A Note on the Coupled Free Bending and Torsional Vibrations of Beams

Shun-ya YOSHIDA

(Department of Civil Engineering, Faculty of Engineering, Shinshu University)

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Synopsis. In discussing lateral vibrations of beams it is always assumed that the beam vibrates in its plane of symmetry. If it is not the case, the lateral vibrations will usually be coupled with torsional vibrations.

This paper deals with the natural vibrations of beams in which the shear-center axis is not collinear with the centroidal axis. Fundamental expressions are derived from energy considerations which, in turn, are based upon assumed normal elastic deflection curves in bending and torsion. The Rayleigh-Ritz method is employed to determine the natural circular frequencies. Frequency equations thus obtained involve some dimensionless values which depend upon the various physical characteristics and the end conditions of the beam under consideration.

Introduction. Consider the natural vibrations of the beam as shown in Fig. 1 in which the longitudinal axis G which passes through the mass centers of the elementary sections is not collinear with the longitudinal axis C about which the beam tends to twist under the influence of an applied torsional couple.

This axis C, here we call it a shear-center axis[6], may be defined by the property that it is the only axis along which transverse loads applied to the beam will produce flexure without torsion.

These two axes just described are not usually collinear in most prismatic beams having nonsymmetrical sections, as well as in built-up beams whose structural members are not symmetrically placed.

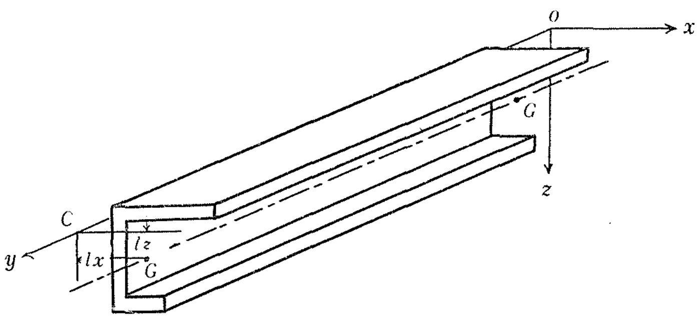

Fig. 1. Type of Beams in which Centroidal Axis is not Collinear with Shear-Center Axis.

The normal modes of vibration of such a beam involve simultaneous displacements in flexure and torsion. Accordingly, the natural frequencies of vibration in the several normal modes differ from the frequencies computed for vibration in pure flexure and pure torsion, respectively.[(1)(2)(3)(4)(5)]

These modes of vibration are discussed by Timoshenko[(1)], Clyne F. Garland[(7)] and the others[(8)], but it is assumed that the flexural rigidity of the beam, say, in the x–y plane is very much greater than that in the y–z plane. Thus, the x component of the motion is neglected and the total motion is considered to be composed of the z and θ components. This assumption may be reasonable when the beams are much stiffer in one plane than in the other. But in general types of beam as used in rigid frames, bridge trusses etc., it is considered that the above assumption may not always be reasonable.

In the following discussion, the writer deals with beams which may be deflected in arbitrary directions.

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(1) The Expressions for Energies

In the free vibrations of a beam of the type shown in Fig. 1, the displacement of any section is considered to be the resultant of the following three components.

$$\left.\begin{aligned} z&=(a_1Y_1+a_2Y_2+a_3Y_3+\cdots\cdots)\sin p_nt,\\ \theta&=(\varphi_1Y_1'+\varphi_2Y_2'+\varphi_3Y_3'+\cdots\cdots)\sin p_nt,\\ x&=(b_1Y_1''+b_2Y_2''+b_3Y_3''+\cdots\cdots)\sin p_nt,\end{aligned}\right\}\quad\cdots\cdots(1)$$

where z denotes the vertical component of displacement at any section; θ denotes the angular displacement at any section; x denotes the horizontal component of displacement at any section; Y_1, $Y_2,\cdots\cdots$, Y'_1, $Y'_2,\cdots\cdots$, Y''_1, $Y''_2,\cdots\cdots$ are functions of y which satisfy the end conditions for any particular beam; and a_1, a_2, $a_3,\cdots\cdots$, φ_1, φ_2, $\varphi_3,\cdots\cdots$, b_1, b_2, $b_3,\cdots\cdots$ are the amplitudes of the respective functions Y_n.

As the energies of vibration we take potential energy and kinetic energy relating to bending and twisting of the beam, neglecting the effects of axial tension, compression or shear.

