27–Figure e^n–Table and its Applications

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Synopsis. This article is an extension of my previous one**) to more precise values, in which e^n's are of 27 decimal places. With an ordinary hand-operated calculating machine, this table enables us to compute efficiently $\log N$, e^ν, N^ν, $\sinh N$, $\sinh^{-1} N$, etc. to 27 decimal places. Examples are added for illustration.

1. The principle of the present procedure of finding precise logarithm of a given number is as follows.

Let us suppose to find natural logarithm of a number, N say; and write

$$\log_e N = n + \varepsilon, \qquad \text{or} \qquad N = e^n \cdot e^\varepsilon, \tag{1}$$

ε being a small fraction. By Taylor's expansion theorem, e^ε may be expanded in power series of ε, in virtue of which the equation last written takes the form

$$\varepsilon = \left(\frac{N}{e^n} - 1\right) - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{6} - \cdots. \tag{2}$$

N is divided by an e^n which is the most approximate to N, and the quotient is divided by the second e^n which is again the most approximate to the first quotient; such divisions, or often multiplications, are performed several times. The last quotient thus obtained becomes almost equal to unity. ε in equation (2) will then be found easily, by the procedure of iteration. The sum of fractional values of the above n's and ε gives the wanted logarithm.

All numbers, large or small, whose logarithms are desired, can be reduced to those of first place, since for example

$$\log_e 326.8 = \frac{2}{M} + \log_e 3.268,$$

where $\frac{1}{M} = \log_e 10 = 2.302\,585 \cdots$ (cf. equation (7)).

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**) B. Tanimoto; "18-Figure e^n-Table and its Applications," Journal of the Faculty of Engineering, Shinshu University, 1956.

2. With the ordinary calculating machine, capable of 10–11–21 places, multiplication of higher precision is effected by the formula

$$(A + Bx + Cx^2 + \cdots)(A' + B'x + C'x^2 + \cdots)$$
$$= AA' + (AB' + BA')x + (AC' + BB' + CA')x^2 + \cdots, \quad (x = 10^{-9}). \quad (3)$$

The product AA' is obtained in the right dial of the machine, and the left half of AA' is recorded on calculation sheet. Then the remaining right half of AA' is shifted to the left part of the dial by the transfer lever, and then AB' and BA' are superposed. The left half of the dial is recorded. By such operations, any two numbers can be multiplied, however precise they may be.

Division of higher precision can also be effected by putting

$$\frac{A' + B'x + C'x^2 + \cdots}{A + Bx + Cx^2 + \cdots} = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad (x = 10^{-9}),$$

and, clearing and rearranging,

$$(A' + B'x + C'x^2 + \cdots) - Aa_0 - (Ba_0 + Aa_1)x$$
$$- (Ca_0 + Ba_1 + Aa_2)x^2 - \cdots = 0, \qquad (x = 10^{-9}), \qquad (4)$$

a_0, a_1, a_2, ... being lows of nine consecutive figures of the wanted quotient. To perform this division, $A' + B'x$ is first set on the right dial. This is divided by A and the first nine figures of the quotient, a_0, is obtained on the left dial. The remaining figures on the right dial is shifted to the left part of it. Then C' is supplemented on the right dial succeeding to the above figures. From this low of figures, the product Ba_0 is subtracted. The resulting low of figures is divided by A; then a_1 is obtained on the left dial. By such procedures, any division can be performed, however precise the numerator and denominator may be.

At times the bell sounds in the course of subtracting Ba_0, and the right dial becomes negative. This inconvenience can be ridden over by putting

$$a_0' = a_0 + nx, \qquad \text{or} \qquad a_0 = a_0' - nx,$$

a_0' being the apparent quotient, a_0 the proper one, and so n one or two units in the last digit of a_0' or a_0. By substitution into equation (4), we have

$$(A' + B'x + C'x^2 + \cdots) - Aa_0' + (- Ba_0' + nA - Aa_1)x$$
$$+ (- Ca_0 + nB - Ba_1 - Aa_2)x^2 + \cdots = 0, \qquad (5)$$

n being chosen so as to make the negative right dial positive.

