 2, 3, \cdots$	$-\frac{1}{n\pi}\xi^3\cos n\pi\xi + \frac{3}{(n\pi)^2}\xi^2\sin n\pi\xi + \frac{6}{(n\pi)^3}\xi\cos n\pi\xi - \frac{6}{(n\pi)^4}\sin n\pi\xi$
4	$1, 2, 3, \cdots$	$-\frac{1}{n\pi}\xi^4\cos n\pi\xi + \frac{4}{(n\pi)^2}\xi^3\sin n\pi\xi + \frac{12}{(n\pi)^3}\xi^2\cos n\pi\xi - \frac{24}{(n\pi)^4}\xi\sin n\pi\xi + \frac{24}{(n\pi)^5}(1-\cos n\pi\xi)$
$\cdots$	…………	……………………

r	n	$\int_0^\xi \rho^r \sin n\pi\rho \, d\rho$
0, 2, 4, … $2s$	1, 2, 3, …	$(2s)!\Big[-\frac{\xi^{2s}\cos n\pi\xi}{(2s)!\,n\pi}+\frac{\xi^{2s-1}\sin n\pi\xi}{(2s-1)!\,(n\pi)^2}+\frac{\xi^{2s-2}\cos n\pi\xi}{(2s-2)!\,(n\pi)^3}$ $-\frac{\xi^{2s-3}\sin n\pi\xi}{(2s-3)!\,(n\pi)^4}-\frac{\xi^{2s-4}\cos n\pi\xi}{(2s-4)!\,(n\pi)^5}+\frac{\xi^{2s-5}\sin n\pi\xi}{(2s-5)!\,(n\pi)^6}$ $+\cdots+\frac{(-1)^{s-1}\xi\sin n\pi\xi}{1!\,(n\pi)^{2s}}+\frac{(-1)^s(1-\cos n\pi\xi)}{0!\,(n\pi)^{2s+1}}\Big]$
1, 3, 5, … $2s+1$	1, 2, 3, …	$(2s+1)!\Big[-\frac{\xi^{2s+1}\cos n\pi\xi}{(2s+1)!\,n\pi}+\frac{\xi^{2s}\sin n\pi\xi}{(2s)!\,(n\pi)^2}+\frac{\xi^{2s-1}\cos n\pi\xi}{(2s-1)!\,(n\pi)^3}$ $-\frac{\xi^{2s-2}\sin n\pi\xi}{(2s-2)!\,(n\pi)^4}-\frac{\xi^{2s-3}\cos n\pi\xi}{(2s-3)!\,(n\pi)^5}+\frac{\xi^{2s-4}\sin n\pi\xi}{(2s-4)!\,(n\pi)^6}$ $+\cdots+\frac{(-1)^{s-1}\xi\cos n\pi\xi}{1!\,(n\pi)^{2s+1}}+\frac{(-1)^s\sin n\pi\xi}{0!\,(n\pi)^{2s+2}}\Big]$

VIII. $\int_0^\xi \rho^r \cos n\pi\rho \, d\rho$

r	n	$\int_0^\xi \rho^r \cos n\pi\rho \, d\rho$
0	1, 2, 3, …	$\frac{1}{n\pi}\sin n\pi\xi$
1	1, 2, 3, …	$\frac{1}{n\pi}\xi\sin n\pi\xi-\frac{1}{(n\pi)^2}(1-\cos n\pi\xi)$
2	1, 2, 3, …	$\frac{1}{n\pi}\xi^2\sin n\pi\xi+\frac{2}{(n\pi)^2}\xi\cos n\pi\xi-\frac{2}{(n\pi)^3}\sin n\pi\xi$
3	1, 2, 3, …	$\frac{1}{n\pi}\xi^3\sin n\pi\xi+\frac{3}{(n\pi)^2}\xi^2\cos n\pi\xi-\frac{6}{(n\pi)^3}\xi\sin n\pi\xi+\frac{6}{(n\pi)^4}(1-\cos n\pi\xi)$
4	1, 2, 3, …	$\frac{1}{n\pi}\xi^4\sin n\pi\xi+\frac{4}{(n\pi)^2}\xi^3\cos n\pi\xi-\frac{12}{(n\pi)^3}\xi^2\sin n\pi\xi$ $-\frac{24}{(n\pi)^4}\xi\cos n\pi\xi+\frac{24}{(n\pi)^5}\sin n\pi\xi$
…	…………	……………………