Then the potential energy of the beam may be expressed as the sum of the energies stored in the beam due to the displacements z, θ, x, respectively, or

$$V=\frac{1}{2}K_x\int_0^l\left(\frac{\partial^2 z}{\partial y^2}\right)^2dy+\frac{1}{2}C\int_0^l\left(\frac{\partial\theta}{\partial y}\right)^2dy+\frac{1}{2}K_z\int_0^l\left(\frac{\partial^2 x}{\partial y^2}\right)^2dy,\quad\cdots\cdots(2)$$

where V is the total potential energy stored in the beam, K_x being the flexural rigidity about x axis, K_z being the flexural rigidity about z axis, C being the torsional rigidity and l being the length of the beam.

Substituting expressions (1) into expression (2), the potential energy is written as

$$V=\frac{1}{2}\left\{K_x \Sigma\, a_i^2 Q_i + C\,\Sigma\, \varphi_i^2 R_i + K_z\, \Sigma\, b_i^2 S_i\right\}\sin^2 p_n t, \quad \cdots\cdots(2a)$$

where Q_i, R_i and S_i denote the integrals $\int_0^l\left(\frac{d^2Y_i}{dy^2}\right)^2 dy$, $\int_0^l\left(\frac{dY_i'}{dy}\right)dy$ and $\int_0^l\left(\frac{d^2Y_i''}{dy^2}\right)^2 dy$ respectively.

Or still more briefly

$$V=\frac{1}{2}\beta\sin^2 p_n t, \quad \cdots\cdots(2b)$$

where β denotes the quantity within the brace in equation (2a).

The kinetic energy of the element is expressed as the sum of the kinetic energy due to the translation of the mass center and that due to the rotation about the mass center.

Thus, integrating over the length of the beam, the expression for the total kinetic energy becomes

$$T=\frac{1}{2}\rho A\int_0^l \dot{z}_G^2\, dy+\frac{1}{2}\rho I_G\int_0^l \dot{\theta}^2 dy+\frac{1}{2}\rho A\int_0^l \dot{x}_G^2\, dy, \quad \cdots\cdots(3)$$

where T is the total kinetic energy at the displacement z, θ and x, ρ being the mass density of the material of which the beam is made, A being the cross-sectional area of the beam, $\dot{z}_G$ and $\dot{x}_G$ being z component and x component of the velocity of the mass center, $\dot{\theta}$ being the instantaneous angular velocity of rotation of the section and I_G being the polar moment of inertia of the section with respect to the gravity axis.

For small values of θ

$$\left.\begin{aligned} z_G&=z+e_x\theta, \\ x_G&=x+e_z\theta. \end{aligned}\right\} \quad \cdots\cdots(4)$$

Hence, by differentiating equation (4) with respect to time

$$\left.\begin{aligned} \dot{z}_G&=\dot{z}+e_x\dot{\theta}, \\ \dot{x}_G&=\dot{x}+e_z\dot{\theta}. \end{aligned}\right\} \quad \cdots\cdots(5)$$

Substituting values of equation (5) into equation (3) the expression for the kinetic energy becomes

$$T=\frac{1}{2}\rho A\left[\int_0^l(\dot{z}^2+2e_x\dot{z}\dot{\theta}+e_x^2\dot{\theta}^2)dy+\frac{I_G}{A}\int_0^l\dot{\theta}^2 dy + \int_0^l(\dot{x}^2+2e_z\dot{x}\dot{\theta}+e_z^2\dot{\theta}^2)dy\right]. \quad \cdots\cdots(3a)$$

From equation (1), by differentiation

$$\left.\begin{aligned} \dot{z}&=(a_1Y_1+a_2Y_2+a_3Y_3+\cdots\cdots)\,p_n\cos p_n t, \\ \dot{\theta}&=(\varphi_1Y_1'+\varphi_2T_2'+\varphi_3Y_3'+\cdots\cdots)\,p_n\cos p_n t, \\ \dot{x}&=(b_1Y_1''+b_2Y_2''+b_3Y_3''+\cdots\cdots)\,p_n\cos p_n t. \end{aligned}\right\} \quad \cdots\cdots(1a)$$