3. The e^{u}-Table attached was prepared as in the following way. First,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

was computed to 30 decimal places, the result being*)

$$e = 2.718\,281\,828{,}459\,045\,235{,}360\,287\,471{,}353.$$

Secondly, the pivotal values

$$\sqrt[3]{e},\quad \sqrt[5]{e},\quad \sqrt[7]{e},\quad \sqrt[9]{e},\quad \sqrt[11]{e},\quad \text{and}\quad (\sqrt{e})^3$$

were computed**); these values being

$$\sqrt[3]{e} = 1.395\,612\,425{,}086\,089\,528{,}628\,125\,319,$$

$$\sqrt[5]{e} = 1.221\,402\,758{,}160\,169\,833{,}921\,071\,995,$$

$$\sqrt[7]{e} = 1.153\,564\,994{,}895\,107\,753{,}461\,339\,625,$$

$$\sqrt[9]{e} = 1.117\,519\,068{,}741\,863\,648{,}622\,059\,717,$$

$$\sqrt[11]{e} = 1.095\,169\,439{,}874\,664\,284{,}656\,241\,323,$$

$$(\sqrt{e})^3 = 4.481\,689\,070{,}338\,064\,822{,}602\,055\,460.$$

Thirdly, square roots of these values were extracted successively.

To extract square root of a given number, we put

$$A + Bx + Cx^2 + \cdots = (a_0 + a_1x + a_2x^2 + \cdots)^2, \quad (x = 10^{-9}),$$

from which we obtain

$$(A + Bx + Cx^2 + \cdots) - a_0^2 - 2a_0a_1x - (a_1^2 + 2a_0a_2)x^2 - \cdots = 0.$$

This formula enables us to get precise square root of a given number.

Several examples which follow will serve as illustration.

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*) This is correct to the last place of decimals, since three figures more were retained in the course of computation of the series.

**) Reference may be made to my book: "Numerical Computations and Methods of Handling a Calculating Machine" (in Japanese), 1959, Kanehara Book Publication Co., Ltd., Tokyo.

4. Example 1. To find $\log_e 2$. Referring to the e^n–Table,

$$2 \div e^{\frac{3}{4}} = 0.944\,733\,105{,}482\,029\,414{,}276\,093\,102 = N_1,$$

$$N_1 \times e^{\frac{1}{20}} = 0.993\,170\,607{,}582\,819\,029{,}409\,600\,186 = N_2,$$

$$N_2 \times e^{\frac{1}{160}} = 0.999\,397\,362{,}219\,079\,577{,}178\,881\,259 = N_3,$$

$$N_3 \times e^{\frac{1}{1\,536}} = 1.000\,048\,223{,}389\,322\,235{,}209\,714\,892 = N_4,$$

$$N_4 \div e^{\frac{1}{20\,480}} = 0.999\,999\,394{,}101\,795\,532{,}475\,131\,602 = N_5,$$

$$N_5 \times e^{\frac{1}{1\,572\,864}} = 1.000\,000\,029{,}884\,490\,026{,}791\,928\,638 = N_6.$$

A rough approximation to ε is then obtained thus :

$$\varepsilon_1 = (N_6 - 1) - \frac{\varepsilon^2}{2} = 0.0^6\,029{,}884\,490\,026{,}792 - 0.0^{15}\,446{,}541$$

$$= 0.0^6\,029{,}884\,489\,580{,}251.$$

Then the accurate value of ε is

$$\varepsilon_2 = (N_6 - 1) - \frac{\varepsilon_1^{\,2}}{2} - \frac{\varepsilon_1^{\,3}}{6}$$

$$= 0.0^6\,029{,}884\,490\,026{,}791\,928\,638$$

$$- 0.0^{15}\,446{,}541\,358\,736 - 0.0^{21}\,004\,448$$

$$= 0.0^6\,029{,}884\,489\,580{,}250\,565\,454.$$

Hence the wanted logarithm amounts to

$$\log_e 2 = \frac{3}{4} - \frac{1}{20} - \frac{1}{160} - \frac{1}{1\,536} + \frac{1}{20\,480} - \frac{1}{1\,572\,864} + \varepsilon_2$$

$$= 0.693\,147\,180{,}559\,945\,309{,}417\,232\,120. \qquad (6)$$

An automatic computer, FACOM 128, afforded *)

$$\log_e 2 = 0.693\,147\,180{,}559\,945\,309{,}417\,232\,121{,}4\cdots;$$

and so our result, equation (6), is true except for the last digit.