r	n	$\int_0^{\xi} \rho^r \cos n\pi\rho \, d\rho$
0, 2, 4, $\cdots$ $2s$	$1, 2, 3, \cdots$	$(2s)!\Big[\frac{\xi^{2s}\sin n\pi\xi}{(2s)!\,n\pi} + \frac{\xi^{2s-1}\cos n\pi\xi}{(2s-1)!\,(n\pi)^2} - \frac{\xi^{2s-2}\sin n\pi\xi}{(2s-2)!\,(n\pi)^3} - \frac{\xi^{2s-3}\cos n\pi\xi}{(2s-3)!\,(n\pi)^4} + \frac{\xi^{2s-4}\sin n\pi\xi}{(2s-4)!\,(n\pi)^5} + \frac{\xi^{2s-5}\cos n\pi\xi}{(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^{s-1}\xi\cos n\pi\xi}{1!\,(n\pi)^{2s}} + \frac{(-1)^s\sin n\pi\xi}{0!\,(n\pi)^{2s+1}}\Big]$
1, 3, 5, $\cdots$ $2s+1$	$1, 2, 3, \cdots$	$(2s+1)!\Big[\frac{\xi^{2s+1}\sin n\pi\xi}{(2s+1)!\,n\pi} + \frac{\xi^{2s}\cos n\pi\xi}{(2s)!\,(n\pi)^2} - \frac{\xi^{2s-1}\sin n\pi\xi}{(2s-1)!\,(n\pi)^3} - \frac{\xi^{2s-2}\cos n\pi\xi}{(2s-2)!\,(n\pi)^4} + \frac{\xi^{2s-3}\sin n\pi\xi}{(2s-3)!\,(n\pi)^5} + \frac{\xi^{2s-4}\cos n\pi\xi}{(2s-4)!\,(n\pi)^6} - \cdots + \frac{(-1)^s\xi\sin n\pi\xi}{1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}(1-\cos n\pi\xi)}{0!\,(n\pi)^{2s+2}}\Big]$

IX. $\int_{\xi}^{\eta} \rho^r \sin n\pi\rho \, d\rho$

r	n	$\int_{\xi}^{\eta} \rho^r \sin n\pi\rho \, d\rho$
0	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(\cos n\pi\xi - \cos n\pi\eta)$
1	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(\xi\cos n\pi\xi - \eta\cos n\pi\eta) - \frac{1}{(n\pi)^2}(\sin n\pi\xi - \sin n\pi\eta)$
2	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(\xi^2\cos n\pi\xi - \eta^2\cos n\pi\eta) - \frac{2}{(n\pi)^2}(\xi\sin n\pi\xi - \eta\sin n\pi\eta) - \frac{2}{(n\pi)^3}(\cos n\pi\xi - \cos n\pi\eta)$
3	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(\xi^3\cos n\pi\xi - \eta^3\cos n\pi\eta) - \frac{3}{(n\pi)^2}(\xi^2\sin n\pi\xi - \eta^2\sin n\pi\eta) - \frac{6}{(n\pi)^3}(\xi\cos n\pi\xi - \eta\cos n\pi\eta) + \frac{6}{(n\pi)^4}(\sin n\pi\xi - \sin n\pi\eta)$
4	$1, 2, 3, \cdots$	$\frac{1}{n\pi}(\xi^4\cos n\pi\xi - \eta^4\cos n\pi\eta) - \frac{4}{(n\pi)^2}(\xi^3\sin n\pi\xi - \eta^3\sin n\pi\eta) - \frac{12}{(n\pi)^3}(\xi^2\cos n\pi\xi - \eta^2\cos n\pi\eta) + \frac{24}{(n\pi)^4}(\xi\sin n\pi\xi - \eta\sin n\pi\eta) + \frac{24}{(n\pi)^5}(\cos n\pi\xi - \cos n\pi\eta)$
$\cdots$	....	....