Substituting these values into equation (3a)

$$T=\frac{1}{2}\rho A\Big\{\Big[\Sigma a_i^2U_i-2\Sigma a_ia_kU_{ik}\Big]+2e_x\Sigma a_i\varphi_kU_{ik'}$$
$$+\Big[(e_x^2+e_z^2+\frac{I_G}{A})(\Sigma\varphi_i^2U_{i'}+2\Sigma\varphi_i\varphi_kU_{i'k'})\Big]$$
$$+\Big[\Sigma b_i^2U_{i'}+2\Sigma b_i\,b_k\,U_{i'k'}\Big]+2e_z\,\Sigma\varphi_ib_kU_{i'k''}\Big\}p_n{}^2\cos^2p_n\,t,\quad\cdots\cdots\cdots\cdots(3b)$$

where U_i, $U_{i'}$, $U_{i''}$, ………denote the integrals $\int_0^l Y_i^2\,dy$, $\int_0^l Y_i'^2dy$, $\int_0^l Y_i''^2dy$, ……… respectively and U_{ik}, $U_{ik'}$, $U_{i'k''}$, ……… denote the integrals $\int_0^l Y_iY_kdy$, $\int_0^l Y_iY_k'dy$, $\int_0^l Y_i'Y_k''dy$, ……… respectively.

Or, more briefly

$$T=\frac{1}{2}\alpha p_n^2\cos^2p_n\,t,\quad\cdots\cdots\cdots\cdots(3c)$$

where α denotes ρA times the quantity within the brace in equation (3b).

Here we apply the Rayleigh-Ritz method to evaluate the natural frequencies of vibration. Equating the maximum values of the potential and kinetic energies, as obtained from equations (2b) and (3c), respectively, and solving for the frequency

$$p_n^2=\frac{\beta}{\alpha}.\quad\cdots\cdots\cdots\cdots(6)$$

Therefore, the values of p_n^2 obtained from equation (6) depend upon the assumed elastic curves of the beam in motion. If the assumed elastic curve is not the exact one, the lowest computed value of natural frequency will be higher than the true fundamental frequency of the beam. Or, to state it a little differently, if several elastic curves are assumed in succession, the one yielding the lowest value of frequency is nearest correct.

In order to obtain the closest approximation possible, the coefficients a_1, a_2, …, φ_1, φ_2, …and b_1, b_2, …must be so chosen that the fundamental frequency computed from equation (6) be a minimum. This value of p_n^2 may be found by equating to zero the partial derivative of β/α with respect to each of the coefficients a_1, a_2, ………, φ_1, φ_2, ……… and b_1, b_2, …respectively. Thus the following simultaneous equations will be obtained.

$$\left.\begin{array}{l}\frac{\partial\beta}{\partial a_1}-p_n^2\frac{\partial\alpha}{\partial a_1}=0,\\ \frac{\partial\beta}{\partial a_2}-p_n^2\frac{\partial\alpha}{\partial a_2}=0,\\ \cdots\cdots\cdots\cdots\\ \frac{\partial\beta}{\partial\varphi_1}-p_n^2\frac{\partial\alpha}{\partial\varphi_1}=0,\\ \frac{\partial\beta}{\partial\varphi_2}-p_n^2\frac{\partial\alpha}{\partial\varphi_2}=0,\\ \cdots\cdots\cdots\cdots\\ \frac{\partial\beta}{\partial b_1}-p_n^2\frac{\partial\alpha}{\partial b_1}=0,\end{array}\right\}\quad\cdots\cdots\cdots\cdots(7)$$

$$\frac{\partial \beta}{\partial b_2} - p_n^2 \frac{\partial \alpha}{\partial b_2} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These equations are seen to be homogeneous and linear in the coefficients a_1, a_2, ······ φ_1, φ_2, ······ and b_1, b_2, ······, and are equal in number to the number of coefficients. By setting the determinant of these equations equal to zero, eliminating the coefficients, and expanding, the frequency equation may be obtained. The frequency equation yields the values of the natural frequencies in the several normal modes.

(2) The Frequency Equation for Three Normal Modes.