5. Example 2. To find $\frac{1}{M} = \log_e 10$.

*) This is due to Mr. S. YAMASHITA, operator of the FACOM 128.

$$10 \div e^2 = 1.353\,352\,832{,}366\,126\,918{,}939\,994\,950 = N_1,$$

$$N_1 \div e^{\frac{1}{3}} = 0.969\,719\,678{,}644\,050\,628{,}099\,066\,593 = N_2,$$

$$N_2 \times e^{\frac{1}{32}} = 1.000\,501\,885{,}563\,147\,616{,}988\,530\,778 = N_3,$$

$$N_3 \div e^{\frac{1}{2\,048}} = 1.000\,013\,478{,}501\,546\,536{,}449\,758\,742 = N_4,$$

$$N_4 \div e^{\frac{1}{81\,920}} = 1.000\,001\,271{,}380\,270\,553{,}895\,812\,631 = N_5,$$

$$N_5 \div e^{\frac{1}{786\,432}} = 0.999\,999\,999{,}813\,707\,142{,}368\,677\,302 = N_6.$$

$$\varepsilon = (N_6 - 1) - \frac{\varepsilon^2}{2} = -\,0.0^9{,}186\,292\,857{,}631\,322\,698 - 17\,352\,514$$

$$= -\,0.0^9{,}186\,292\,857{,}648\,675\,212.$$

$$\frac{1}{M} = \log_e 10 = 2 + \frac{1}{3} - \frac{1}{32} + \frac{1}{2\,048} + \frac{1}{81\,920} + \frac{1}{786\,432} + \varepsilon$$

$$= 2.302\,585\,092{,}994\,045\,684{,}017\,991\,454. \qquad (7)$$

This is in perfect accordance with the value computed by the FACOM 128, the latter part of which is $\cdots 991\,453{,}68\cdots$.

The reciprocal of equation (7) is

$$M = \frac{1}{\log_e 10} = 0.434\,294\,481{,}903\,251\,827{,}651\,128\,919. \qquad (8)$$

6. Example 3. To find $\log_e \pi$, where

$$\pi = 3.141\,592\,653{,}589\,793\,238{,}462\,643\,383.$$

$$\pi \div e = 1.155\,727\,349{,}790\,921\,717{,}910\,093\,183 = N_1,$$

$$N_1 \div e^{\frac{1}{7}} = 1.001\,874\,497{,}670\,597\,732{,}311\,176\,831 = N_2,$$

$$N_2 \div e^{\frac{1}{512}} = 0.999\,919\,621{,}222\,804\,341{,}507\,818\,373 = N_3,$$

$$N_3 \times e^{\frac{1}{12\,283}} = 1.000\,000\,998{,}201\,088\,852{,}709\,259\,500 = N_4,$$

$$N_4 \div e^{\frac{1}{1\,048\,576}} = 1.000\,000\,044{,}526\,275\,235{,}378\,467\,282 = N_5.$$

$$\varepsilon_1 = (N_5 - 1) - \frac{\varepsilon^2}{2} = 0.0^6\,044{,}526\,275\,235{,}378 - 991{,}295$$

$$= 0.0^6\,044{,}526\,274\,244{,}083.$$

$$\varepsilon_2 = (N_5 - 1) - \frac{\varepsilon_1{}^2}{2} - \frac{\varepsilon_1{}^3}{6}$$

$$= 0.0^6\,044{,}526\,275\,235{,}378\,467\,282 - 991{,}294\,549\,030 - 14\,713$$

$$= 0.0^6\,044{,}526\,274\,244{,}083\,903\,539.$$

$$\log_e \pi = 1 + \frac{1}{7} + \frac{1}{512} - \frac{1}{12\,288} + \frac{1}{1\,048\,576} + \varepsilon$$

$$= 1.\,144\,729\,885{,}849\,400\,174{,}143\,427\,349. \qquad (9)$$

7. Example 4. To find $e^{\sqrt{2}}$, where

$$\sqrt{2} = 1.\,414\,213\,562{,}373\,095\,048{,}801\,688\,724.$$

By referring to fractional values of n in the e^n–Table, we have

$$\sqrt{2} = n + \varepsilon = \frac{3}{2} - \frac{1}{12} - \frac{1}{384} + \frac{1}{7\,168} + \frac{1}{90\,112} + \frac{1}{2\,097\,152} + \varepsilon,$$