r	n	$\int_\xi^\eta \rho^r \sin n\pi\rho\, d\rho$
0, 2, 4, ⋯ 2s	1, 2, 3, ⋯	$(2s)!\Big[\frac{\xi^{2s}\cos n\pi\xi-\eta^{2s}\cos n\pi\eta}{(2s)!\,n\pi}-\frac{\xi^{2s-1}\sin n\pi\xi-\eta^{2s-1}\sin n\pi\eta}{(2s-1)!\,(n\pi)^2}-\frac{\xi^{2s-2}\cos n\pi\xi-\eta^{2s-2}\cos n\pi\eta}{(2s-2)!\,(n\pi)^3}+\frac{\xi^{2s-3}\sin n\pi\xi-\eta^{2s-3}\sin n\pi\eta}{(2s-3)!\,(n\pi)^4}+\frac{\xi^{2s-4}\cos n\pi\xi-\eta^{2s-4}\cos n\pi\eta}{(2s-4)!\,(n\pi)^5}-\frac{\xi^{2s-5}\sin n\pi\xi-\eta^{2s-5}\sin n\pi\eta}{(2s-5)!\,(n\pi)^6}-\cdots+\frac{(-1)^s(\xi\sin n\pi\xi-\eta\sin n\pi\eta)}{1!\,(n\pi)^{2s}}+\frac{(-1)^s(\cos n\pi\xi-\cos n\pi\eta)}{0!\,(n\pi)^{2s+1}}\Big]$
1, 3, 5, ⋯ 2s + 1	1, 2, 3, ⋯	$(2s+1)!\Big[\frac{\xi^{2s+1}\cos n\pi\xi-\eta^{2s+1}\cos n\pi\eta}{(2s+1)!\,n\pi}-\frac{\xi^{2s}\sin n\pi\xi-\eta^{2s}\sin n\pi\eta}{(2s)!\,(n\pi)^2}-\frac{\xi^{2s-1}\cos n\pi\xi-\eta^{2s-1}\cos n\pi\eta}{(2s-1)!\,(n\pi)^3}+\frac{\xi^{2s-2}\sin n\pi\xi-\eta^{2s-2}\sin n\pi\eta}{(2s-2)!\,(n\pi)^4}+\frac{\xi^{2s-3}\cos n\pi\xi-\eta^{2s-3}\cos n\pi\eta}{(2s-3)!\,(n\pi)^5}-\frac{\xi^{2s-4}\sin n\pi\xi-\eta^{2s-4}\sin n\pi\eta}{(2s-4)!\,(n\pi)^6}-\cdots+\frac{(-1)^s(\xi\cos n\pi\xi-\eta\cos n\pi\eta)}{1!\,(n\pi)^{2s+1}}+\frac{(-1)^{s+1}(\sin n\pi\xi-\sin n\pi\eta)}{0!\,(n\pi)^{2s+2}}\Big]$

X. $\int_\xi^\eta \rho^r \cos n\pi\rho\, d\rho$

r	n	$\int_\xi^\eta \rho^r \cos n\pi\rho\, d\rho$
0	1, 2, 3, ⋯	$-\frac{1}{n\pi}(\sin n\pi\xi-\sin n\pi\eta)$
1	1, 2, 3, ⋯	$-\frac{1}{n\pi}(\xi\sin n\pi\xi-\eta\sin n\pi\eta)-\frac{1}{(n\pi)^2}(\cos n\pi\xi-\cos n\pi\eta)$
2	1, 2, 3, ⋯	$-\frac{1}{n\pi}(\xi^2\sin n\pi\xi-\eta^2\sin n\pi\eta)-\frac{2}{(n\pi)^2}(\xi\cos n\pi\xi-\eta\cos n\pi\eta)+\frac{2}{(n\pi)^3}(\sin n\pi\xi-\sin n\pi\eta)$
3	1, 2, 3, ⋯	$-\frac{1}{n\pi}(\xi^3\sin n\pi\xi-\eta^3\sin n\pi\eta)-\frac{3}{(n\pi)^2}(\xi^2\cos n\pi\xi-\eta^2\cos n\pi\eta)+\frac{6}{(n\pi)^3}(\xi\sin n\pi\xi-\eta\sin n\pi\eta)+\frac{6}{(n\pi)^4}(\cos n\pi\xi-\cos n\pi\eta)$