In applying the same methed as used in this analysis to the determination of the natural frequencies of a vibrating string, Timoshenko[(1)] shows us that when only one term is used to express the assumed elastic curve, the computed value for the fundamental frequency differs from the exact value by 0.66 per cent. In another example ······ the case of a vibrating wedge of constant width, with the thick end built in and the other end free ······ Timoshenko[(1)] also demonstrates that the error in the computed value of the fundamental frequency is about 3 per cent when only one term is used to express the elastic curve. Therefore it is seen that the method gives very satisfactory results, even when only one term is used. Moreover, it should be recognized that when a larger number of terms is used, the extra labour of computation may not be worth the increased accuracy which will be gained. Thus we express the instantaneous displacement of the beam by the components

$$\left.\begin{aligned} z &= a_1 Y_1 \sin p_n t, \\ \theta &= \varphi_1 Y_1' \sin p_n t, \\ x &= b_1 Y_1'' \sin p_n t. \end{aligned}\right\} \qquad \cdots\cdots (1b)$$

From equation (2b)

$$\beta = K_x a_1^2 Q_1 + C\varphi_1^2 R_1 + K_z b_1^2 S_1, \qquad \cdots\cdots (8)$$

and from equation (3c)

$$\alpha = \rho A\Big[a_1^2 U_1 + 2e_x a_1 \varphi_1 U_{11'} + \left(e_x^2 + e_z^2 + \frac{I_G}{A}\right) \varphi_1^2 U_{1'} + b_1^2 U_{1''} + 2e_z \varphi_1 b_1 U_{1'1''}\Big] \qquad \cdots\cdots (9)$$

Taking partial derivatives with respect to a_1, φ_1 and b_1, respectively, and denoting

$$\eta = \frac{U_{1'}}{U_1}, \quad \gamma = \frac{U_{11'}}{U_1}, \quad \phi = \frac{U_{1''}}{U_1}, \quad \xi = \frac{U_{1'1''}}{U_1}, \qquad \cdots\cdots (10)$$

the following expressions are obtained.

$$\frac{\partial \beta}{\partial a_1} = 2K_x Q_1 a_1, \quad \frac{\partial \beta}{\partial \varphi_1} = 2CR_1 \varphi_1, \quad \frac{\partial \beta}{\partial b_1} = 2K_z S_1 b_1, \quad \frac{\partial \alpha}{\partial a_1} = 2\rho A U_1 [a_1 + e_x \gamma \varphi_1],$$

$$\frac{\partial \alpha}{\partial \varphi_1}=2\rho AU_1\ [e_x\gamma a_1+e_z\xi b_1+\ (e_x^2+e_z^2+\frac{I_G}{A})\eta\varphi_1],$$

$$\frac{\partial \alpha}{\partial b_1}=2\rho AU_1\ [e_z\xi\varphi_1+\psi b_1].$$

Substituting these terms into equation (7) and rearranging into determinant form

$$\begin{vmatrix} 2[K_xQ_1-\rho AU_1p_n^2]\ a_1 & -2\rho AU_1e_x\gamma p_n^2\varphi_1 & 0 \\ -2\rho AU_1e_x\gamma p_n^2a_1 & 2[CR_1-\rho AU_1\ (e_x^2+e_z^2+\frac{I_G}{A})p_n^2\eta]\varphi_1 & -2\rho AU_1e_z\xi p_n^2b_1 \\ 0 & -2\rho AU_1e_z\xi p_n^2\varphi_1 & 2[K_zS_1-\rho AU_1\psi p_n^2]b_1 \end{vmatrix}=0. \quad \cdots(11)$$

Dividing the first column by $2\rho AU_1a_1$, the second column by $2\rho AU_1\varphi_1$ and the third column by $2\rho AU_1b_1$, the determinant is reduced to the form

$$\begin{vmatrix} \frac{K_xQ_1}{\rho AU_1}-p_n^2 & -e_x\gamma p_n^2 & 0 \\ -e_x\gamma p_n^2 & \frac{CR_1}{\rho AU_1}-(e_x^2+e_z^2+\frac{I_G}{A})\eta p_n^2 & -e_z\xi p_n^2 \\ 0 & -e_z\xi p_n^2 & \frac{K_zS_1}{\rho AU_1}-\psi p_n^2 \end{vmatrix}=0. \quad \cdots\cdots(11a)$$

Now the quantity $\left[\frac{K_xQ_1}{\rho AU_1}\right]^{\frac{1}{2}}$ expresses the natural frequency of flexural vibration for a uniform beam. This quantity is therefore the natural frequency of the beam under consideration for the particular case in which e_x and e_z are zeros. Then the natural frequency for this particular case may be denoted by p_0 and the relationship

$$p_0^2=\frac{K_xQ_1}{\rho AU_1} \quad \cdots\cdots(12)$$

may be substituted into equation (11a) and each term in the determinant divided by p_0^2.