so that

$$\varepsilon = -\,0.\,0^6\,020{,}693\,770\,946{,}531\,103\,483.$$

Then we have

$$e^{\frac{3}{2}} \div e^{\frac{1}{12}} = 4.\,123\,352\,997{,}269\,820\,753{,}397\,328\,867 = N_1,$$

$$N_1 \div e^{\frac{1}{384}} = 4.\,112\,629\,068{,}349\,070\,260{,}453\,568\,380 = N_2,$$

$$N_2 \times e^{\frac{1}{7\,168}} = 4.\,113\,202\,856{,}847\,386\,040{,}299\,223\,139 = N_3,$$

$$N_3 \times e^{\frac{1}{90\,112}} = 4.\,113\,248\,502{,}551\,395\,019{,}996\,931\,557 = N_4,$$

$$N_4 \times e^{\frac{1}{2\,097\,152}} = 4.\,113\,250\,463{,}901\,589\,582{,}152\,828\,708 = N_5 = e^n.$$

On the other hand we have

$$e^{\varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{6}$$

$$= 1 - 0.\,0^6\,020{,}693\,770\,946{,}531\,103\,483 + 214{,}116\,077\,994 - 738$$

$$= 0.\,9^6\,979{,}306\,229\,267{,}584\,973\,773.$$

Hence we obtain

$$e^{\sqrt{2}} = e^{n} \times e^{\varepsilon} = 4.\,113\,250\,378{,}782\,927\,517{,}173\,584\,865. \qquad (10)$$

8. Example 5. To find $\sinh\sqrt{2}$ and $\cosh\sqrt{2}$.

From equation (10),

$$e^{-\sqrt{2}} = \frac{1}{e^{\sqrt{2}}} = 0.\,243\,116\,734{,}434\,214\,210{,}804\,862\,140.$$

Hence we obtain

$$\sinh\sqrt{2} = \tfrac{1}{2}(e^{\sqrt{2}} - e^{-\sqrt{2}}) = 1.\,935\,066\,822{,}174\,356\,653{,}184\,361\,362,$$

$$\cosh\sqrt{2} = \tfrac{1}{2}(e^{\sqrt{2}} + e^{-\sqrt{2}}) = 2.\,178\,183\,556{,}608\,570\,863{,}989\,223\,502.$$

These values are checked by

$$\cosh^2\sqrt{2} - \sinh^2\sqrt{2} = +\,0.\,0^{27},8\cdots,$$

which is computed by a continuous operation of the machine.

9. Example 6. To find $\sinh^{-1}\sqrt{2}$ and $\cosh^{-1}\sqrt{2}$.

we have in general

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1}) = \log N,$$

$$\cosh^{-1} x = \log(x + \sqrt{x^2-1}) = \log N'.$$

In this case

$$N = 3.\,146\,264\,369{,}941\,972\,342{,}329\,135\,066,$$

$$N' = 2.\,414\,213\,562{,}373\,095\,048{,}801\,688\,724.$$

Then we can proceed as before, and arrive at the results

$$\sinh^{-1}\sqrt{2} = 1.\,146\,215\,834{,}780\,588\,843{,}900\,393\,654,$$

$$\cosh^{-1}\sqrt{2} = 0.\,881\,373\,587{,}019\,543\,025{,}232\,609\,320.$$

10. Precise value of N^{ν} can also be computed by using the e^n–Table, N and ν denoting given numbers. In fact, if we put

$$P = N^{\nu}$$

and take logarithms of both members :

$$\log_e P = \nu \log_e N, \quad = \alpha \text{ say,}$$

we then have

$$P = e^{\alpha},$$

which is the required result, $\log_e N$ and e^{α} being computed by the preceding procedures.