r	n	$\int_\xi^\eta \rho^r \cos n\pi\rho \, d\rho$
4	1, 2, 3, ⋯	$-\frac{1}{n\pi}(\xi^4 \sin n\pi\xi - \eta^4 \sin n\pi\eta) - \frac{4}{(n\pi)^2}(\xi^3 \cos n\pi\xi - \eta^3 \cos n\pi\eta) + \frac{12}{(n\pi)^3}(\xi^2 \sin n\pi\xi - \eta^2 \sin n\pi\eta) + \frac{24}{(n\pi)^4}(\xi \cos n\pi\xi - \eta \cos n\pi\eta) - \frac{24}{(n\pi)^5}(\sin n\pi\xi - \sin n\pi\eta)$
⋯	…………	……………………
0, 2, 4, ⋯ 2s	1, 2, 3, ⋯	$(2s)!\Big[-\frac{\xi^{2s} \sin n\pi\xi - \eta^{2s} \sin n\pi\eta}{(2s)!\, n\pi} - \frac{\xi^{2s-1} \cos n\pi\xi - \eta^{2s-1} \cos n\pi\eta}{(2s-1)!\, (n\pi)^2} + \frac{\xi^{2s-2} \sin n\pi\xi - \eta^{2s-2} \sin n\pi\eta}{(2s-2)!\, (n\pi)^3} + \frac{\xi^{2s-3} \cos n\pi\xi - \eta^{2s-3} \cos n\pi\eta}{(2s-3)!\, (n\pi)^4} - \frac{\xi^{2s-4} \sin n\pi\xi - \eta^{2s-4} \sin n\pi\eta}{(2s-4)!\, (n\pi)^5} - \frac{\xi^{2s-5} \cos n\pi\xi - \eta^{2s-5} \cos n\pi\eta}{(2s-5)!\, (n\pi)^6} + \cdots + \frac{(-1)^s(\xi \cos n\pi\xi - \eta \cos n\pi\eta)}{1!\, (n\pi)^{2s}} + \frac{(-1)^{s+1}(\sin n\pi\xi - \sin n\pi\eta)}{0!\, (n\pi)^{2s+1}}\Big]$
1, 3, 5, ⋯ 2s + 1	1, 2, 3, ⋯	$(2s+1)!\Big[-\frac{\xi^{2s+1} \sin n\pi\xi - \eta^{2s+1} \sin n\pi\eta}{(2s+1)!\, n\pi} - \frac{\xi^{2s} \cos n\pi\xi - \eta^{2s} \cos n\pi\eta}{(2s)!\, (n\pi)^2} + \frac{\xi^{2s-1} \sin n\pi\xi - \eta^{2s-1} \sin n\pi\eta}{(2s-1)!\, (n\pi)^3} + \frac{\xi^{2s-2} \cos n\pi\xi - \eta^{2s-2} \cos n\pi\eta}{(2s-2)!\, (n\pi)^4} - \frac{\xi^{2s-3} \sin n\pi\xi - \eta^{2s-3} \sin n\pi\eta}{(2s-3)!\, (n\pi)^5} - \frac{\xi^{2s-4} \cos n\pi\xi - \eta^{2s-4} \cos n\pi\eta}{(2s-4)!\, (n\pi)^6} + \cdots + \frac{(-1)^{s+1}(\xi \sin n\pi\xi - \eta \sin n\pi\eta)}{1!\, (n\pi)^{2s+1}} + \frac{(-1)^{s+1}(\cos n\pi\xi - \cos n\pi\eta)}{0!\, (n\pi)^{2s+2}}\Big]$

XI. $\int_\eta^1 \rho^r \sin n\pi\rho \, d\rho$

r	n	$\int_\eta^1 \rho^r \sin n\pi\rho \, d\rho$
0	1, 3, 5, ⋯	$\frac{1}{n\pi}(\cos n\pi\eta + 1)$
	2, 4, 6, ⋯	$\frac{1}{n\pi}(\cos n\pi\eta - 1)$
1	1, 3, 5, ⋯	$\frac{1}{n\pi}(\eta \cos n\pi\eta + 1) - \frac{1}{(n\pi)^2} \sin n\pi\eta$