After this operation is performed and the determinant is expanded, simplified and rearranged, the following cubic equation is obtained.

$$\left[e_x^2\gamma^2\psi+e_z^2\xi^2-\left(e_x^2+e_z^2+\frac{I_G}{A}\right)\eta\psi\right]\left(\frac{p_n^2}{p_0^2}\right)^3+\left[\frac{CR_1}{K_xQ_1}\psi-\frac{K_zS_1}{K_xQ_1}e_x^2\gamma^2-e_z^2\xi^2\right.$$
$$\left.+\left(e_x^2+e_z^2+\frac{I_G}{A}\right)\left(\psi+\frac{K_zS_1}{K_xQ_1}\right)\eta\right]\left(\frac{p_n^2}{p_0^2}\right)^2-\left[\frac{CR_1}{K_xQ_1}\psi+\frac{CR_1K_zS_1}{(K_xQ_1)^2}\right.$$
$$\left.+\left(e_x^2+e_z^2+\frac{I_G}{A}\right)\frac{K_zS_1}{K_xQ_1}\eta\right]\left(\frac{p_n}{p_0}\right)+\frac{CR_1K_zS_1}{(K_xQ_1)^2}=0. \quad \cdots\cdots(13)$$

Furthermore we introduce the following additional dimensionless quantities

$$\lambda=\frac{CR_1A}{K_xQ_1I_G\eta},\quad \varepsilon_x=\frac{e_x^2A}{I_G},\quad \varepsilon_z=\frac{e_z^2A}{I_G},\quad \kappa=\frac{1}{\psi}\frac{K_zS_1}{K_xQ_1}. \quad \cdots\cdots(14)$$

After all the frequency equation is

$$\left[1+\varepsilon_x\left(1-\frac{r^2}{\eta}\right)+\varepsilon_z\left(1-\frac{\xi^2}{\phi\eta}\right)\right]\left(\frac{p_n^2}{p_0^2}\right)^3-\left[1+\lambda+\kappa+\varepsilon_x\left(1+\kappa-\frac{r^2}{\eta}\kappa\right)\right.$$
$$\left.+\varepsilon_z\left(1+\kappa-\frac{\xi^2}{\phi\eta}\right)\right]\left(\frac{p_n^2}{p_0^2}\right)^2+\left[(1+\kappa)\lambda+(1+\varepsilon_x+\varepsilon_z)\kappa\right]\left(\frac{p_n^2}{p_0^2}\right)-\lambda\kappa=0, \quad \cdots\cdots\cdots(15)$$

and its three roots will be readily obtained as follows.

Let
$$A_1=\left[1+\varepsilon_x\left(1-\frac{r^2}{\eta}\right)+\varepsilon_z\left(1-\frac{\xi^2}{\phi\eta}\right)\right],$$
$$A_2=-\left[1+\lambda+\kappa+\varepsilon_x\left(1+\kappa-\frac{r^2}{\eta}\kappa\right)+\varepsilon_z\left(1+\kappa-\frac{\xi^2}{\phi\eta}\right)\right],$$
$$A_3=\left[(1+\kappa)\lambda+(1+\varepsilon_x+\varepsilon_z)\kappa\right],$$
$$A_4=-\lambda\kappa,$$
$$p=\frac{1}{9A_1^2}(-A_2^2+3A_1A_3),$$
$$q=\frac{1}{54A_1^3}(2A_2^3-9A_1A_2A_3+27A_1^2A_4).$$

Case (i): when $q^2+p^3>0$.

$$\frac{p_1^2}{p_0^2}=u+v-\frac{A_2}{3A_1},\quad \frac{p_2^2}{p_0^2}=u\omega_1+v\omega_2-\frac{A_2}{3A_1},\quad \frac{p_3^2}{p_0^2}=u\omega_2+v\omega_1-\frac{A_2}{3A_1},$$

where $u=\sqrt[3]{-q+\sqrt{q^2+p^3}}$, $v=\sqrt[3]{-q-\sqrt{q^2+p^3}}$.