Example 7. As an example, we shall get the precise value of $P = \pi^{\sqrt{2}}$. Taking logarithms of both members, and referring to the preceding values, we have

$$\log_e P = \sqrt{2} \log_e \pi$$

$$= 1.414\,213\,562{,}373\,095\,048{,}801\,688\,724$$

$$\times\ 1.144\,729\,885{,}849\,400\,174{,}143\,426\,439$$

$$= 1.618\,892\,529{,}822\,026\,668{,}464\,528\,924, \qquad = \alpha.$$

$$\alpha = n + \varepsilon = \frac{3}{2} + \frac{1}{9} + \frac{1}{128} - \frac{1}{32\,768} - \frac{1}{1\,835\,008} + \varepsilon,$$

$$\varepsilon = -\,0.0^6\,018{,}754\,207\,210{,}503\,725\,044.$$

$$e^n = \exp\left(\frac{3}{2} + \frac{1}{9} + \frac{1}{128} - \frac{1}{32\,768} - \frac{1}{1\,835\,008}\right)$$

$$= 5.047\,497\,362{,}032\,721\,660{,}552\,626\,122.$$

$$e^{\varepsilon} = 1 + \varepsilon + \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{6}$$

$$= 1 - 0.0^6\,018{,}754\,207\,210{,}503\,725\,044 + 175{,}860\,144\,048 - 1\,099$$

$$= 0.9^6\,981{,}245\,792\,965{,}356\,417\,905.$$

$$P = e^n \times e^{\varepsilon} = \pi^{\sqrt{2}} = 5.047\,497\,267{,}370\,911\,126{,}173\,634\,091.$$

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e^n–Table

n	n	e^n
2	2. 000	7. 389 056 098,930 650 227,230 427 461
$^3/_2$	1. 500	4. 481 689 070,338 064 822,602 055 460
1	1. 000	2. 718 281 828,459 045 235,360 287 471
$^3/_4$	0. 750	2. 117 000 016,612 674 668,545 369 820
$^1/_2$	0. 500	1. 648 721 270,700 128 146,848 650 788
$^3/_8$	0. 375	1. 454 991 414,618 201 336,053 793 692
$^1/_3$	0. 333 333 333,333 333 333,333 333 333	1. 395 612 425,086 089 528,628 125 319
$^1/_4$	0. 250	1. 284 025 416,687 741 484,073 420 568
$^1/_5$	0. 200	1. 221 402 758,160 169 833,921 071 995
$^3/_{16}$	0. 187 500	1. 206 230 249,420 980 710,655 586 010
$^1/_6$	0. 166 666 666,666 666 666,666 666 667	1. 181 360 412,865 645 980,305 112 152
$^1/_7$	0. 142 857 142,857 142 857,142 857 143	1. 153 564 994,895 107 753,461 339 625
$^1/_8$	0. 125	1. 133 148 453,066 826 316,829 007 228
$^1/_9$	0. 111 111 111,111 111 111,111 111 111	1. 117 519 068,741 863 648,622 059 717
$^1/_{10}$	0. 100	1. 105 170 918,075 647 624,811 707 827
$^3/_{32}$	0. 093 750	1. 098 285 140,307 825 848,650 209 934
$^1/_{11}$	0. 090 909 090,909 090 909,090 909 091	1. 095 169 439,874 664 284,656 241 323
$^1/_{12}$	0. 083 333 333,333 333 333,333 333 333	1. 086 904 049,521 228 888,638 279 697
$^1/_{14}$	0. 071 428 571,428 571 428,571 428 571	1. 074 041 430,716 295 856,924 373 890
$^1/_{16}$	0. 062 500	1. 064 494 458,917 859 429,563 390 595
$^1/_{20}$	0. 050	1. 051 271 096,376 024 039,697 517 636
$^3/_{64}$	0. 046 875	1. 047 991 002,016 632 702,276 738 292
$^1/_{22}$	0. 045 454 545,454 545 454,545 454 545	1. 046 503 435,194 870 387,414 574 926
$^1/_{24}$	0. 041 666 666,666 666 666,666 666 667	1. 042 546 905,189 991 386,335 842 113
$^1/_{28}$	0. 035 714 285,714 285 714,285 714 286	1. 036 359 701,414 666 092,729 721 995
$^1/_{32}$	0. 031 250	1. 031 743 407,499 102 670,938 747 815
$^1/_{40}$	0. 025	1. 025 315 120,524 428 840,678 021 030
$^3/_{128}$	0. 023 437 500	1. 023 714 316,602 357 916,968 850 533
$^1/_{44}$	0. 022 727 272,727 272 727,272 727 273	1. 022 987 504,906 521 518,269 099 826
$^1/_{48}$	0. 020 833 333,333 333 333,333 333 333	1. 021 051 862,145 107 390,399 522 115
$^1/_{56}$	0. 017 857 142,857 142 857,142 857 143	1. 018 017 534,924 947 200,307 354 762
$^1/_{64}$	0. 015 625	1. 015 747 708,586 685 747,458 535 073
$^1/_{80}$	0. 012 500	1. 012 578 451,540 634 376,676 921 551
$^3/_{256}$	0. 011 718 750	1. 011 787 683,559 331 491,514 113 844
$^1/_{88}$	0. 011 363 636,363 636 363,636 363 636	1. 011 428 447,744 338 224,409 788 839