r	n	$\int_\eta^1 \rho^r \sin n\pi\rho\, d\rho$
1	2, 4, 6, …	$\frac{1}{n\pi}(\eta\cos n\pi\eta - 1) - \frac{1}{(n\pi)^2}\sin n\pi\eta$
2	1, 3, 5, …	$\frac{1}{n\pi}(\eta^2\cos n\pi\eta + 1) - \frac{2}{(n\pi)^2}\eta\sin n\pi\eta - \frac{2}{(n\pi)^3}(\cos n\pi\eta + 1)$
	2, 4, 6, …	$\frac{1}{n\pi}(\eta^2\cos n\pi\eta - 1) - \frac{2}{(n\pi)^2}\eta\sin n\pi\eta - \frac{2}{(n\pi)^3}(\cos n\pi\eta - 1)$
3	1, 3, 5, …	$\frac{1}{n\pi}(\eta^3\cos n\pi\eta + 1) - \frac{3}{(n\pi)^2}\eta^2\sin n\pi\eta - \frac{6}{(n\pi)^3}(\eta\cos n\pi\eta + 1) + \frac{6}{(n\pi)^4}\sin n\pi\eta$
	2, 4, 6, …	$\frac{1}{n\pi}(\eta^3\cos n\pi\eta - 1) - \frac{3}{(n\pi)^2}\eta^2\sin n\pi\eta - \frac{6}{(n\pi)^3}(\eta\cos n\pi\eta - 1) + \frac{6}{(n\pi)^4}\sin n\pi\eta$
4	1, 3, 5, …	$\frac{1}{n\pi}(\eta^4\cos n\pi\eta + 1) - \frac{4}{(n\pi)^2}\eta^3\sin n\pi\eta - \frac{12}{(n\pi)^3}(\eta^2\cos n\pi\eta + 1) + \frac{24}{(n\pi)^4}\eta\sin n\pi\eta + \frac{24}{(n\pi)^5}(\cos n\pi\eta + 1)$
	2, 4, 6, …	$\frac{1}{n\pi}(\eta^4\cos n\pi\eta - 1) - \frac{4}{(n\pi)^2}\eta^3\sin n\pi\eta - \frac{12}{(n\pi)^3}(\eta^2\cos n\pi\eta - 1) + \frac{24}{(n\pi)^4}\eta\sin n\pi\eta + \frac{24}{(n\pi)^5}(\cos n\pi\eta - 1)$
…	…………	………………………
0, 2, 4, … 2s	1, 3, 5, …	$(2s)!\Big[\frac{\eta^{2s}\cos n\pi\eta + 1}{(2s)!\,n\pi} - \frac{\eta^{2s-1}\sin n\pi\eta}{(2s-1)!\,(n\pi)^2} - \frac{\eta^{2s-2}\cos n\pi\eta + 1}{(2s-2)!\,(n\pi)^3} + \frac{\eta^{2s-3}\sin n\pi\eta}{(2s-3)!\,(n\pi)^4} + \frac{\eta^{2s-4}\cos n\pi\eta + 1}{(2s-4)!\,(n\pi)^5} - \frac{\eta^{2s-5}\sin n\pi\eta}{(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^s\,\eta\sin n\pi\eta}{1!\,(n\pi)^{2s}} + \frac{(-1)^s(\cos n\pi\eta + 1)}{0!\,(n\pi)^{2s+1}}\Big]$
	2, 4, 6, …	$(2s)!\Big[\frac{\eta^{2s}\cos n\pi\eta - 1}{(2s)!\,n\pi} - \frac{\eta^{2s-1}\sin n\pi\eta}{(2s-1)!\,(n\pi)^2} - \frac{\eta^{2s-2}\cos n\pi\eta - 1}{(2s-2)!\,(n\pi)^3} + \frac{\eta^{2s-3}\sin n\pi\eta}{(2s-3)!\,(n\pi)^4} + \frac{\eta^{2s-4}\cos n\pi\eta - 1}{(2s-4)!\,(n\pi)^5} - \frac{\eta^{2s-5}\sin n\pi\eta}{(2s-5)!\,(n\pi)^6} - \cdots + \frac{(-1)^s\,\eta\sin n\pi\eta}{1!\,(n\pi)^{2s}} + \frac{(-1)^s(\cos n\pi\eta - 1)}{0!\,(n\pi)^{2s+1}}\Big]$