Case (ii) : when $q^2+p^3=0$.

$$\frac{p_1^2}{p_0^2}=2\sqrt[3]{-q}-\frac{A_2}{3A_1},\quad \frac{p_2^2}{p_0^2}=\frac{p_3^2}{p_0^2}=-\sqrt[3]{-q}.$$

Case (iii) : when $q^2+p^3<0$.

$$\frac{p_1^2}{p_0^2}=2\sqrt{-p}\cos\left(\frac{u}{3}\right)-\frac{A_2}{3A_1},$$
$$\frac{p_2^2}{p_0^2}=2\sqrt{-p}\cos\left(\frac{u}{3}+\frac{2\pi}{3}\right)-\frac{A_2}{3A_1},$$
$$\frac{p_3^2}{p_0^2}=2\sqrt{-p}\cos\left(\frac{u}{3}+\frac{4\pi}{3}\right)-\frac{A_2}{3A_1},$$

where $\cos u=\frac{q}{p\sqrt{-p}}$, when $0<u<\pi$.

(3) The Frequency Equations for Special Cases

(i) When the beam has one plane of symmetry.

For instance, when the cross section of the beam is symmetrical about x-axis (Fig. 2), put $\varepsilon_z=0$ in equation (15). Then the frequency equation is

$$\left[1+\varepsilon_x\left(1-\frac{r^2}{\eta}\right)\right]\left(\frac{p_n^2}{p_0^2}\right)^3-\left[1+\lambda+\kappa+\varepsilon_x\left(1+\kappa-\frac{r^2}{\eta}\kappa\right)\right]\left(\frac{p_n^2}{p_0^2}\right)^2$$
$$+\left[(1+\kappa)\lambda+(1+\varepsilon_x)\kappa\right]\left(\frac{p_n^2}{p_0^2}\right)-\lambda\kappa=0. \quad \cdots\cdots\cdots(16)$$

(ii) In case (i), when K_z is very much greater than K_x.
Dividing equation (16) by κ and putting $\kappa=\infty$

$$\left[1+\varepsilon_x\left(1-\frac{\gamma^2}{\eta}\right)\right]\left(\frac{p_n^2}{p_0^2}\right)^2-(1+\lambda+\varepsilon_x)\left(\frac{p_n^2}{p_0^2}\right)+\lambda=0. \quad\cdots\cdots(17)$$

This equation is the same one as has been obtained by Garland.[7]

(iii) When the shear-center axis is collinear with the centroidal axis (Fig. 3).
Putting $\varepsilon_x=\varepsilon_z=0$ in equation (15)

$$\left(\frac{p_n^2}{p_0^2}\right)^3-(1+\lambda+\kappa)\left(\frac{p_n^2}{p_0^2}\right)^2+\left[(1+\kappa)\lambda+\kappa\right]\left(\frac{p_n}{p_0}\right)-\lambda\kappa=0. \quad\cdots\cdots(18)$$

Fig. 2.

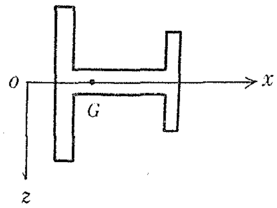
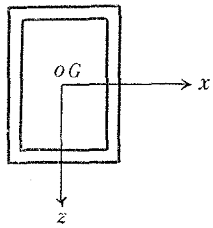


Fig. 3.



(4) Discussions about the Frequencies

It is noted that the three roots of equation (18) are 1.0, λ and κ respectively, or