n	n	e^n
1/96	0. 010 416 666,666 666 666,666 666 667	1. 010 471 109,010 597 784,146 745 853
1/112	0. 008 928 571,428 571 428,571 428 571	1. 008 968 550,017 763 041,428 317 680
1/128	0. 007 812 500	1. 007 843 097,206 447 977,693 453 560
1/160	0. 006 250	1. 006 269 572,003 762 010,144 984 985
3/512	0. 005 859 375	1. 005 876 574,714 478 322,766 682 988
1/176	0. 005 681 818,181 818 181,818 181 818	1. 005 697 990,325 295 531,466 644 204
1/192	0. 005 208 333,333 333 333,333 333 333	1. 005 221 920,279 595 665,834 949 221
1/224	0. 004 464 285,714 285 714,285 714 286	1. 004 474 265,483 075 002,521 322 048
1/256	0. 003 906 250	1. 003 913 889,338 347 573,443 609 604
1/320	0. 003 125	1. 003 129 887,902 739 148,639 306 275
3/1 024	0. 002 929 687,500	1. 002 933 983,228 446 757,953 008 292
1/352	0. 002 840 909,090 909 090,909 090 909	1. 002 844 948,297 240 779,129 044 445
1/384	0. 002 604 166,666 666 666,666 666 667	1. 002 607 560,454 037 103,823 952 579
1/448	0. 002 232 142,857 142 857,142 857 143	1. 002 234 635,942 639 433,246 463 762
1/512	0. 001 953 125	1. 001 955 033,591 002 812,046 518 898
1/640	0. 001 562 500	1. 001 563 721,339 156 307,921 160 852
3/2 048	0. 001 464 843,750	1. 001 465 917,157 666 808,009 877 319
1/704	0. 001 420 454,545 454 545,454 545 455	1. 001 421 463,868 855 159,225 868 898
1/768	0. 001 302 083,333 333 333,333 333 333	1. 001 302 931,411 886 511,622 675 996
1/896	0. 001 116 071,428 571 428,571 428 571	1. 001 116 694,468 052 227,399 671 589
1/1 024	0. 000 976 562,500	1. 000 977 039,492 416 535,242 845 293
1/1 280	0. 000 781 250	1. 000 781 555,255 269 634,169 066 129
3/4 096	0. 000 732 421,875	1. 000 732 690,161 397 099,950 751 232
1/1 408	0. 000 710 227,272 727 272,727 272 727	1. 000 710 479,543 836 474,570 439 063
1/1 536	0. 000 651 041,666 666 666,666 666 667	1. 000 651 253,640 291 259,502 726 621
1/1 792	0. 000 558 035,714 285 714,285 714 286	1. 000 558 191,445 181 376,862 810 137
1/2 048	0. 000 488 281,250	1. 000 488 400,478 694 473,126 173 624
1/2 560	0. 000 390 625	1. 000 390 701,303 880 390,166 043 156
3/8 192	0. 000 366 210,937 500	1. 000 366 278,000 911 574,085 842 186
1/2 816	0. 000 355 113,636 363 636,363 636 364	1. 000 355 176,696 675 306,693 811 009
1/3 072	0. 000 325 520,833 333 333,333 333 333	1. 000 325 573,820 989 173,006 328 510
1/3 584	0. 000 279 017,857 142 857,142 857 143	1. 000 279 056,786 245 713,538 476 715
1/4 096	0. 000 244 140,625	1. 000 244 170,429 747 854,937 005 234
1/5 120	0. 000 195 312,500	1. 000 195 331,574 728 152,193 169 310
3/16 384	0. 000 183 105,468 750	1. 000 183 122,233 579 571,457 356 971
1/5 632	0. 000 177 556,818 181 818,181 818 182	1. 000 177 572,582 326 656,268 346 455
1/6 144	0. 000 162 760,416 666 666,666 666 667	1. 000 162 773,662 861 925,759 229 682