r	n	$\int_\eta^1 \rho^r \sin n\pi\rho \, d\rho$
1, 3, 5, … $2s+1$	1, 3, 5, ⋯	$(2s+1)!\left[\frac{\eta^{2s+1}\cos n\pi\eta+1}{(2s+1)!\,n\pi}-\frac{\eta^{2s}\sin n\pi\eta}{(2s)!\,(n\pi)^2}-\frac{\eta^{2s-1}\cos n\pi\eta+1}{(2s-1)!\,(n\pi)^3}+\frac{\eta^{2s-2}\sin n\pi\eta}{(2s-2)!\,(n\pi)^4}+\frac{\eta^{2s-3}\cos n\pi\eta+1}{(2s-3)!\,(n\pi)^5}-\frac{\eta^{2s-4}\sin n\pi\eta}{(2s-4)!\,(n\pi)^6}-\cdots+\frac{(-1)^s(\eta\cos n\pi\eta+1)}{1!\,(n\pi)^{2s+1}}+\frac{(-1)^{s+1}\sin n\pi\eta}{0!\,(n\pi)^{2s+2}}\right]$
	2, 4, 6, ⋯	$(2s+1)!\left[\frac{\eta^{2s+1}\cos n\pi\eta-1}{(2s+1)!\,n\pi}-\frac{\eta^{2s}\sin n\pi\eta}{(2s)!\,(n\pi)^2}-\frac{\eta^{2s-1}\cos n\pi\eta-1}{(2s-1)!\,(n\pi)^3}+\frac{\eta^{2s-2}\sin n\pi\eta}{(2s-2)!\,(n\pi)^4}+\frac{\eta^{2s-3}\cos n\pi\eta-1}{(2s-3)!\,(n\pi)^5}-\frac{\eta^{2s-4}\sin n\pi\eta}{(2s-4)!\,(n\pi)^6}-\cdots+\frac{(-1)^s(\eta\cos n\pi\eta-1)}{1!\,(n\pi)^{2s+1}}+\frac{(-1)^{s+1}\sin n\pi\eta}{0!\,(n\pi)^{2s+2}}\right]$

XII. $\int_\eta^1 \rho^r \cos n\pi\rho \, d\rho$

r	n	$\int_\eta^1 \rho^r \cos n\pi\rho \, d\rho$
0	1, 3, 5, ⋯	$-\frac{1}{n\pi}\sin n\pi\eta$
	2, 4, 6, ⋯	$-\frac{1}{n\pi}\sin n\pi\eta$
1	1, 3, 5, ⋯	$-\frac{1}{n\pi}\eta\sin n\pi\eta-\frac{1}{(n\pi)^2}(\cos n\pi\eta+1)$
	2, 4, 6, ⋯	$-\frac{1}{n\pi}\eta\sin n\pi\eta-\frac{1}{(n\pi)^2}(\cos n\pi\eta-1)$
2	1, 3, 5, ⋯	$-\frac{1}{n\pi}\eta^2\sin n\pi\eta-\frac{2}{(n\pi)^2}(\eta\cos n\pi\eta+1)+\frac{2}{(n\pi)^3}\sin n\pi\eta$
	2, 4, 6, ⋯	$-\frac{1}{n\pi}\eta^2\sin n\pi\eta-\frac{2}{(n\pi)^2}(\eta\cos n\pi\eta-1)+\frac{2}{(n\pi)^3}\sin n\pi\eta$
3	1, 3, 5, ⋯	$-\frac{1}{n\pi}\eta^3\sin n\pi\eta-\frac{3}{(n\pi)^2}(\eta^2\cos n\pi\eta+1)+\frac{6}{(n\pi)^3}\eta\sin n\pi\eta+\frac{6}{(n\pi)^4}(\cos n\pi\eta+1)$