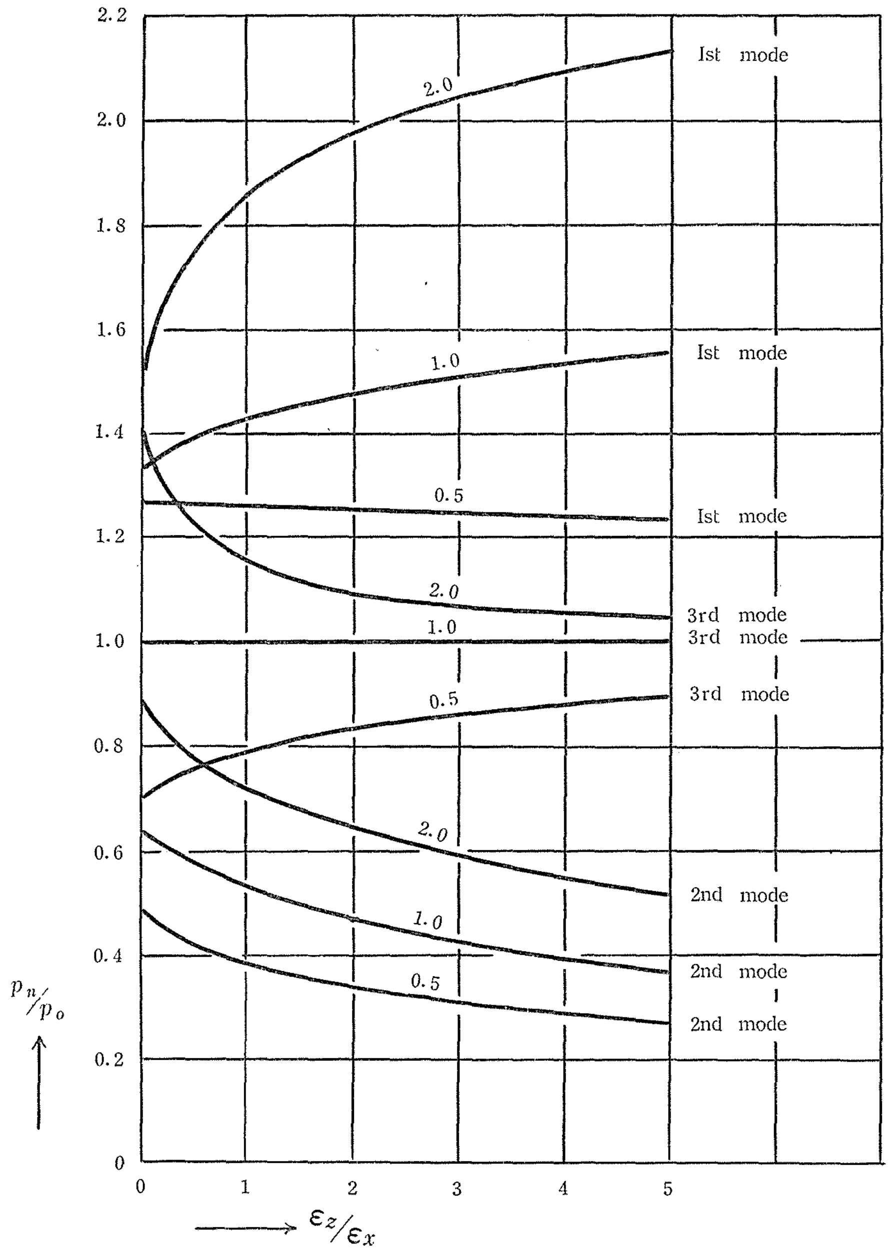
$$p_1=p_0=\left(\frac{K_xQ_1}{\rho AU_1}\right)^{\frac{1}{2}},\quad p_2=p_0\lambda^{\frac{1}{2}}=\left(\frac{CR_1}{\rho I_G\eta U_1}\right)^{\frac{1}{2}},\quad p_3=p_0\kappa^{\frac{1}{2}}=\left(\frac{K_zS_1}{\rho A\phi U_1}\right)^{\frac{1}{2}}.$$

This is explained by the fact that in the actual beam, if the shear-center axis is collinear with the centroidal axis, two of the normal modes of vibration are those of pure flexure, and the other one is that of pure torsion. Thus, it will be inferred that the frequencies of a beam in which the effects of eccentricity are not neglected differ from those in pure flexure or in pure torsion. That is to say, the natural frequencies of beams in which the shear-center axis is not collinear with the centroidal axis depend upon the distance of these two axes. This will be shown in a numerical example as follows. Omitting the process of calculations, only the results are presented graphycally in Fig. 4.

(5) Summary

The natural frequencies of the beam in which the shear-center axis is not collinear with the centroidal axis are shown to differ from those in pure flexural or pure torsional vibrations, and the normal mode of vibration of this beam consists of simultaneous vibrations in flexure and torsion. Thus,

Fig. 4. Relations between p_n/p_0 and $\varepsilon_z/\varepsilon_x$ when $\eta=1$, $\gamma=0.8$, $\kappa=\lambda$. Numbers on Curves Denote Values of λ.



it is seen that computations of the natural frequencies of such a beam, in which the effects of the eccentricity are neglected, are apt to lead eroneous results.

When the type of beam, end conditions and load distribution are known, the values of natural frequency of the beam can be obtained from the frequency equation (15). Higher degree of accuracy will be attained by using a sufficient number of terms in the expressions (1), but, as mentioned previously, for most practical problems satisfactory values of frequency may be obtained by using only one or two terms. The absolute amplitudes are of course arbitrary since they depend upon the initial displacement of the beam, but the amplitude ratios will be found by substituting the values of natural frequency obtained from the frequency equation into equation (11).

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