n	n	e^n
$^1/_{7\,168}$	0. 000 139 508,928 571 428,571 428 571	1. 000 139 518,660 394 557,623 287 386
$^1/_{8\,192}$	0. 000 122 070,312 500	1. 000 122 077,763 383 771,076 503 520
$^1/_{10\,240}$	0. 000 097 656,250	1. 000 097 661,018 526 806,249 987 854
$^3/_{32\,768}$	0. 000 091 552,734 375	1. 000 091 556,925 454 486,389 489 607
$^1/_{11\,264}$	0. 000 088 778,409 090 909,090 909 091	1. 000 088 782,350 010 491,441 468 399
$^1/_{12\,288}$	0. 000 081 380,208 333 333,333 333 333	1. 000 081 383,519 792 315,986 470 888
$^1/_{14\,336}$	0. 000 069 754,464 285 714,285 714 286	1. 000 069 756,897 184 926,377 448 912
$^1/_{16\,384}$	0. 000 061 035,156 250	1. 000 061 037,018 933 045,421 779 121
$^1/_{20\,480}$	0. 000 048 828,125	1. 000 048 829,317 112 298,298 300 215
$^3/_{65\,536}$	0. 000 045 776,367 187 500	1. 000 045 777,414 941 383,836 928 725
$^1/_{22\,528}$	0. 000 044 389,204 545 454,545 454 545	1. 000 044 390,189 760 772,222 050 687
$^1/_{24\,576}$	0. 000 040 690,104 166 666,666 666 667	1. 000 040 690,932 020 183,657 631 603
$^1/_{28\,672}$	0. 000 034 877,232 142 857,142 857 143	1. 000 034 877,840 360 589,079 356 176
$^1/_{32\,768}$	0. 000 030 517,578 125	1. 000 030 518,043 791 024,295 451 285
$^1/_{40\,960}$	0. 000 024 414,062 500	1. 000 024 414,360 525 649,210 960 899
$^3/_{131\,072}$	0. 000 022 888,183 593 750	1. 000 022 888,445 530 222,523 482 656
$^1/_{45\,056}$	0. 000 022 194,602 272 727,272 727 273	1. 000 022 194,848 574 734,483 328 681
$^1/_{49\,152}$	0. 000 020 345,052 083 333,333 333 333	1. 000 020 345,259 045 309,018 451 585
$^1/_{57\,344}$	0. 000 017 438,616 071 428,571 428 571	1. 000 017 438,768 124 977,681 309 291
$^1/_{65\,536}$	0. 000 015 258,789 062 500	1. 000 015 258,905 478 413,948 140 044
$^1/_{81\,920}$	0. 000 012 207,031 250	1. 000 012 207,105 756 109,135 064 060
$^3/_{262\,144}$	0. 000 011 444,091 796 875	1. 000 011 444,157 280 743,328 546 060
$^1/_{90\,112}$	0. 000 011 097,301 136 363,636 363 636	1. 000 011 097,362 711 637,664 841 308
$^1/_{98\,304}$	0. 000 010 172,526 041 666,666 666 667	1. 000 010 172,577 781 985,143 956 897
$^1/_{114\,688}$	0. 000 008 719,308 035 714,285 714 286	1. 000 008 719,346 048 991,079 626 470
$^1/_{131\,072}$	0. 000 007 629,394 531 250	1. 000 007 629,423 635 154,471 743 185
$^1/_{163\,840}$	0. 000 006 103,515 625	1. 000 006 103,534 251 489,387 979 968
$^3/_{524\,288}$	0. 000 005 722,045 898 437,500	1. 000 005 722,062 269 373,356 979 944
$^1/_{180\,224}$	0. 000 005 548,650 568 181,818 181 818	1. 000 005 548,665 961 971,853 648 136
$^1/_{196\,608}$	0. 000 005 086,263 020 833,333 333 333	1. 000 005 086,275 955 891,022 240 882
$^1/_{229\,376}$	0. 000 004 359,654 017 857,142 857 143	1. 000 004 359,663 521 162,530 935 596
$^1/_{262\,144}$	0. 000 003 814,697 265 625	1. 000 003 814,704 541 591,866 050 788
$^1/_{327\,680}$	0. 000 003 051,757 812 500	1. 000 003 051,762 469 117,610 032 579
$^1/_{360\,448}$	0. 000 002 774,325 284 090,909 090 909	1. 000 002 774,329 132 534,859 008 257
$^1/_{393\,216}$	0. 000 002 543,131 510 416,666 666 667	1. 000 002 543,134 744 178,347 596 906
$^1/_{458\,752}$	0. 000 002 179,827 008 928,571 428 571	1. 000 002 179,829 384 753,192 151 058
$^1/_{524\,288}$	0. 000 001 907,348 632 812,500	1. 000 001 907,350 451 803,060 028 725