r	n	$\int_{\eta}^{1} \rho^r \cos n\pi\rho \, d\rho$
3	$2, 4, 6, \cdots$	$-\frac{1}{n\pi}\eta^3 \sin n\pi\eta - \frac{3}{(n\pi)^2}(\eta^2 \cos n\pi\eta - 1) + \frac{6}{(n\pi)^3}\eta \sin n\pi\eta + \frac{6}{(n\pi)^4}(\cos n\pi\eta - 1)$
4	$1, 3, 5, \cdots$	$-\frac{1}{n\pi}\eta^4 \sin n\pi\eta - \frac{4}{(n\pi)^2}(\eta^3 \cos n\pi\eta + 1) + \frac{12}{(n\pi)^3}\eta^2 \sin n\pi\eta + \frac{24}{(n\pi)^4}(\eta \cos n\pi\eta + 1) - \frac{24}{(n\pi)^5} \sin n\pi\eta$
	$2, 4, 6, \cdots$	$-\frac{1}{n\pi}\eta^4 \sin n\pi\eta - \frac{4}{(n\pi)^2}(\eta^3 \cos n\pi\eta - 1) + \frac{12}{(n\pi)^3}\eta^2 \sin n\pi\eta + \frac{24}{(n\pi)^4}(\eta \cos n\pi\eta - 1) - \frac{24}{(n\pi)^5} \sin n\pi\eta$
$\cdots$	$\cdots\cdots\cdots\cdots$	$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$0, 2, 4, \cdots 2s$	$1, 3, 5, \cdots$	$(2s)!\left[-\frac{\eta^{2s}\sin n\pi\eta}{(2s)!\,n\pi} - \frac{\eta^{2s-1}\cos n\pi\eta + 1}{(2s-1)!\,(n\pi)^2} + \frac{\eta^{2s-2}\sin n\pi\eta}{(2s-2)!\,(n\pi)^3} + \frac{\eta^{2s-3}\cos n\pi\eta + 1}{(2s-3)!\,(n\pi)^4} - \frac{\eta^{2s-4}\sin n\pi\eta}{(2s-4)!\,(n\pi)^5} - \frac{\eta^{2s-5}\cos n\pi\eta + 1}{(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^s(\eta\cos n\pi\eta + 1)}{1!\,(n\pi)^{2s}} + \frac{(-1)^{s+1}\sin n\pi\eta}{0!\,(n\pi)^{2s+1}}\right]$
	$2, 4, 6, \cdots$	$(2s)!\left[-\frac{\eta^{2s}\sin n\pi\eta}{(2s)!\,n\pi} - \frac{\eta^{2s-1}\cos n\pi\eta - 1}{(2s-1)!\,(n\pi)^2} + \frac{\eta^{2s-2}\sin n\pi\eta}{(2s-2)!\,(n\pi)^3} + \frac{\eta^{2s-3}\cos n\pi\eta - 1}{(2s-3)!\,(n\pi)^4} - \frac{\eta^{2s-4}\sin n\pi\eta}{(2s-4)!\,(n\pi)^5} - \frac{\eta^{2s-5}\cos n\pi\eta - 1}{(2s-5)!\,(n\pi)^6} + \cdots + \frac{(-1)^s(\eta\cos n\pi\eta - 1)}{1!\,(n\pi)^{2s}} + \frac{(-1)^{s+1}\sin n\pi\eta}{0!\,(n\pi)^{2s+1}}\right]$
$1, 3, 5, \cdots 2s+1$	$1, 3, 5, \cdots$	$(2s+1)!\left[-\frac{\eta^{2s+1}\sin n\pi\eta}{(2s+1)!\,n\pi} - \frac{\eta^{2s}\cos n\pi\eta + 1}{(2s)!\,(n\pi)^2} + \frac{\eta^{2s-1}\sin n\pi\eta}{(2s-1)!\,(n\pi)^3} + \frac{\eta^{2s-2}\cos n\pi\eta + 1}{(2s-2)!\,(n\pi)^4} - \frac{\eta^{2s-3}\sin n\pi\eta}{(2s-3)!\,(n\pi)^5} - \frac{\eta^{2s-4}\cos n\pi\eta + 1}{(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s+1}\eta\sin n\pi\eta}{1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}\cos n\pi\eta + 1}{0!\,(n\pi)^{2s+2}}\right]$

r	n	$\int_{\eta}^{1} \rho^r \cos n\pi\rho \, d\rho$
1, 3, 5, $\cdots$ $2s+1$	$2, 4, 6, \cdots$	$(2s+1)!\Big[-\frac{\eta^{2s+1}\sin n\pi\eta}{(2s+1)!\, n\pi} - \frac{\eta^{2s}\cos n\pi\eta - 1}{(2s)!\,(n\pi)^2} + \frac{\eta^{2s-1}\sin n\pi\eta}{(2s-1)!\,(n\pi)^3} + \frac{\eta^{2s-2}\cos n\pi\eta - 1}{(2s-2)!\,(n\pi)^4} - \frac{\eta^{2s-3}\sin n\pi\eta}{(2s-3)!\,(n\pi)^5} - \frac{\eta^{2s-4}\cos n\pi\eta - 1}{(2s-4)!\,(n\pi)^6} + \cdots + \frac{(-1)^{s+1}\eta\sin n\pi\eta}{1!\,(n\pi)^{2s+1}} + \frac{(-1)^{s+1}(\cos n\pi\eta - 1)}{0!\,(n\pi)^{2s+2}}\Big]$