n	n	e^n
$^1/_{655\ 360}$	0. 000 001 525,878 906 250	1. 000 001 525,880 070 403,810 388 521
$^1/_{720\ 896}$	0. 000 001 387,162 642 045,454 545 455	1. 000 001 387,163 604 155,997 156 601
$^1/_{786\ 432}$	0. 000 001 271,565 755 208,333 333 333	1. 000 001 271,566 563 648,410 904 139
$^1/_{917\ 504}$	0. 000 001 089,913 504 464,285 714 286	1. 000 001 089,914 098 420,225 107 943
$^1/_{1\ 048\ 576}$	0. 000 000 953,674 316 406,250	1. 000 000 953,674 771 153 745 446 789
$^1/_{1\ 310\ 720}$	0. 000 000 762,939 453 125	1. 000 000 762,939 744 163,378 582 220
$^1/_{1\ 441\ 792}$	0. 000 000 693,581 321 022,727 272 727	1. 000 000 693,581 561 550,307 317 019
$^1/_{1\ 572\ 864}$	0. 000 000 635,782 877 604,166 666 667	1. 000 000 635,783 079 714,143 226 669
$^1/_{1\ 835\ 008}$	0. 000 000 544,956 752 232,142 857 143	1. 000 000 544,956 900 721,100 732 198
$^1/_{2\ 097\ 152}$	0. 000 000 476,837 158 203,125	1. 000 000 476,837 271 889,980 791 655
$^1/_{2\ 621\ 440}$	0. 000 000 381,469 726 562,500	1. 000 000 381,469 799 322,085 393 694
$^1/_{2\ 883\ 584}$	0. 000 000 346,790 660 511,363 636 364	1. 000 000 346,790 720 643,251 696 377
$^1/_{3\ 145\ 728}$	0. 000 000 317,891 438 802,083 333 333	1. 000 000 317,891 489 329,572 119 248
$^1/_{3\ 670\ 016}$	0. 000 000 272,478 376 116,071 428 571	1. 000 000 272,478 413 238,307 525 666
$^1/_{4\ 194\ 304}$	0. 000 000 238,418 579 101,562 500	1. 000 000 238,418 607 523,274 189 159
$^1/_{5\ 242\ 880}$	0. 000 000 190,734 863 281,250	1. 000 000 190,734 881 471,145 191 941
$^1/_{5\ 767\ 168}$	0. 000 000 173,395 330 255,681 818 182	1. 000 000 173,395 345 288,652 964 303
$^1/_{6\ 291\ 456}$	0. 000 000 158,945 719 401,041 666 667	1. 000 000 158,945 732 032,913 193 886
$^1/_{7\ 340\ 032}$	0. 000 000 136,239 188 058,035 714 286	1. 000 000 136,239 197 338,594 317 101
$^1/_{8\ 388\ 608}$	0. 000 000 119,209 289 550,781 250	1. 000 000 119,209 296 656,208 889 946
$^1/_{10\ 485\ 760}$	0. 000 000 095,367 431 640,625	1. 000 000 095,367 436 188,098 653 425

Note. The computation of preparing the above e^n-Table was carried out by two or three persons; each proceeded on independently, as he checked his own preceding step. Several persons were engaged in the proof-reading; and this work was done for the printed sheets.